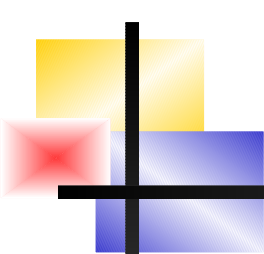


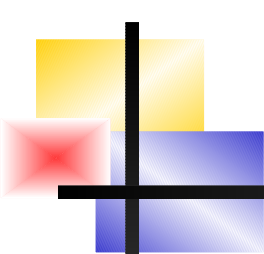
# **On the Energy-Probability Relationship in Statistical Physics of Infinite-Volume Systems**

S. Dachian

(joint work with B.S. Nahapetian)



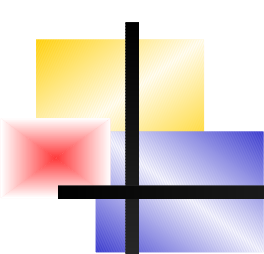
# Setup



# Setup

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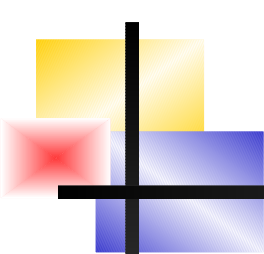
- ◆  $\mathbb{Z}^d$  —  $d$ -dimensional integer *lattice*,  $d \geq 1$ .



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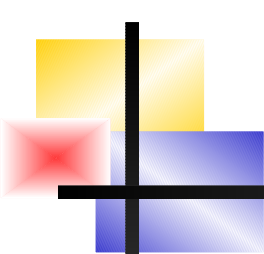
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# Setup

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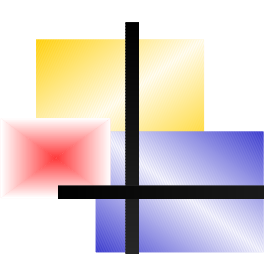
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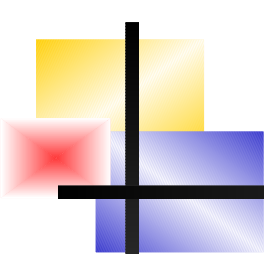
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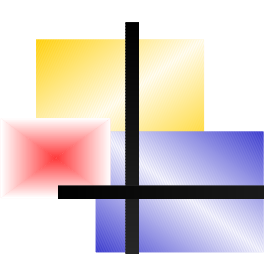
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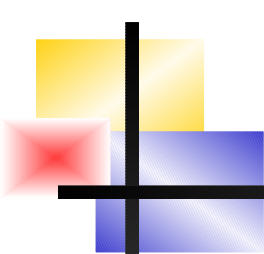
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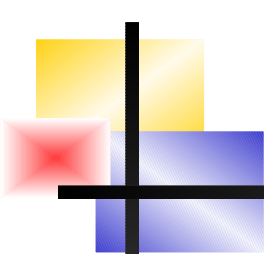
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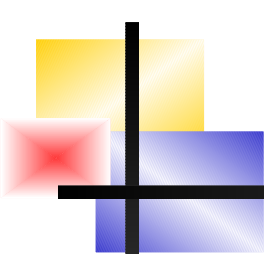
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# Setup

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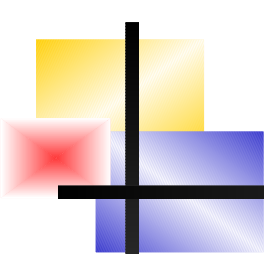
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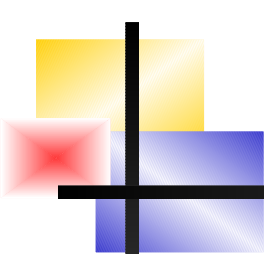
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# Random fields

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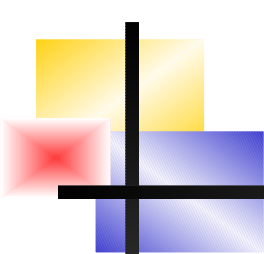


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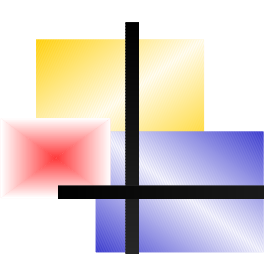


