Stochastic Models, Free Boundaries, and Numerical Methods in Financial Bubble Analysis

Financial Bubble Analysis

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Stochastic and Analytic Methods in Mathematical Physics

Abstract, at a glance

Based on joint work with, supported by the Higher Education and Science Committee of the Ministry of Education, Science, Culture and Sports of the Republic of Armenia (Research Project No. 25RL-1A040).

A. Arakelyan, R. Barkhudaryan, V. Khalatyan, M. Poghosyan The Financial Bubble Model with Lévy Jump Processes.
arXiv:2606.25104 [math.AP] , 2026.

- Speculative bubbles arise from **investor disagreement** and **limits to arbitrage**: prices may exceed fundamental value through the option to resell an asset to a more optimistic investor.
- We extend the **Berestycki–Monneau–Scheinkman** model by letting disagreement evolve through a **Lévy jump process**, capturing sudden shifts in market sentiment and financial shocks.
- Optimal stopping theory + Itô–Lévy calculus $\Rightarrow$ a **non-local PIDE with a moving obstacle** for the bubble premium.
- We build a **viscosity solution** framework – comparison, existence,

From Harrison–Kreps to Berestycki–Monneau–Scheinkman

- Scheinkman–Xiong (2003): continuous-time *diffusion* model of investor disagreement.
- Chen–Kohn (2011): mean-reverting SDE for the disagreement process.
- **Berestycki–Monneau–Scheinkman (2014, BMS):** macroscopic reduction $\Rightarrow$ the bubble value solves a parabolic obstacle PDE; the free boundary is the optimal resale threshold.

Economic model (Berestycki–Monneau–Scheinkman (2014, BMS):)

Two groups A, B disagree about dividend growth; belief gap $g^C = \hat{f}^C - \hat{f}^{\hat{C}}$ follows

$$dg_t^C = -\rho\, g_t^C\, dt + \sigma\, dW_t^g, \qquad \rho > \lambda \geq 0.$$

No short sales, transaction cost $c > 0$; reservation price of a buyer in C (resale to group $\hat{C}$):

$$p_t^C = \sup_{\tau \leq T} \mathbb{E}_t^C \Big\{ e^{-r(\tau - t)} (p_\tau^{\hat{C}} - c\, \mathbf{1}_{\{\tau < T\}}) + \int_t^T e^{-r(s-t)} dD_s \Big\} =$$

$$\mathbb{E}_t^C \Big[\int_t^T e^{-r(s-t)} dD_s \Big] + q(g_t^C, t).$$

$q \geq 0$ is the **bubble**: the resale option value from future disagreement.

The model assumes that trading agents "agree to disagree" and there should be restrictions on short selling. As a result asset prices may become higher than their fundamental values. Other important part of the model is overconfidence, i.e. the agent believe that his information is more correct for a disagreement as that of others.

$$\begin{cases} \min(u_t - \frac{1}{2}\sigma^2 u_{xx} + \rho x u_x + ru, \\ \qquad\qquad u(x) - u(-x) - \psi(x)) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = 0, & x \in \mathbb{R}. \end{cases} \tag{1}$$

Financial Derivative Pricing: The Black–Scholes equation is a famous parabolic PDE used to determine the fair price of options based on stock prices and time.

$$u_t - \frac{1}{2}\sigma^2 u_{xx} + \rho x u_x + ru = 0$$

The stationary version of this model in one dimension was introduced and solved by Scheinkman and Xiong [1].

Scheinkman JA, Xiong W. Overconfidence and speculative bubbles.
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Barkhudaryan R, Juráš M, Salehi, M Iterative scheme for an elliptic non-local free boundary problem.
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One-phase (or classical) obstacle problem

- membrane $u(x), \quad x \in \Omega \subset \mathbb{R}^n$;
- $u(x) = g(x), \quad x \in \partial\Omega$;
- $\psi : \Omega \to \mathbb{R}$,
- $u(x) \geq \psi(x)$, $x \in \Omega$.

