On the boundary points of the numerical range

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Abstract

Some relations between the Hilbert space operator spectrum and its numerical range are established.

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Let A be a (linear, bounded) operator, acting in a Hilbert space H. The numerical range of A (or the field of values) is the set $W(A) = \{\langle Ax, x\rangle : \|x\| = 1\}$. According to classical Hausdorff-Toeplitz theorem $W(A)$ is convex and the closure $\overline{W(A)}$ of $W(A)$ contains the spectrum SpA of A. For some classes of operators (including normal operators) $\overline{W(A)}$ coincides with the convex hull $ch(SpA)$ of SpA. Denote by ∂D the boundary of an arbitrary set D.

We start by an elementary result.

Lemma 1. Let $\lambda \in SpA$ and $|\lambda| = \|A\|$. Then there exists a sequence of unit vectors $\{x_n\}$ such that

$$\|(A - \lambda I) x_n\| + \left\|(A - \lambda I)^* x_n\right\| \to 0. \tag{1}$$

Proposition 1. Let $\lambda \in \partial W(A)$. Then $\lambda \in SpA$ if and only if there exists a sequence of unit vectors $\{x_n\}$, satisfying (1).

Proof. The sufficiency of this condition is obvious. We show now that this condition is necessary. Let n be the perpendicular to the support line to $\overline{W}(A)$ at the point λ. Take a point μ on n, such that $\mu \notin \overline{W}(A)$. As $SpA \subset \overline{W}(A)$, the point μ belongs to the resolvent set $\rho(A) = \mathbb{C}\backslash SpA$. According to the classical result of M. Stone [3] for the resolvent $R_\mu(A) = (A - \mu I)^{-1}$ the inequality

$$\|R_\mu(A)\| \leqslant \frac{1}{dist(\mu, W(A))},$$

where

$$dist(\mu, W(A)) = \inf_{s \in W(A)} |s - \mu|$$

is satisfied. By choice of μ we have $dist\,(\mu, W\,(A)) = |\lambda - \mu|$ and

$$\|R_\mu\,(A)\| \leqslant \frac{1}{|\lambda - \mu|}.$$

As for any operator

$$\|R_\mu\,(A)\| \geqslant \frac{1}{dist\,(\mu, SpA)} \quad \left(= \frac{1}{|\lambda - \mu|}\right),$$

hence

$$\|R_\mu\,(A)\| = \frac{1}{|\lambda - \mu|}.$$

From the spectral mapping theorem

$$\frac{1}{\lambda - \mu} \in SpR_\mu\,(A)\,,$$

so by Lemma 1 we have

$$\left\|\left(R_\mu\,(A) - \frac{I}{\lambda - \mu}\right) x_n\right\| + \left\|\left(R_\mu\,(A) - \frac{I}{\lambda - \mu}\right)^* x_n\right\| \to 0.$$

As

$$A - \lambda I = (\mu - \lambda)\,(A - \mu I)\left(R_\mu\,(A) - \frac{I}{\lambda - \mu}\right)$$

and

$$(A - \lambda I)^* = \left(\overline{\mu} - \overline{\lambda}\right)(A - \mu I)^*\left(R_\mu\,(A) - \frac{I}{\lambda - \mu}\right)^*$$

hence

$$\|(A - \lambda I)\,x_n\| + \left\|(A - \lambda I)^*\,x_n\right\| \to 0.\square$$

Proposition 2. Let $\overline{W\,(A)} = ch\,(SpA)$ and λ be an extreme point of $\overline{W\,(A)}$. Then there exists a sequence of unit elements $\{x_n\}$ satisfying (1).

Proof. According to the partial inversion of Krein-Milman theorem, any extreme point of $\overline{W}\,(A)$ belongs to ∂SpA and the proof is completed, recalling Proposition 1.

Corollary. Let m and M be the lower and upper bounds of a selfadjoint operator A

$$m = \inf_{\|x\|=1} \langle Ax, x\rangle\,,\ \ M = \sup_{\|x\|=1} \langle Ax, x\rangle\,.$$

Then there exists a sequence of unit elements $\{x_n\}$ such that

$$\lim_{n\to\infty} \|(A - mI)\,x_n\| = 0.$$

If $\in W(A)$, then there exists an element $x \neq \theta$ such that $Ax = mx$. The same assertions are true for M.

Only the second part needs a proof. One may translate the operator A and consider the non-negative operator $B = A - mI$. We prove that the condition $\langle Bx, x\rangle = 0$ implies $Bx = \theta$. Indeed

$$\left\langle B\left(x - \frac{Bx}{\|B\|}\right), x - \frac{Bx}{\|B\|}\right\rangle = -2\frac{\|Bx\|^2}{\|B\|} + \frac{\langle B^2x, Bx\rangle}{\|B\|^2} \leqslant -\frac{1}{\|B\|}\|Bx\|^2 \leqslant 0. \quad (2)$$

By similar arguments may be proved the following assertion.
Proposition 3. Let $\lambda \in \partial W(A)$. Then there exists a number $\varphi \in [0;2\pi)$ and a sequence of unit vectors $\{x_n\}$ such that $\left(e^{-i\varphi}(A-\lambda I)+e^{i\varphi}(A-\lambda I)^*\right)x_n \to \theta$. If $\lambda \in \partial W(A) \cap W(A)$, then there exists a nonzero element x such that

$$\left(e^{-i\varphi}(A-\lambda I) + e^{i\varphi}(A-\lambda I)^*\right)x = 0.$$

Definition Let $F \subset \mathbb{C}$ be a closed set. A point b is said to be a bare point of F if there is a circumference through b such that no point of F lies outside of that circle.

This notion is introduced by Orland in [2] and is widely used by specialists in the operator theory. More convenient and conceptual term is the farthest point.

Proposition 4. Let for any $\mu \in \mathbb{C}$
$\overline{W(A-\mu I)} = ch\left(Sp(A-\mu I)\right)$ and λ be a bare point of $W(A)$. Then there exists an element $x \neq \theta$ such that $Ax = \lambda x$, $A^*x = \overline{\lambda}x$, i.e. λ is a normal eigenvalue and x is a normal eigenelement.

Remark. In finite dimensional space the condition $\lambda \in SpA \cap \partial W(A)$ implies that λ is a normal eigenvalue. The next example shows that this is not the case in the infinite dimensional space.

Example. Let V be the Volterra integration operator, acting in the space $L^2(0;1)$ by the formula

$$(Vf)(x) = \int_0^x f(t)\,dt.$$

Evidently $SpV = \{0\}$ and for the function $g(x) = \cos \pi x$ one has $\langle Vg, g\rangle = 0$. It is easy to see that for any function $f \in L^2(0;1)$ the inequality $\mathrm{Re}\,\langle Vf, f\rangle \geqslant 0$ is satisfied, implying $0 \in \partial W(V)$.

References

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