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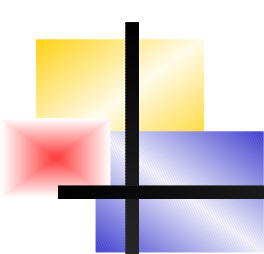
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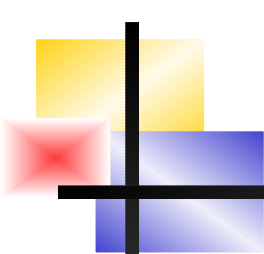
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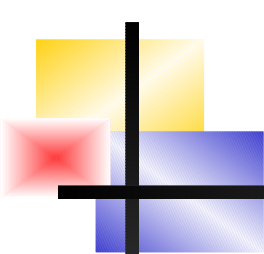
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- ◆ Kolmogorov's description:  $P \longleftrightarrow P$ .



# Conditional distribution of a random field

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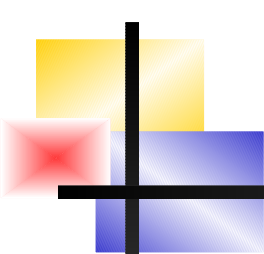
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- ◆  $Q = \{q_V^{\bar{x}}, V \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus V}\}$  — *(version of) conditional distribution* of  $P$  if

$$q_V^{\bar{x}} = Q_V^{\bar{x}}$$

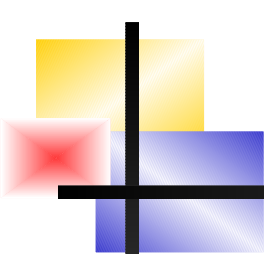
for all  $V \in \mathbb{Z}^d$  and almost all  $\bar{x} \in X^{\mathbb{Z}^d \setminus V}$ .



# Specifications

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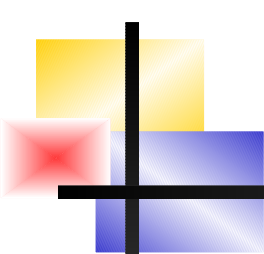
[Dobrushin 68]



# Specifications

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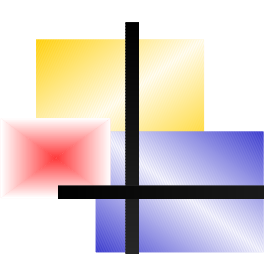
[Dobrushin 68] — description of random fields by means of conditional distributions



# Specifications

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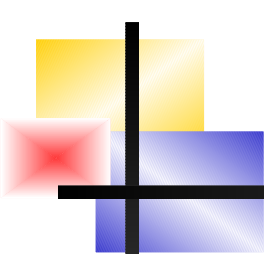
[Dobrushin 68] — description of random fields by means of *consistent* families of probability distributions in finite volumes indexed by infinite boundary conditions



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[Dobrushin 68] — description of random fields by means of *consistent* families of probability distributions in finite volumes indexed by infinite boundary conditions: *specifications*.

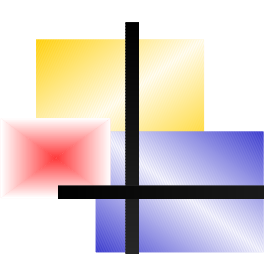


# Specifications

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[Dobrushin 68] — description of random fields by means of *consistent* families of probability distributions in finite volumes indexed by infinite boundary conditions: *specifications*.

**Definition.** A family  $\mathcal{Q} = \{q_V^{\bar{x}}, V \Subset \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus V}\}$  is called *specification* if its elements are consistent in the Dobrushin sense



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$$q_V^{\bar{x}}(xy) = (q_V^{\bar{x}})_{V \setminus I}(x) q_I^{\bar{x}x}(y)$$

for all  $I \subset V \in \mathbb{Z}^d$ ,  $x \in X^{V \setminus I}$ ,  $y \in X^I$  and  $\bar{x} \in X^{\mathbb{Z}^d \setminus V}$ .