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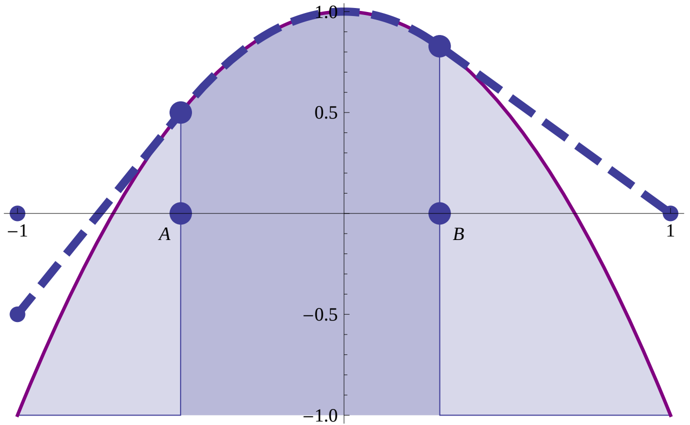
One-phase obstacle problem

Determination of the equilibrium state of the elastic membrane with the presence of the the obstacle.


1.0
0.5
−0.5
−1.0
−1
1
A
B

Mathematical formulation of the one-phase obstacle problem

One-phase obstacle problem

One-phase obstacle problem is equivalent to minimizing the following functional

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx + \int_\Omega \lambda(x)u(x)dx$$

over the set of all possible "deformations" of the membrane:

$$u \in \mathbb{K} = \{v \in H^1(\Omega) : v - g \in H_0^1(\Omega), v(x) \geq \psi(x) \;\text{ a.e. } \;\text{ in } \Omega\}.$$

Reformulation of the problem

Fully nonlinear form

$$\begin{cases} \mathcal{F}[u] = 0 & \text{â} \quad \Omega \\ u = g & \text{îà} \quad \partial\Omega. \end{cases} \tag{2}$$

Reformulation of the problem

Fully nonlinear form

$$\begin{cases} \mathcal{F}[u] = 0 & \text{â} \quad \Omega \\ u = g & \text{íà} \quad \partial\Omega. \end{cases} \tag{2}$$

- in one-phase case

$$\mathcal{F}[u] = \min\{-\Delta u + f, u - \psi\} = 0,$$

- non-local obstacle problem

$$\mathcal{F}[u] = \min(Mu, u(x) - u(-x) - \psi(x)). \tag{3}$$

To deal with the problem we first recall the definition of the so-called viscosity solution.

Definition (Viscosity sub/super solution of the equation (3))

A function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of (3) on Ω, (that is, of the first equation in (3)), if u is upper semi-continuous (resp. lower semi-continuous), and if for any function $\varphi \in C^{2,1}(\Omega)$ and any point $x_0 \in \Omega$ such that $u(x_0) = \varphi(x_0)$ and

$$u \leq \varphi \text{ on } \Omega \text{ (resp. } u \geq \varphi \text{ on } \Omega)$$

then

$$\min(-(L\varphi)(x) + f(x), \varphi(x) - \varphi(-x) - \psi(x)) \leq 0,$$

(resp. $\min(-(L\varphi)(x) + f(x), \varphi(x) - \varphi(-x) - \psi(x)) \geq 0$).

Definition (Viscosity solution of the equation (3))

A function $u : \Omega \to R$ is a viscosity solution of (3) on Ω, if and only if u^* is a viscosity subsolution and u_* is a viscosity supersolution on Ω, where

$$u^*(x) = \limsup_{y \to x} u(y),$$

and

$$u_*(x) = \liminf_{y \to x} u(y).$$

$$\begin{cases} \min(-L^i u_i - f_i, \min\limits_{j\neq i}(u_i - u_j + \psi_{ij})) = 0, \\ u_i|_{\partial\Omega} = g_i, \qquad i = 1, \ldots, m. \end{cases} \tag{4}$$

