A random copy of a segment within a domain

N.G. Aharonyan, H.S. Harutyunyan, V.K. Ohanyan

Yerevan State University, Armenia
E-mail: *narine78@ysu.am, hrach87@web.am, victo@aua.am*

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Let S be a segment of length l and $\mathbf{D}$ be a bounded convex domain in the Euclidean plane $\mathbf{R}^2$. We consider a randomly moving copy $L(\omega)$ of S, under the condition that it hits $\mathbf{D}$, that is a random segment from the following set:

$$\Omega = \{\text{the set of all positions of the segment } S \text{ intersecting domain } \mathbf{D}\}.$$

Denote by $|L|(\omega)$ $(\omega \in \Omega)$ the length of $L(\omega) \cap \mathbf{D}$. A random segment $L(\omega)$ we can specified by the coordinates (g, t), where $g \in \mathbf{G}$ ($\mathbf{G}$ is the space of all lines in $\mathbf{R}^2$) is the line containing the line L, while t is one dimensional coordinate of the center of L on the line g. Note, that the origin on the line g is one of the intersection point of g with domain $\mathbf{D}$. The set of all possible positions of the segment $L(\omega)$ in (g, t)-coordinates is the following domain:

$$\Omega = \{(g,t) : L(g,t) \cap \mathbf{D} \neq \emptyset\} = \left\{(g,t) : g \in [\mathbf{D}], t \in \left[-\frac{l}{2}, \chi(g) + \frac{l}{2}\right]\right\},$$

where $\chi(g) = g \cap \mathbf{D}$ is the chord of intersection of line g with the domain $\mathbf{D}$, and

$$[\mathbf{D}] = \{g \in \mathbf{G} : g \cap \mathbf{D} \neq \emptyset\}.$$

In the space $\mathbf{G} \times \mathbf{R}$ ($\mathbf{R}$ is the real axis) we determine the measure m in the following way:

$$m(dL) = dg\, dt,$$

where dg is the locally finite measure on the space $\mathbf{G}$ which invariant with respect to the group of all Euclidean motions (translations and rotations), while dt is one dimensional Lebesgue measure on g. Measure $m(\cdot)$ is called the kinematic measure in the group of all Euclidean motions in the plane $\mathbf{R}^2$ (see [1]—[3]). A random segment in Ω is one with distribution proportional to restriction of m on Ω. Therefore

$$P(B) = \frac{m(B)}{m(\Omega)} \qquad \text{for any Borel set} \qquad B \subset \Omega.$$

Furthermore, let

$$B_{\mathbf{D}}^{x} = \{(g,t) \in \Omega : \ |L|(g,t) \le x\}, \ \ x \in \mathbf{R}.$$

The distribution function of $|L|$ has the following form:

$$F_{|L|}(x) = P(|L| \le x) = \frac{m(B_{\mathbf{D}}^{x})}{m(\Omega)} = \frac{1}{m(\Omega)} \int_{B_{\mathbf{D}}^{x}} dg\, dt.$$

We have (see [1]—[3])

$$m(\Omega) = \int_{[\mathbf{D}]} dg \int_{-\frac{l}{2}}^{\chi(g)+\frac{l}{2}} dt = \int_{[\mathbf{D}]} (\chi(g) + l)\, dg = \pi\, ||\mathbf{D}|| + l\, |\partial \mathbf{D}|,$$

where $||\mathbf{D}||$ is the area of $\mathbf{D}$, while $|\partial \mathbf{D}|$ is the perimeter of $\mathbf{D}$ (i.e. the length of the boundary of $\mathbf{D}$).
It is obvious that $F_{|L|}(x) = 0$, if $x \le 0$ and $F_{|L|}(x) = 1$, if $x \ge l$. In the case $0 \le x < l$, we have

$$F_{|L|}(x) = \frac{1}{m(\Omega)} \left[\int_{\chi(g) \le x} \chi(g)\, dg + l\, |\partial D|\, F(x) + 2x\, |\partial D|\, (1 - F(x)) \right],$$

where $F(x)$ is the chord length distribution function for the domain $\mathbf{D}$.
By definition (see [2] and [10]):

$$F(x) = \frac{1}{|\partial D|} \int_{\chi(g) \le x} dg.$$

In [10] an elementary expression for the chord length distribution function of a regular polygon are obtained. Similar expressions in the cases of a regular triangle, a square, a regular pentagon, and a regular hexagon have been obtained in [4] — [7], respectively.

Hence, we have derived the following formula for the distribution function for random variable $|L(\omega)|$.

Theorem 1. *Distribution function of the random variable $|L(\omega)|$ has the following form:*

$$F_{|L|}(x) = \begin{cases} 0 & \textit{if} \quad x \le 0 \\ \frac{|\partial \mathbf{D}|}{\pi\, ||\mathbf{D}|| + +l\, |\partial \mathbf{D}|} \left[\int_0^x u f(u)\, du + \right. & \\ \left. + l\, F(x) + 2x\, (1 - F(x)) \right] & \textit{if} \quad 0 < x < l \\ 1 & \textit{if} \quad x \ge l \end{cases} \qquad (1)$$

where $f(x)$ is the chord length density function of $\mathbf{D}$ (that is $f(x) = F'(x)$).

Therefore, (1) gives a relation between distribution functions $F(x)$ and $F_{|L|}(x)$.

It is obvious that at point α if we add the value of $F_L(l)$ and the jump $J(l)$ of the function at the point l, we get $1 = F_L(l) + J(l)$. In the present paper we present the graphs of $F_{|L|}(x)$ for regular n-gons (for $n = 3, 4, 5, 6, 7$) with the side equal to 1 and $l = 1$.

It is not difficult to verify (see formula (1)) that if l tends to ∞ distribution function $F_{|L|}(x)$ tends to $F(x)$. Moreover, if in formula (1) we substitute $x = 0$ we obtain $F_L(0) = 0$, that is $F_{|L|}(x)$ is continuous at $x = 0$. Nevertheless, $F_{|L|}(x)$ can be discontinuous at point $x = l$ and the corresponding jump of $F_{|L|}$ is equal to

$$J(l) = \frac{1}{m(\Omega)} \int_{\chi(g)>l} (\chi(g) - l)\, dg.$$

Therefore, at the point $x = l$ the function $F_{|L|}(x)$ has a jump

$$J(l) = \frac{1}{m(\Omega)} \left[\pi\, ||\mathbf{D}|| - G(l) - l\, |\partial D|\, (1 - F(l))\right].$$

Note that several models of using explicit forms of distribution function $F(x)$ in crystallography see works [8], [9].

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