Convergence Acceleration of Fourier Series by the Roots of the Laguerre Polynomial

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Abstract

We apply the idea of Fourier-Pade approximation for accelerating the convergence of the truncated Fourier series. The resultant rational linear approximation of the smooth function f on $[-1,1]$ is constructed by the roots of the Laguerre polynomial that depends on the smoothness of the approximated function. Numerical results outlined the quality of approximations.

Key Words: rational approximation, convergence acceleration, Laguerre polynomial

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1. Basic Formulae Denote

$$S_N(f) = \sum_{n=-N}^{N} f_n e^{i\pi nx}, \; f_n = \frac{1}{2}\int_{-1}^{1} f(x) e^{-i\pi nx} dx. \tag{1}$$

For a finite sequence of complex numbers $\theta := \{\theta_k\}_{k=-p}^{p}, \; p \geq 1$ we put

$$\Delta_n^0(\theta) = f_n, \; \Delta_n^k(\theta) = \Delta_n^{k-1}(\theta) + \theta_{k\, sgn(n)} \Delta_{(|n|-1)sqn(n)}^{k-1}(\theta), \; k \geq 1,$$

where $sgn(n) = 1$ if $n \geq 0$ and $sgn(n) = -1$ if $n < 0$.

By $R_N(f)$ denote the approximation error of the truncated Fourier series

$$R_N(f) = f(x) - S_N(f) = R_N^+(f) + R_N^-(f),$$

where

$$R_N^+(f) = \sum_{n=N+1}^{\infty} f_n e^{i\pi nx}, \quad R_N^-(f) = \sum_{n=-\infty}^{-N-1} f_n e^{i\pi nx}.$$

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Applying the Abel transformation, we get

$$R_N^+(f) = -e^{i\pi(N+1)x}\sum_{k=1}^{p}\frac{\theta_k\Delta_N^{k-1}(\theta)}{\prod_{s=1}^{k}(1+\theta_s e^{i\pi x})} + \frac{1}{\prod_{k=1}^{p}(1+\theta_k e^{i\pi x})}\sum_{n=N+1}^{\infty}\Delta_n^p(\theta)e^{i\pi n x}.$$

Similar expansion of $R_N^-(f)$ reduces to the following approximation ([2])

$$S_{p,N}(f) := \sum_{n=-N}^{N} f_n e^{i\pi n x} - e^{i\pi(N+1)x}\sum_{k=1}^{p}\frac{\theta_k\Delta_N^{k-1}(\theta)}{\prod_{s=1}^{k}(1+\theta_s e^{i\pi x})} - e^{-i\pi(N+1)x}\sum_{k=1}^{p}\frac{\theta_{-k}\Delta_{-N}^{k-1}(\theta)}{\prod_{s=1}^{k}(1+\theta_{-s} e^{-i\pi x})}$$

with the error

$$R_{p,N}(f) = f(x) - S_{p,N}(f) = R_{p,N}^+(f) + R_{p,N}^-(f),$$

where

$$R_{p,N}^{\pm}(f) := \frac{1}{\prod_{k=1}^{p}(1+\theta_{\pm k}e^{\pm i\pi x})}\sum_{n=N+1}^{\infty}\Delta_{\pm n}^p(\theta)e^{\pm i\pi n x}.$$

If θ is the solution of system

$$\Delta_n^p(\theta) = 0,\ \ n = -N-1,\cdots,-N-p; N+1,\cdots,N+p, \tag{2}$$

then approximation $S_{p,N}(f)$ coincides with the Fourier-Pade approximation [1].

In this paper we introduce an alternative approach for determining the parameters θ_k when the approximated function is smooth on $[-1,1]$. The resultant approximates f by means of rational functions but realizes linear approximation.

We put

$$A_k(f) = f^{(k)}(1) - f^{(k)}(-1).$$

Further we suppose that

$$\theta_k = \theta_{-k} = 1 - \frac{\tau_k}{N},\ \ k = 1,\cdots,p. \tag{3}$$

By $\gamma_k(p)$, $k = 0,\cdots,p$ we denote the coefficients of the polynomial

$$\prod_{k=1}^{p}(1+\tau_k x) \equiv \sum_{k=0}^{p}\gamma_k(p)x^k.$$

First we need the following Lemma.