In addition, we assume the following boundary compatibility condition:

$$g_i - g_j + \psi_{ij} \geq 0, \quad i \neq j \geq 0 \text{ on } \partial\Omega. \tag{5}$$

$$\begin{cases} \min(-Lu(x)+f(x), u(x)-u(-x)-\psi(x))=0, & x\in\Omega, \\ u(x)=g(x), & x\in\partial\Omega, \end{cases} \tag{6}$$

Then, (6) can be written as

$$\begin{cases} \min(-Lu_1+f_1, u_1-u_2+\psi_1)=0 \\ \min(-Lu_2+f_2, u_2-u_1+\psi_2)=0, \end{cases} \tag{7}$$

with the boundary conditions $u^i\big|_{\partial\Omega}=g^i$.
We assume further $\psi_1+\psi_2\geq 0$ in Ω. A simpler situation occurs when there is $\gamma>0$ such that $\psi_1+\psi_2\geq\gamma$ in Ω. In this last case, (7) has a unique solution.

Theorem

If there are no loops of zero switching costs, i.e. $\psi_1+\psi_2\geq 0$, then problems (6) and (7) are equivalent.

The gap: markets jump

- BMS (and its extensions) assume the disagreement process g_t evolves along continuous diffusion paths.
- Empirically: sentiment shocks, liquidity crises, and news arrive as abrupt jumps $\Rightarrow$ heavy-tailed returns that a pure diffusion cannot reproduce (Cont–Tankov).
- Wehowar (2018) extended Chen–Kohn to a Lévy setting and observed that classical $C^{2,1}$ solutions need *not* exist.

This talk

Let the disagreement process be driven by an **independent Lévy jump process** on top of the mean-reverting diffusion. We derive the resulting **non-local obstacle PIDE**, build a **viscosity solution theory** for it (comparison, existence, uniqueness) that pins down its **free boundary**, design a **provably convergent** monotone scheme, and test it on four Lévy models.

Optimal stopping formulation

Let g_t be the state variable (disagreement between two investor groups). Risk-neutral valuation with constant discount $r > 0$:

$$V(g,t) = \sup_{\tau \in [t,T]} \mathbb{E}\left[e^{-r(\tau - t)} \Phi(g_\tau, \tau) \,\middle|\, g_t = g \right].$$

- τ: stopping time (moment of resale)
- Φ: payoff obtained when the trade is executed

Net resale payoff. With fixed transaction cost $c > 0$ and a decaying time profile $\alpha \in C^1$, $\alpha \geq 0$, $\alpha' \leq 0$, $\alpha(T) = 0$:

$$\psi(g,t) := g\,\alpha(t) - c.$$

Payoff couples to the solution itself

The seller receives cash net of costs, *plus* the continuation value seen from the buyer's side, at the mirrored state $-g_\tau$:

$$\Phi_{\text{Bubble}}(g_\tau, \tau) = \left[V(-g_\tau, \tau) + \psi(g_\tau, \tau)\right]^+.$$

Complementarity condition

$$\psi(g, t) + \psi(-g, t) = -2c < 0.$$

The two groups can never find it optimal to exercise simultaneously — exercise regions are disjoint, no instantaneous trading loop. *This is exactly what makes the problem a genuine obstacle problem*, with a well-posed free boundary.

$$u(g, t) := \sup_{\tau \in [t, T]} \mathbb{E}\left[e^{-r(\tau - t)}\left[u(-g_\tau, \tau) + \psi(g_\tau, \tau)\right]^+ \,\middle|\, g_t = g\right].$$

Main result: the non-local obstacle PIDE

Theorem (Bubble non-local PIDE)

The bubble value $u(g,t)$ satisfies

$$\min\big(-\mathcal{L}^{\nu} u(g,t),\ u(g,t) - u(-g,t) - \psi(g,t)\big) = 0, \qquad u(g,T) = 0,$$

where $\mathcal{L}^{\nu} u := u_t + \mathcal{A}u - ru$ and

$$\mathcal{A}u := -\rho g\, u_g + \tfrac{\sigma^2}{2} u_{gg} + \int_{\mathbb{R}\setminus\{0\}} \big[u(g+\gamma) - u(g) - \gamma\, u_g\, \mathbf{1}_{\{|z|\le 1\}}\big]\, \nu(\mathrm{d}z).$$

