On spectrum of Sturm-Liouville operator

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Abstract

In this paper the Sturm - Liouville operator on the finite interval is considered. For some special cases of the boundary conditions, a group of invariant transforms, which preserve the operator's spectrum is constructed. This result permits us to find out conditions, when the spectrum is discrete. The influence of this group, on the inverse problem, is discussed, too.

Keywords: spectrum, Sturm-Liouville operator, inverse problem.
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The following results are proved.

Theorem 1. Let the operator

$$-y''(x) = \lambda p(x)y(x) \quad -1 < x < 1,$$

with the boundary conditions

$$y(-1) - ay'(-1) = 0, \quad y(1) + by'(1) = 0,$$

has a discrete spectrum. If $2a + a^2 = 0$ and $2b + b^2 = 0$, then for arbitrary $x_0 \in (-1, 1)$ the potential function

$$\frac{(1 - x_0^2)^2}{(1 - xx_0)^4} p\left(\frac{x - x_0}{1 - xx_0}\right)$$

generates an operator with the same spectrum as $p(x)$ does.

If the additional condition $a = b$ holds the potential function $p(-x)$ generates an operator with the same spectrum, too.

Theorem 2. Let $p(x) \geq 0, \quad -1 \leq x \leq 1$, be a measurable function. Then the spectrum of the operator

$$-y''(x) = \lambda p(x)y(x), \quad -1 \leq x \leq 1,$$

with the boundary conditions

$$y(-1) = y(1) = 0,$$

in the space of functions for which

$$\int_{-1}^{1} |y'(x)|^2 \, dx < \infty$$

is discrete if and only if

$$\sup_{|x|<1} \int_{2|t-x|<1-|x|} \sqrt{p(t)} dt < \infty$$

and

$$\lim_{|x|\to 1-} \int_{2|t-x|<1-|x|} \sqrt{p(t)} dt = 0.$$

Theorem 3. Let $p(x) \geq 0, \quad -1 \leq x \leq 1$, be a measurable function. Then the spectrum of the operator

$$-y''(x) = \lambda p(x) y(x), \quad -1 \leq x \leq 1,$$

with the boundary conditions

$$y(-1) = y(1) = 0,$$

in the space of functions for which

$$\int_{-1}^{1} \frac{y^2(x)}{(1-x^2)^2} dx < \infty$$

is discrete if and only if

$$\sup_{|x|<1} \left((1-|x|)^3 \int_{2|t-x|<1-|x|} p^2(t) dt \right) < \infty.$$

and

$$\lim_{|x|\to 1-} \left((1-|x|)^3 \int_{2|t-x|<1-|x|} p^2(t) dt \right) = 0.$$

Theorem 4. Let the real numbers a, b satisfy the condition

$$(1-(a+1)^2)(1-(b+1)^2) > 0.$$

Let $x_0 = (a-b)/(a+b+ab) \in (-1,1)$, (we put $x_0 = 0$ if $a = b$). Let $p(x) > 0$ be a continuous differentiable function satisfying the condition

$$(1-x^2)^2 p(x) = (1-y^2)^2 p(y), \quad x, y \in (-1,1), \quad \frac{x+y}{1+xy} = x_0.$$

Let $\lambda_0 < \lambda_1 < \lambda_2 < \dots$ be the spectrum of the operator

$$-y''(x) = \lambda p(x)y(x), \quad y(-1) - ay'(-1) = 0, \quad y(1) + by'(1) = 0.$$

Then there is no other potential function $p(x)$ satisfying the conditions given above and generating the operator with the same spectrum.

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