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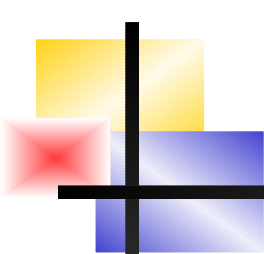
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Equivalent form of Dobrushin's consistency conditions:

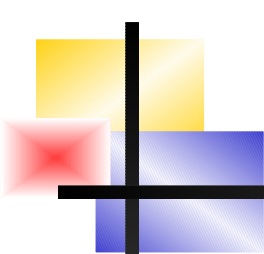
$$q_V^{\bar{x}}(xy) q_{V \setminus I}^{\bar{x}y}(u) = q_V^{\bar{x}}(uy) q_{V \setminus I}^{\bar{x}y}(x)$$

for all  $I \subset V \in \mathbb{Z}^d, x, u \in X^{V \setminus I}, y \in X^I$  and  $\bar{x} \in X^{\mathbb{Z}^d \setminus V}$ .



# Gibbsian specifications

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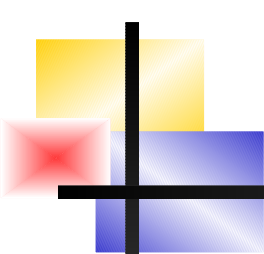


# Gibbsian specifications

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$Q$  is a *Gibbsian specification* if  $q_V^{\bar{x}}$  is defined by the Gibbs formula

$$q_V^{\bar{x}}(x) = \frac{\exp\{-H_V^{\bar{x}}(x)\}}{\sum_{z \in X^V} \exp\{-H_V^{\bar{x}}(z)\}}, \quad x \in X^V,$$



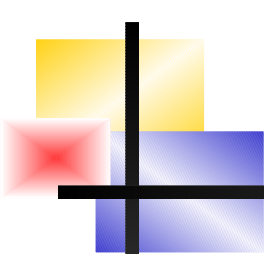
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$$H_V^{\bar{x}}(x) = \lim_{\Lambda \uparrow \mathbb{Z}^d \setminus V} \sum_{\emptyset \neq I \subset V} \sum_{J \subset \Lambda} \Phi_{I \cup J}(x_I \bar{x}_J), \quad x \in X^V,$$



# Gibbsian specifications

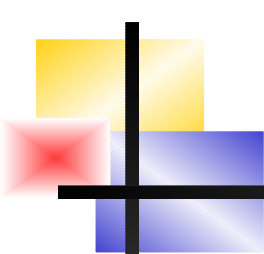
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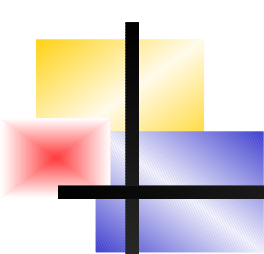
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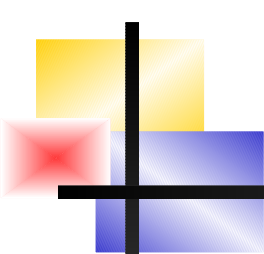
**Ruelle's problem:** how broad the class of Gibbsian specifications is ?  
[Ruelle 78, Appendix B (Open problems), B.1]



# Dobrushin's theorems and problem

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[Dobrushin 68]

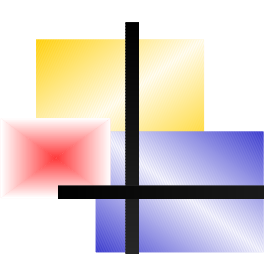


# Dobrushin's theorems and problem

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[Dobrushin 68]

**Theorem.** Let  $Q = \{q_V^{\bar{x}}, V \neq \emptyset, \bar{x} \in X^{\mathbb{Z}^d \setminus V}\}$  be some specification.



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[Dobrushin 68]

**Theorem.** Let  $Q = \{q_V^{\bar{x}}, V \neq \emptyset, \bar{x} \in X^{\mathbb{Z}^d \setminus V}\}$  be some specification.

- 1) If  $Q$  is *quasilocal* (i.e., if  $q_V^{\bar{x}}(x)$  is quasilocal w.r.t.  $\bar{x}$ ), then there exists a random field  $P$  *compatible* with  $Q$  (i.e., having  $Q$  as a conditional distribution).

