Iterative Scheme for Elliptic Optimal Multiple Switching Problem

R.H. Barkhudaryan, T.L. Gharibyan, M.P. Poghosyan

Institute of Mathematics NAS Armenia
Yerevan State University
E-mail: *rafayel@instmath.sci.am, gtatev@instmath.sci.am, michael@ysu.am*

Key Words: Iterative scheme, Free boundary, optimal switching
Mathematics Subject Classification 2000: 65M12

Introduction and main results. The paper considers so-called *elliptic optimal switching problem*, which is the elliptic counterpart of parabolic optimal switching problem, considered in [1]. The problem can be described as a system of two interconnected obstacle-type elliptic problems in a given bounded domain $\Omega \subset \mathbb{R}^n$: find a pair (u, v) satisfying

$$\begin{cases} \min\Big\{ -\Delta u - f_1; u - \big(v - \psi_1\big)\Big\} = 0, & in \quad \Omega \\ \min\Big\{ -\Delta v - f_2; v - \big(u - \psi_2\big)\Big\} = 0, & in \quad \Omega \\ u = g_1 \quad and \quad v = g_2 & on \quad \partial\Omega \end{cases} \tag{1}$$

where $f_1, f_2, \psi_1, \psi_2 : \Omega \to \mathbb{R}$ are strictly positive functions and $g_1, g_2 : \partial\Omega \to \mathbb{R}$. We assume that ψ_1 and ψ_2 are continuously differentiable in Ω.

Our purpose is to construct a sequence of functions which will converge to the viscosity solution of (1). We start by u_0, v_0, which are the solutions of the following Dirichlet problems:

$$\begin{cases} -\Delta u_0 - f_1 = 0, \\ u_0\big|_{\partial\Omega} = g_1, \end{cases} \qquad and \qquad \begin{cases} -\Delta v_0 - f_2 = 0, \\ v_0\big|_{\partial\Omega} = g_2 : \end{cases} \tag{2}$$

Then we define u_k, v_k inductively by the following recurrent relations

$$\begin{cases} \min\{(-\Delta u_k - f_1), (u_k - (v_{k-1} - \psi_1))\} = 0, & u_k\big|_{\partial\Omega} = g_1, \\ \min\{(-\Delta v_k - f_2), (v_k - (u_k - \psi_2))\} = 0, & v_k\big|_{\partial\Omega} = g_2. \end{cases} \tag{3}$$

[1] *The authors thanks Goran Gustafsson foundation for visit appointment to KTH.*

As it follows from the definition of u_k and v_k, in every step we are solving obstacle problem with known obstacle: in the k-th step we solve the obstacle problem with $v_{k-1} - \psi_1$ as an obstacle, and this gives us u_k, then, we solve another obstacle problem with already known obstacle $u_k - \psi_2$ to find the value of v_k.

The main result of the paper states that the constructed sequence converges to the viscosity solution of (1):

Theorem 1. *The sequences* $\{u_k\}$ *and* $\{v_k\}$ *defined by* (3) *are increasing and bounded sequences and if denote by* u *and* v *the pointwise limits of* $\{u_k\}$ *and* $\{v_k\}$*, respectively, then the pair* (u, v) *is a viscosity solutions of* (1).

Numerical Results.

The following two examples (one and two dimensional, respectively) were solved numerically by the proposed iterative scheme using finite difference discretization and PSOR (Projected Successive Over-Relaxation) algorithm.

Example 1. We consider the following problem:

$$\begin{cases} \min\Big\{u(x) - \big(v(x) - 1\big); -u''(x) - \psi_1(x)\Big\} = 0, \quad x \in (0,1) \\ \min\Big\{v(x) - \big(u(x) - 1\big); -v''(x) - \psi_2(x)\Big\} = 0, \quad x \in (0,1) \\ u(0) = 0.7,\ u(1) = 0.2, \quad v(0) = 0,\ v(1) = 0.5, \end{cases} \tag{4}$$

where

$$\psi_1(x) = 70 \sin 6x \quad \text{and} \quad \psi_2(x) = 20\left(x^2 - \frac{1}{2}\right).$$

The iterative algorithm gives the solution for (4) represented in Figure 1. The graph of the function u is the orange solid line, and the orange dotted line is the obstacle for the second equation - that is, $u - 1$. The blue solid line represents the graph of v, and the blue dotted line is $v - 1$.

It is easy to see on this picture, that the solutions u and v consist of two parts: the first part, where they touch the dotted obstacle $v - 1$ or $u - 1$ (the so-called *coincidence sets*), respectively, and the second part, where they are above the obstacles and hence the solutions to the differential equations $-u'' = \psi_1$ and $-v'' = \psi_2$ (*non-coincidence sets*). The points, that divide the coincidence set from non-coincidence set forms the free boundary of the problem.

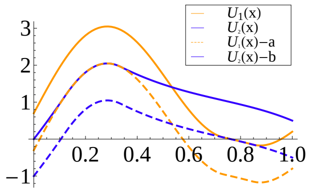

Figure 1: 1D Optimal Switching

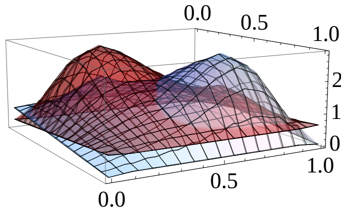

Figure 2: 2D Optimal Switching

Example 2. As a second example, we consider the following two dimensional elliptic optimal switching problem:

$$\begin{cases} \min\left\{u(x,y)-\left(v(x,y)-1\right); -\Delta u(x,y)-\psi_1(x,y)\right\}=0, \\ \min\left\{v(x,y)-\left(u(x,y)-1\right); -\Delta v(x,y)-\psi_2(x,y)\right\}=0, \\ u(0,y)=0.2, \quad u(1,y)=0.6, \\ u(x,0)=u(x,1)=0.4x+0.2, \\ v(0,y)=0.7, \quad v(1,y)=0.1, \\ v(x,0)=v(x,1)=-0.6x+0.7, \end{cases} \tag{5}$$

where $x, y \in (0,1)$,

$$\psi_1(x,y)=120e^{-20(x-1/4)^2-20(y-1/4)^2}$$

and

$$\psi_2(x,y)=(165x+y)e^{-20(x-3/4)^2-20(y-3/4)^2}.$$

The solution represented in Figure 2 has been obtained for (5).

References

[1] T. Arnarson, B. Djehiche, M. Poghosyan, and H. Shahgholian. A PDE approach to regularity of solutions to finite horizon optimal switching problems. *Nonlinear Anal.*, 71(12):6054–6067, 2009.