Holomorphic Bloch spaces on the unit ball in C^n

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Let C^n be the n dimensional complex Euclidean space. For $z=(z_1,..,z_n)$, $\zeta=(\zeta_1,...,\zeta_n)$ in C^n we define the inner product as follows:$< z,\zeta >= z_1\overline{\zeta}_1 + ... + z_n\overline{\zeta}_n$.

Let $B^n = \{z \in C^n,\ |z| < 1\}$ be the unit ball in C^n and let $S^n = \{z \in C^n,\ |z| = 1\}$ be the boundary of B^n. We denote by $H(B^n)$ the set of holomorphic functions on B^n .

Let $f \in H(B^n)$, then $f(z) = \sum_m a_m z^m\ (z \in B^n)$, where the summation is over all multi-indices $m = (m_1,..,m_n)$, each m_k is a nonnegative integer and $z^m = z_1^{m_1}\dots z_n^{m_n}$. Putting $f_k(z) = \sum_{|m|=k} a_m z^m$ for each $k \geq 0$, $|m| = m_1 + ... + m_n$, then the Taylor series of f has the following form

$$f(z) = \sum_{k=0}^{\infty} f_k(z) \tag{1}$$

which is called the homogeneous expansion of f. It is clear, that each f_k is homogeneous polynomial of degre k.

An important notion in the study of holomorphic function spaces is the notion of fractional differential operators. In this paper we consider one type of them. For a holomorphic function f with homogeneus expansion (1) and for $\alpha > -1$ we define the fractional differential as follows:

$$D^\alpha f(z) = \sum_{k=0}^{\infty} (k+1)^\alpha f_k(z),\ z \in B^n$$

and the inverse operator $D^{-\alpha}$ we define in the standard sense:

$$D^{-\alpha} D^\alpha f(z) = f(z).$$

The Bloch space plays a very important role in classical geometric function theory. The one dimensional case of the holomorphic Bloch space is well investigated. The aim of this paper is the study of the Bloch space on the unit ball in C^n. We give a new generalization of them and consider the weighted

case which is new also in the one dimensional case. Note that the polydisc case has been already investigated (see for example [1], [2]).

Let S be the class of all non-negative measurable functions ω on $(0,1)$, for which there exist positive numbers $M_\omega,\ q_\omega,\ m_\omega,\ (m_\omega, q_\omega \in (0,1)$, such that

$$m_\omega \leq \frac{\omega(\lambda r)}{\omega(r)} \leq M_\omega,$$

for all $r \in (0,1)$ and $\lambda \in [q_\omega, 1]$. For properties of functions from S, see [3].

One of the applications is the description of the $(A^p(\omega))^*$ in case $0<p\leq 1$ via Bloch spaces. Here $A^p(\omega)$ is the ω generalisation of $A^p(\alpha)$ space in the case of unit ball in C^n and is defined as the class of holomorphic functions f for which

$$\|f\|^p_{A^p(\omega)} = \int_{B^n} |f(z)|^p \omega(1-|z|) d\nu(z) < +\infty,$$

where $d\nu(z)$ is volume measure on B^n, normalized so that $\nu(B^n) = 1$ and $0 < \beta_\omega < 1$.

The corresponding space of measurable functions we denote by $L^p(\omega)$.

Definition 1. *Let $f \in H(B^n)$, $\omega \in S$ and $0 < \alpha_\omega < 1$. A function f belongs to the Bloch space $B^n_\omega \equiv B_\omega$ if*

$$M_f = \sup_{z\in B^n} \left\{ \frac{(1-|z|^2)}{\omega(1-|z|)} |Df(z)| \right\} < +\infty \tag{2}$$

It is easy to see that B_ω is a Banach space with respect to the norm $\|\cdot\|$. Let $L^\infty_\omega = L^\infty_\omega(B^n)$ be the class of measurable functions on B^n, for which

$$\|f\|_{L^\infty_\omega} = \sup_{z\in B^n} \{|f(z)|\omega^{-1}(1-|z|^2)\} < +\infty.$$

The next theorem gives a description of the analytic part of L^∞_ω.