# Dobrushin's theorems and problem

[Dobrushin 68]

**Theorem.** Let  $Q = \{q_V^{\bar{x}}, V \neq \emptyset, \bar{x} \in X^{\mathbb{Z}^d \setminus V}\}$  be some specification.

1) If  $Q$  is *quasilocal* (i.e., if  $q_V^{\bar{x}}(x)$  is quasilocal w.r.t.  $\bar{x}$ ), then there exists a random field  $P$  *compatible* with  $Q$  (i.e., having  $Q$  as a conditional distribution).

2) If, moreover,  $Q$  satisfies *Dobrushin's uniqueness condition*

$$\frac{1}{2} \sup_{t \in \mathbb{Z}^d} \sum_{s \in \mathbb{Z}^d \setminus t} \sup_{\bar{x}, \bar{y} : \bar{x}_{\mathbb{Z}^d \setminus t} = \bar{y}_{\mathbb{Z}^d \setminus t}} \sum_{x \in X^t} |q_t^{\bar{x}}(x) - q_t^{\bar{y}}(x)| < 1,$$

then the random field  $P$  compatible with  $Q$  is unique.

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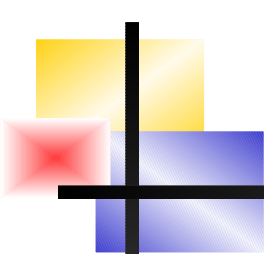
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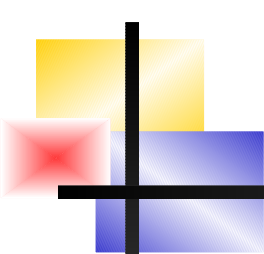
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**Dobrushin's problem:**  $Q \overset{?}{\longleftrightarrow} Q^{(1)} = \{q_t^{\bar{x}}, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus t}\}.$



# Energy $\longleftrightarrow$ probability in finite volume

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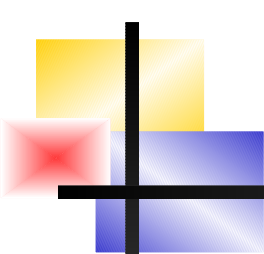


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**Definition.** A function  $\Delta_V = \{\Delta_V(x, u), x, u \in X^V\}$  is called *transition energy in volume  $V$*  if

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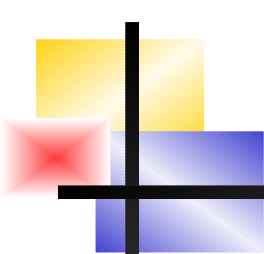
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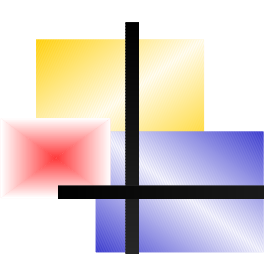
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where  $u \in X^V$  is arbitrary and  $\Delta_V$  is some transition energy in volume  $V$ , uniquely determined by

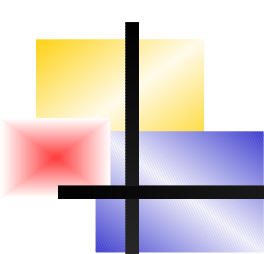
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**Remark.** We have a one-to-one relationship  $P_V \longleftrightarrow \Delta_V$ .



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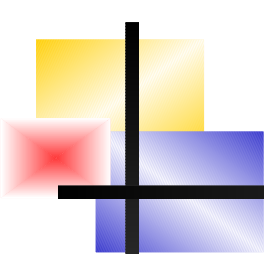
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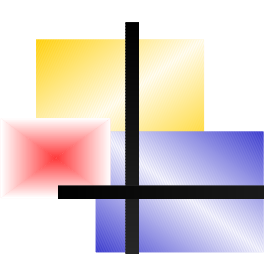
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$$P_V(x) = \frac{\exp\{\Delta_V(x, u)\}}{\sum_{z \in X^V} \exp\{\Delta_V(z, u)\}} = \frac{\exp\{-H_V(x)\}}{\sum_{z \in X^V} \exp\{-H_V(z)\}}, \quad x \in X^V.$$