- Derived via the Itô–Lévy product rule on $e^{-r(s-t)}u(g_s,s)$: continuation region $\Rightarrow$ local martingale $\Rightarrow$ $\mathcal{L}^{\nu} u = 0$; exercise region $\Rightarrow$ supermartingale $\Rightarrow$ $-\mathcal{L}^{\nu} u \ge 0$.
- The non-local integral is the new ingredient: it evaluates u at distant points $g + \gamma(g,z)$ and destroys the purely local (differential) character of the BMS free-boundary operator.

Why viscosity solutions?

- Free boundary $\Rightarrow$ u need not be $C^{2,1}$ across the contact set.
- Lévy jumps sharpen the difficulty: Wehowar (2018) shows classical solutions need not exist at all in the jump setting.
- We need a weak-solution notion robust to *both* the obstacle *and* the non-local integral – strong enough to still pin down a unique free boundary.

Framework

Crandall–Ishii–Lions viscosity solutions, extended to non-local integro-PDEs by Alvarez–Tourin and Barles–Imbert.

The free boundary problem, precisely

Rewriting Theorem 2.1 in the vocabulary just introduced:

- **Continuation region** $\mathcal{C}(t) := \{g : u(g,t) > u(-g,t) + \psi(g,t)\}$: here $-\mathcal{L}^\nu u = 0$, the PIDE holds with equality.
- **Exercise (contact) set** $\mathcal{E}(t) := \{g : u(g,t) = u(-g,t) + \psi(g,t)\}$: here $-\mathcal{L}^\nu u \geq 0$, the discounted value Z_s is only a supermartingale.
- **Free boundary** $\Gamma(t) := \partial\mathcal{C}(t) \cap \partial\mathcal{E}(t) = \{\pm g^*(t)\}$: the a priori unknown optimal resale threshold.

The non-local twist

Unlike a classical (local) free-boundary problem, $-\mathcal{L}^\nu u$ at a point of $\mathcal{C}(t)$ still reads u everywhere on $\mathbb{R}$ through the jump integral: $\mathcal{C}(t)$ and $\mathcal{E}(t)$ are coupled *globally*, not only across $\Gamma(t)$.

Viscosity sub/supersolutions

Test operator: local terms come from the *test function* φ, the non-local integral and the obstacle keep the *actual* function u:

$$\mathcal{L}^{\nu}_{\varphi}(g_0, t_0; u) := -\varphi_t + \rho g_0 \varphi_g - \frac{\sigma^2}{2}\varphi_{gg} + r\, u(g_0, t_0) - \int_{\mathbb{R}\setminus\{0\}} \big[u(g_0+\gamma, t_0) - u(g_0, t_0) - \gamma\varphi_g \mathbf{1}_{|z|\le 1}\big]\nu(\mathrm{d}z).$$

- **Subsolution** (USC u): at every local max of $u - \varphi$, $\min\big(\mathcal{L}^{\nu}_{\varphi}(g_0, t_0; u),\ u(g_0, t_0) - u(-g_0, t_0) - \psi(g_0, t_0)\big) \le 0.$
- **Supersolution** (LSC v): reverse inequality at local minima of $v - \varphi$.
- **Stability:** a locally bounded family of subsolutions with a common linear-growth bound $\Rightarrow$ the relaxed upper envelope is again a subsolution (used later in Perron's method and in the convergence proof).

Comparison principle — the crux of the theory

Theorem (Comparison)

Assume the Lipschitz/integrability bounds on γ and that r, ρ dominate the jump growth. If u is a subsolution and v a supersolution, both of at most linear growth, then $u \leq v$ on $\mathbb{R} \times [0, T]$.