Lemma 1. *[2] Suppose $f \in C^{q+p}[-1,1]$, $q \geq 0$, $p \geq 1$, and $f^{(q+p)}$ is absolutely continuous on $[-1,1]$. Then, if $A_j(f) = 0$ for $j = 0, \cdots, q-1$ and θ_k are chosen as in (3), the following asymptotic expansion holds as $N \to \infty$, $n \geq N+1$*

$$\Delta_n^p(\theta) = A_q(f)\frac{(-1)^{n+p+1}}{2(i\pi)^{q+1}q!}\sum_{k=0}^{p}\frac{(q+p-k)!(-1)^k\gamma_k(p)}{N^k n^{q+p-k+1}} + o(n^{-q-p-1}).$$

In view of Lemma 1 and system (2) we get the following system for determining the numbers τ_k

$$\sum_{k=0}^{p}\frac{(q+p-k)!(-1)^k\gamma_k(p)}{\left(1+\frac{s}{N}\right)^{q+p-k+1}} = 0, \ \ s = 1, \cdots, p.$$

Expansion into Taylor series in terms of $1/N$ leads to the following equations

$$\sum_{k=0}^{p}(-1)^k\gamma_k(p)(m+q+p-k)! = 0, \ \ m = 0, \cdots, p-1.$$

From here we get

$$\gamma_k(p) = \binom{p}{k}\frac{(q+p)!}{(q+p-k)!}, \ \ k = 0, \cdots, p, \ \ \gamma_0(p) = 1.$$

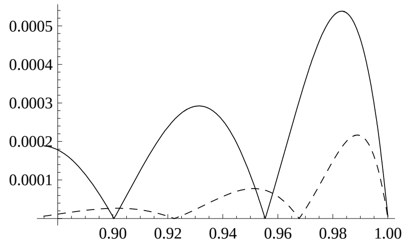

Figure 1: Absolute errors while approximating (4) by the truncated Fourier series (solid line) and rational approximation for $p = 1$ (dashed line) when $N = 16$ and $q = 2$.

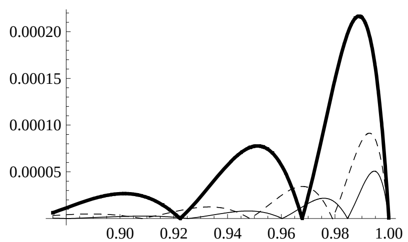

Figure 2: Absolute errors while approximating (4) by the rational approximation for p=1 (thick solid line), $p = 2$ (dashed line) and $p = 3$ (thin solid line) when N=16 and q=2.

Taking into account the definition of γ_k we obtain the equation

$$(p+q)!\sum_{k=0}^{p}\frac{(-1)^k}{(p-k)!\,k!\,(q+k)!}x^k = 0,$$

with Laguerre polynomial ([3]) $L_p^q(x)$ in the left hand side that has positive single roots $\{x_k\}_{k=1}^p$ satisfying the condition $x_k = \tau_k$. Now $S_{p,N}(f)$ with θ_k defined by (3), where τ_k are the roots of the Laguerre polynomial $L_p^q(x)$, realizes rational linear approximation of the smooth function f on $[-1,1]$.

It is easy to check that for $p = 1$ we have $\tau_1 = 1 + q$ and for $p = 2$ we get $\tau_1 = q + 2 + \sqrt{q+2}$ and $\tau_2 = q + 2 - \sqrt{q+2}$.

2. Numerical Illustrations Consider the following simple function

$$f(x) = (1 - x^2)^2 \sin(x - 0.5). \tag{4}$$

Figures 1 and 2 compare truncated Fourier series with the rational approximations at the point $x = 1$. Figure 3 shows the plots of the absolute errors on the interval $[-0.5, 0.5]$ while approximating (4) by the truncated Fourier series and rational approximations.

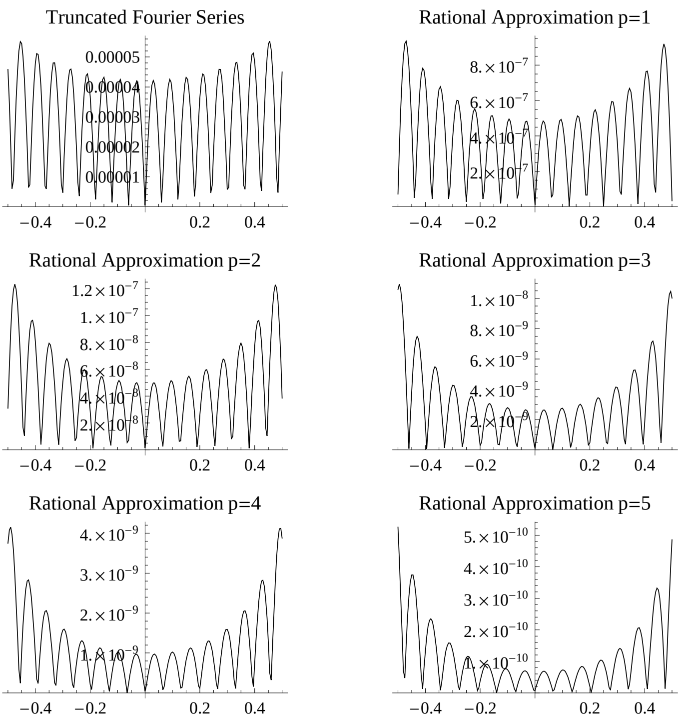

Figure 3: Absolute errors while approximating (4) by the truncated Fourier series and rational approximations on the interval $[-0.5, 0.5]$ for $N = 16$ and $q = 2$.

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