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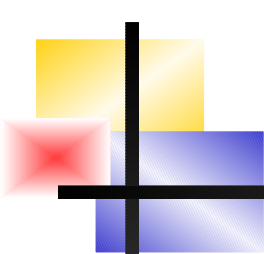
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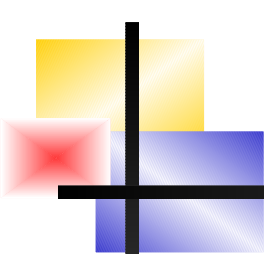
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But we do not have a one-to-one relationship  $P_V \longleftrightarrow H_V$  (as  $H_V$  is defined up to an arbitrary additive constant).



# Energy $\longleftrightarrow$ probability in infinite volume

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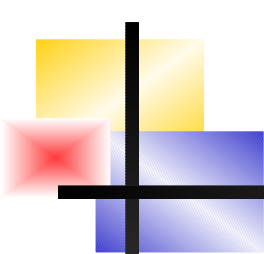


# Energy $\longleftrightarrow$ probability in infinite volume

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**Definition.** A function  $\Delta(\cdot, \cdot)$  defined on pairs of configurations on  $\mathbb{Z}^d$  that differ in at most a finite number of points is called *transition energy (function)* if

$$\Delta(\overline{u}, \overline{w}) = \Delta(\overline{u}, \overline{z}) + \Delta(\overline{z}, \overline{w}), \quad \overline{u}, \overline{w}, \overline{z} \in X^{\mathbb{Z}^d}.$$



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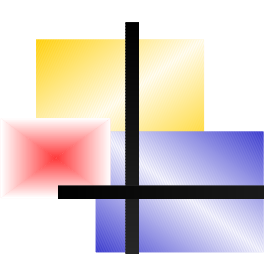
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Moreover:  $\Delta_{V \cup I}^{\bar{x}}(xy, uy) = \Delta(xy\bar{x}, uy\bar{x}) = \Delta_V^{\bar{x}y}(x, u)$ .

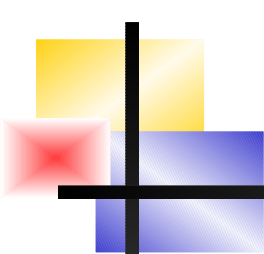


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**Definition.** A family  $\Delta = \{\Delta_V^{\bar{x}}, V \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus V}\}$  is called *transition energy field* if its elements are consistent in the following sense:

$$\Delta_{V \cup I}^{\bar{x}}(xy, uy) = \Delta_V^{\bar{x}y}(x, u).$$

for all  $V, I \in \mathbb{Z}^d$  such that  $V \cap I = \emptyset$  and all  $x, u \in X^V, y \in X^I$  and  $\bar{x} \in X^{\mathbb{Z}^d \setminus (V \cup I)}$ .



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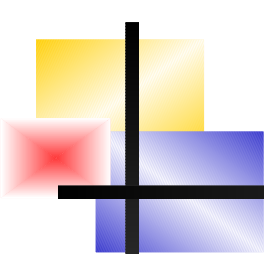
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Equivalent form of consistency conditions:

$$\Delta_{V \cup I}^{\bar{x}}(xy, uv) = \Delta_V^{\bar{x}y}(x, u) + \Delta_I^{\bar{x}u}(y, v)$$

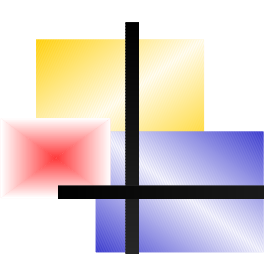
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# Energy $\longleftrightarrow$ probability in infinite volume

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a family  $\mathbf{Q} = \{q_V^{\bar{x}}, V \subseteq \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus V}\}$  is a specification iff