Theorem 1. *Let $f \in H(B^n)$, $\alpha > \alpha_\omega + 1$, $k \in N$ Then $(1-|z|^2)^\alpha D^k f(z) \in L^\infty_\omega$ if and only if $(1-|z|^2)^{\alpha-1} D^{k-1} f(z) \in L^\infty_\omega$.*

Using Theorem 1 one can give an another characterization of B_ω.

Theorem 2. *A function f belongs to B_ω if and only if*

$$\sup_{z\in B^n} \left\{ \frac{(1-|\zeta|^2)^k}{\omega(1-|\zeta|)} |D^k f(z)| \right\} < \infty,$$

for $\alpha > \alpha_\omega$.

Let us consider the following operator

$$P_\alpha f(z) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_{B^n} \frac{(1-|\zeta|^2)^{\alpha-1} f(\zeta)}{(1-<z,\zeta>)^{\alpha+n}} d\nu(\zeta)(\alpha > 0),$$

then we have

Theorem 3. *Let $\alpha > \beta_\omega$. Then P_α is a bounded operator from L^∞_ω to B_ω and if $\alpha > \beta_\omega$ then is P_α onto.*

The next problem in which we are interested is following: our aim is to find the inverse operator of P_α which maps B_ω to L^∞_ω. Furthermore, if this is the case whether $P_\alpha(P^-_\alpha(f))(z) = f(z)$ $(z \in B^n)$ for all $f \in B_\omega$. The solution of this problem is positive. We consider the general operator

$$R_{\alpha,\beta} f(z) = (1-|z|^2)^\beta \int_{B^n} \frac{(1-|w|^2)^{\alpha-1} f(\zeta)}{(1-<z,\zeta>)^{\alpha+\beta+n}} d\nu(\zeta),\ \alpha+\beta > -1.$$

The following theorem is true

Theorem 4. *Let $\alpha > \beta_\omega$ and $\beta > \alpha_\omega$. Then*
a) $P_\alpha(R_{\alpha,\beta}(f)(z) \equiv f(z)$ $(z \in B^n)$ *for all* $f \in B_\omega$
b) *The operator $R_{\alpha,\beta}$ is bounded from B_ω to L^∞_ω, and there exist constants $C_1(\omega)$, $C_1(\omega)$ such that*

$$C_1(\omega)\|f\|_{B_\omega} \le \|R_{\alpha,\beta} f\|_{L^\infty_\omega} \le C_2(\omega)\|f\|_{B_\omega}. \tag{3}$$

c) $f \in B_\omega$ *if and only if* $R_{\alpha,\beta} f \in L^\infty_\omega$.

Next we describe the duals of $A^p(\omega)$ in terms of holomorphic Bloch space in the case if $0 < p \le 1$. We need to establish the following lemmas before proving the duality result. The following theorem describes the continuous linear functionals on $A^p(\omega)$ in the case $0 < p \le 1$

Theorem 5. *Let $0 < p \le 1$, $\omega \in S$. Then the dual of $A^p(\omega)$ under the pairing*

$$< f, g >= \int_{B^n} f(z)\overline{g(z)}(1-|z|^2)^\alpha d\nu(z) \tag{4}$$

is isomorphic to B_{ω^*}, where $\omega^*(t) = \omega^{1/p}(t)t^{(n+1)(1/p-1)-\alpha}$ and $\alpha > \alpha_\omega/p + (n+1)(1/p-1)$.

References

[1] A.V. Harutyunyan(2007). *Bloch spaces of holomorphic functions in the polydisk*. J. Funct. Spaces Appl. 5 , no. 3, 213–230

[2] Z. Zhou(2005). *The Essential Norm of Composition Operator between Generalized Bloch space in Polydiscs and its Applications*. arXiv: math.Fa/0503723v3

[3] E. Seneta (1985). *Functions of Regular Variation [in Russian]*. Nauka, Moscow