Existence via Perron's method: the lower barrier

Subsolution barrier

$$\underline{u}(g,t) := \max\big(0, \psi(g,t)\big).$$

- The *intrinsic*, unexercised payoff: "do nothing clever, just take the resale value if it is already positive."
- Checking $\psi(g_0,t_0) \gtrless 0$ shows $\underline{u}$ meets the obstacle inequality with **equality** on $\{\psi > 0\}$ – even this trivial barrier already carries a piece of the free boundary, at $\{\psi = 0\}$.
- Perron's method needs a matching **upper** barrier to sandwich the true solution:

$$u := \sup\{\, w \text{ subsolution} : \underline{u} \le w \le \tilde{u}\,\}.$$

Existence via Perron's method: the upper barrier

Supersolution barrier

$$\tilde{u}(g,t) = \alpha(t)\Big(k\sqrt{g^2+M} + \tfrac{g}{2} + K\Big).$$

- A single C^2 formula of linear growth — no gluing of a quadratic core to a linear tail. Gluing would create its *own* kink, and the non-local integral does not tolerate a non-smooth barrier.
- Requires the **growth condition**

$$r+\rho \; > \; \sqrt{\lambda}\,L_\gamma + \tfrac{1}{2}L_\gamma^2 :$$

  discount + mean reversion must outrun the growth of the jump variance – not smallness of σ, but control of the jump amplitude's growth.
- Comparison then forces the Perron supremum's upper/lower semicontinuous envelopes to coincide $\Rightarrow$ a genuine viscosity solution, sandwiched between the two explicit barriers.

Uniqueness — and back to the free boundary

Corollary (Uniqueness)

If u and v are both viscosity solutions, comparison gives $u \leq v$ *and*, swapping sub/super roles, $v \leq u \Rightarrow u = v$.

- The value function – and hence the free boundary $g^*(\tau)$ it defines – is **unambiguous**, even though it may fail to be classically $C^{2,1}$.
- Wehowar (2018): once jumps are added, classical solutions can fail to exist at all. Viscosity theory is not a technicality here – it is what makes $g^*(\tau)$ well defined in the first place.
- Preview: numerically, $g^*(\tau)$ turns out to be smooth, strictly decreasing, and *identical* across four very different jump laws (coming up in Section 7).

Discretizing the PIDE: an Implicit–Explicit scheme

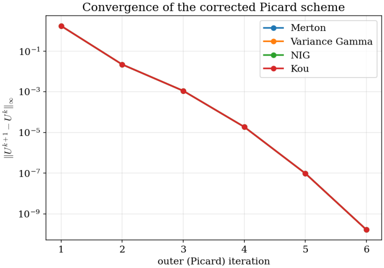
Theorem

Under the Lévy parameter bounds and the growth condition, the discrete solution u^h converges *locally uniformly* to the unique viscosity solution u as $h \to 0$.

- Stable + monotone + consistent $\Rightarrow$ relaxed semi-limits $\bar{u}, \underline{u}$ of u^h are, respectively, a sub- and a supersolution of the continuous PIDE.
- Non-local operator handled by the Barles–Imbert splitting (finite exterior integral by reverse Fatou, interior integral by dominated convergence against the smooth test function).
- Comparison principle forces $\bar{u} = \underline{u} = u$ – *and* forces the discrete contact set to converge to the true free boundary $g^*(\tau)$.
- Extends Barles–Souganidis (bounded domain, local operators) to a non-local operator on an unbounded, linear-growth domain.

Solving it in practice: PSOR + Picard

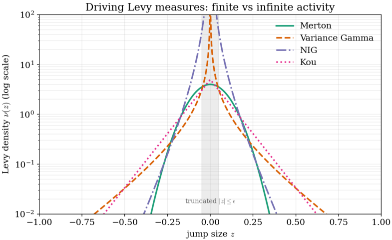
- Both the local matrix D and the dense jump matrix J are treated *implicitly* – still an M-matrix, so **unconditionally monotone**, no CFL needed in practice.
- One time step: a linear complementarity problem $\min(AU^{m+1} - U^m,\ U^{m+1} - \Phi^{m+1}) = 0$, solved by **Projected SOR**.
- The resale coupling $u(-g, \cdot)$ makes the obstacle genuinely nonlinear $\Rightarrow$ outer **Picard** iteration: use the *current* sweep's reflected value, evaluated at the *same* time level (else an $\mathcal{O}(\Delta t)$ bias enters the contact set).