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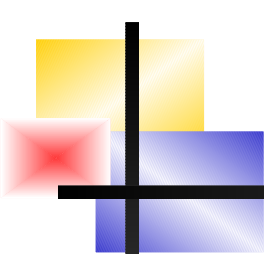
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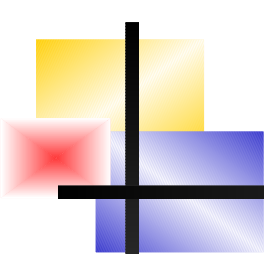
**Proof.** We have

$$\Delta_{V \cup I}^{\bar{x}}(xy, uy) = \Delta_V^{\bar{x}y}(x, u) \iff \frac{q_{V \cup I}^{\bar{x}}(xy)}{q_{V \cup I}^{\bar{x}}(uy)} = \frac{q_V^{\bar{x}y}(x)}{q_V^{\bar{x}y}(u)}.$$



# One-point transition energy fields

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Hence, for  $V = \{t_1, t_2, \dots, t_n\}$  we get

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So the problem of reconstruction of  $\Delta$  by

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For the problem of reconstruction, the natural consistency conditions are:

$$\Delta_t^{\bar{x}y}(x, u) + \Delta_s^{\bar{x}u}(y, v) = \Delta_s^{\bar{x}x}(y, v) + \Delta_t^{\bar{x}v}(x, u).$$

# One-point transition energy fields

**Definition.** A family  $\Delta^{(1)} = \{\Delta_t^{\bar{x}}, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus t}\}$  is called *one point transition energy field* if its elements are consistent in the following sense:

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for all  $t, s \in \mathbb{Z}^d$  such that  $t \neq s$  and all  $x, u \in X^t, y, v \in X^s$  and  $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t, s\}}$ .

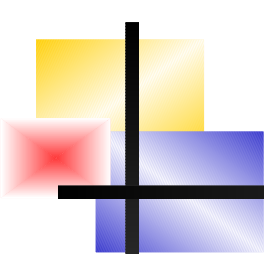
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**Theorem 2.** We have a one-to-one relationship  $\Delta \longleftrightarrow \Delta^{(1)}$ .



# 1-specifications

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$$\Delta_t^{\bar{x}y}(x, u) + \Delta_s^{\bar{x}u}(y, v) = \Delta_s^{\bar{x}x}(y, v) + \Delta_t^{\bar{x}v}(x, u)$$

$$\Updownarrow$$

$$\frac{q_t^{\bar{x}y}(x)}{q_t^{\bar{x}y}(u)} \frac{q_s^{\bar{x}u}(y)}{q_s^{\bar{x}u}(v)} = \frac{q_s^{\bar{x}x}(y)}{q_s^{\bar{x}x}(v)} \frac{q_t^{\bar{x}v}(x)}{q_t^{\bar{x}v}(u)}$$

$$\Updownarrow$$

$$q_t^{\bar{x}y}(x) q_s^{\bar{x}x}(v) q_t^{\bar{x}v}(u) q_s^{\bar{x}u}(y) = q_s^{\bar{x}x}(y) q_t^{\bar{x}y}(u) q_s^{\bar{x}u}(v) q_t^{\bar{x}v}(x)$$

# 1-specifications

$$\Delta_t^{\bar{x}y}(x, u) + \Delta_s^{\bar{x}u}(y, v) = \Delta_s^{\bar{x}x}(y, v) + \Delta_t^{\bar{x}v}(x, u)$$

$$\Updownarrow$$

$$\frac{q_t^{\bar{x}y}(x)}{q_t^{\bar{x}y}(u)} \frac{q_s^{\bar{x}u}(y)}{q_s^{\bar{x}u}(v)} = \frac{q_s^{\bar{x}x}(y)}{q_s^{\bar{x}x}(v)} \frac{q_t^{\bar{x}v}(x)}{q_t^{\bar{x}v}(u)}$$

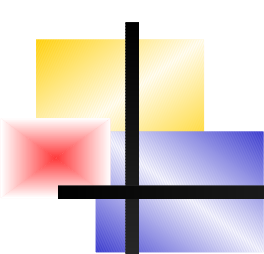
$$\Updownarrow$$

$$q_t^{\bar{x}y}(x) q_s^{\bar{x}x}(v) q_t^{\bar{x}v}(u) q_s^{\bar{x}u}(y) = q_s^{\bar{x}x}(y) q_t^{\bar{x}y}(u) q_s^{\bar{x}u}(v) q_t^{\bar{x}v}(x)$$

**Definition.** A family  $Q^{(1)} = \{q_t^{\bar{x}}, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus t}\}$  is called *1-specification* if its elements are consistent in the following sense:

$$q_t^{\bar{x}y}(x) q_s^{\bar{x}x}(v) q_t^{\bar{x}v}(u) q_s^{\bar{x}u}(y) = q_s^{\bar{x}x}(y) q_t^{\bar{x}y}(u) q_s^{\bar{x}u}(v) q_t^{\bar{x}v}(x)$$

for all  $t, s \in \mathbb{Z}^d$  such that  $t \neq s$  and all  $x, u \in X^t, y, v \in X^s$  and  $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t, s\}}$ .



# 1-specifications

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**Theorem 3.** We have a one-to-one relationship  $Q^{(1)} \longleftrightarrow \Delta^{(1)}$

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$$q_t^{\bar{x}}(x) = \frac{\exp\{\Delta_t^{\bar{x}}(x, u)\}}{\sum_{z \in X^t} \exp\{\Delta_t^{\bar{x}}(z, u)\}}, \quad x \in X^t,$$

where  $u \in X^t$  is arbitrary and  $\Delta^{(1)} = \{\Delta_t^{\bar{x}}, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus t}\}$  is some one-point transition energy field

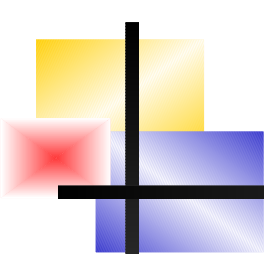
# 1-specifications

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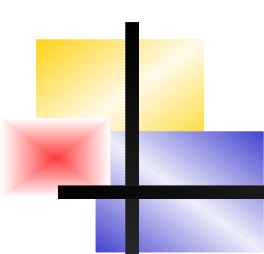
where  $u \in X^t$  is arbitrary and  $\Delta^{(1)} = \{\Delta_t^{\bar{x}}, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus t}\}$  is some one-point transition energy field, uniquely determined by

$$\Delta_t^{\bar{x}}(x, u) = \ln \frac{q_t^{\bar{x}}(x)}{q_t^{\bar{x}}(u)}, \quad x, u \in X^t.$$



# Answer to Dobrushin's problem

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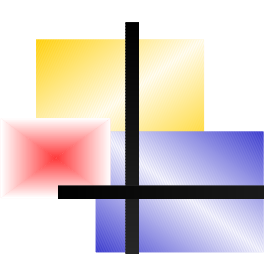
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$$\begin{array}{ccc} Q & & Q^{(1)} \\ \text{Th. 1} \updownarrow & & \updownarrow \text{Th. 3} \\ \Delta & \xleftrightarrow{\text{Th. 2}} & \Delta^{(1)} \end{array}$$

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# Explicit formula

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$$\begin{aligned}
 q_V^{\bar{x}}(x) &= \frac{\exp\{\Delta_V^{\bar{x}}(x, u)\}}{\sum_{z \in X^V} \exp\{\Delta_V^{\bar{x}}(z, u)\}} \\
 &= \frac{\frac{q_{t_1}^{\bar{x}x_{t_2}x_{t_3}\cdots x_{t_n}}(x_{t_1})}{q_{t_1}^{\bar{x}x_{t_2}x_{t_3}\cdots x_{t_n}}(u_{t_1})} \frac{q_{t_2}^{\bar{x}u_{t_1}x_{t_3}\cdots x_{t_n}}(x_{t_2})}{q_{t_2}^{\bar{x}u_{t_1}x_{t_3}\cdots x_{t_n}}(u_{t_2})} \cdots \frac{q_{t_n}^{\bar{x}u_{t_1}u_{t_2}\cdots u_{t_{n-1}}}(x_{t_n})}{q_{t_n}^{\bar{x}u_{t_1}u_{t_2}\cdots u_{t_{n-1}}}(u_{t_n})}}{\sum_{z \in X^V} \frac{q_{t_1}^{\bar{x}x_{t_2}x_{t_3}\cdots x_{t_n}}(x_{t_1})}{q_{t_1}^{\bar{x}x_{t_2}x_{t_3}\cdots x_{t_n}}(u_{t_1})} \frac{q_{t_2}^{\bar{x}u_{t_1}x_{t_3}\cdots x_{t_n}}(x_{t_2})}{q_{t_2}^{\bar{x}u_{t_1}x_{t_3}\cdots x_{t_n}}(u_{t_2})} \cdots \frac{q_{t_n}^{\bar{x}u_{t_1}u_{t_2}\cdots u_{t_{n-1}}}(x_{t_n})}{q_{t_n}^{\bar{x}u_{t_1}u_{t_2}\cdots u_{t_{n-1}}}(u_{t_n})}}
 \end{aligned}$$