Outer Picard update falls below 10^{-9} within six iterations, for every model.

Four Lévy laws, one test bed

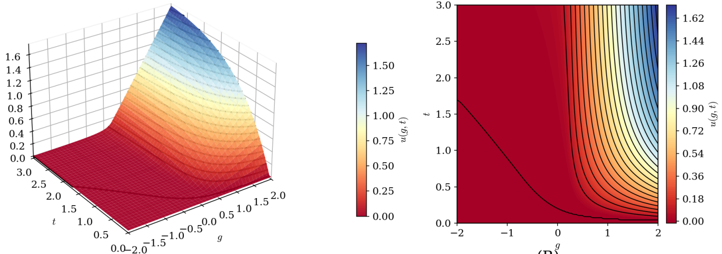
- **Merton:** finite activity, Gaussian jumps.
- **Kou:** finite activity, asymmetric double exponential, heavier tails.
- **Variance Gamma (VG):** infinite activity, non-integrable singularity at 0.
- **NIG:** infinite activity, semi-heavy tails, Bessel-kernel singularity.



All calibrated on $[g_{\min}, g_{\max}] = [-2, 2]$, $T = 3$; small jumps $|z| \leq \varepsilon = 0.05$ excluded from quadrature (singular densities are not integrable there).

The bubble surface and the free boundary

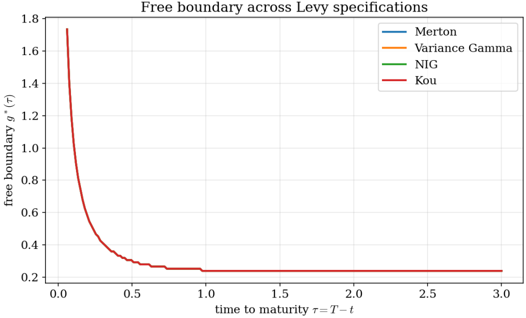
Numerical Solution under the Merton Jump Model



(A) u_h (B) u_h

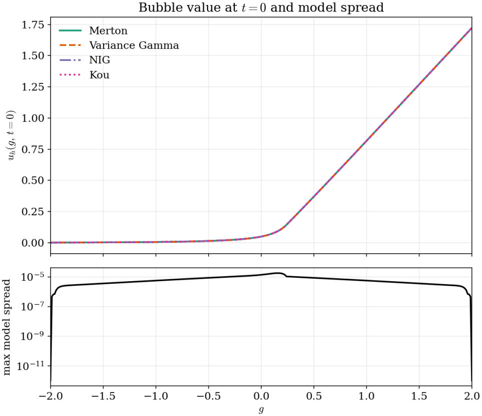
$u \approx 0$ for $g < 0$; rises monotonically for $g > 0$, peaking at $u_h(g_{\max}, 0) \approx 1.7231$.
Black curve: the free boundary $\{u = u(-g, \cdot) + \psi\}$.

Optimal exercise threshold shrinks with horizon



- $g^*(\tau)$ **decreases** as horizon $\tau = T - t$ lengthens: ≈ 0.75 at $\tau = 0.15$ down to a saturation value ≈ 0.24 for $\tau \gtrsim 1.5$.
- Longer resale horizon $\Rightarrow$ larger time profile $\bar{\alpha}(\tau)$ $\Rightarrow$ reselling optimal at *smaller* disagreement.
- All four Lévy specifications **coincide to plotting accuracy** – exactly as Corollary 4.2 promised.

A striking robustness across jump specifications



Despite very different driving measures (finite vs. infinite activity), the bubble values are visually indistinguishable; the max model spread peaks at only $\sim 1.9 \times 10^{-5}$, concentrated near the free boundary – far below discretization accuracy.

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Thank you.