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 q_V^{\bar{x}}(x) &= \frac{\exp\{\Delta_V^{\bar{x}}(x, u)\}}{\sum_{z \in X^V} \exp\{\Delta_V^{\bar{x}}(z, u)\}} \\
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 \end{aligned}$$

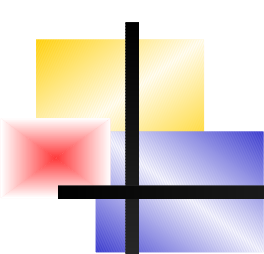
Hence:  $Q$  is quasilocal  $\iff Q^{(1)}$  is quasilocal.

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 \end{aligned}$$

Hence:  $Q$  is quasilocal  $\iff Q^{(1)}$  is quasilocal.

Moreover:  $P$  is compatible with  $Q \iff P$  is *compatible* with  $Q^{(1)}$ .



# Reformulation of Dobrushin's theorems

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# Reformulation of Dobrushin's theorems

**Theorem.** Let  $Q = \{q_V^{\bar{x}}, V \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus V}\}$  be some specification.

1) If  $q_t^{\bar{x}}(x)$  (where  $t \in \mathbb{Z}^d$ ) is quasilocal w.r.t.  $\bar{x}$ , then there exists a random field  $\mathbf{P}$  compatible with  $Q$ .

2) If, moreover,

$$\frac{1}{2} \sup_{t \in \mathbb{Z}^d} \sum_{s \in \mathbb{Z}^d \setminus t} \sup_{\bar{x}, \bar{y} : \bar{x}_{\mathbb{Z}^d \setminus t} = \bar{y}_{\mathbb{Z}^d \setminus t}} \sum_{x \in X^t} |q_t^{\bar{x}}(x) - q_t^{\bar{y}}(x)| < 1,$$

then the random field  $\mathbf{P}$  compatible with  $Q$  is unique.

# Reformulation of Dobrushin's theorems

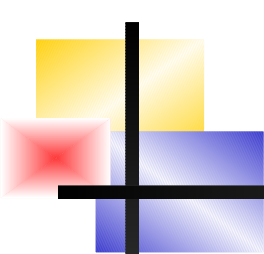
**Theorem.** Let  $Q^{(1)} = \{q_t^{\bar{x}}, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus t}\}$  be some 1–specification.

1) If  $Q^{(1)}$  is quasilocal (i.e., if  $q_t^{\bar{x}}(x)$  is quasilocal w.r.t.  $\bar{x}$ ), then there exists a random field  $\mathbf{P}$  compatible with  $Q^{(1)}$ .

2) If, moreover,

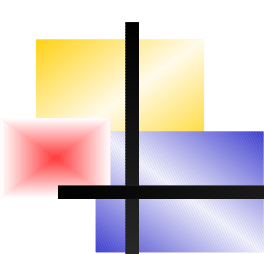
$$\frac{1}{2} \sup_{t \in \mathbb{Z}^d} \sum_{s \in \mathbb{Z}^d \setminus t} \sup_{\bar{x}, \bar{y} : \bar{x}_{\mathbb{Z}^d \setminus t} = \bar{y}_{\mathbb{Z}^d \setminus t}} \sum_{x \in X^t} |q_t^{\bar{x}}(x) - q_t^{\bar{y}}(x)| < 1,$$

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# Further consequences

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A family  $\Delta = \{\Delta_V^{\bar{x}}, V \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus V}\}$  is a transition energy field iff

$$\Delta_V^{\bar{x}}(x, u) = H_V^{\bar{x}}(u) - H_V^{\bar{x}}(x),$$

where the family  $H = \{H_V^{\bar{x}}, V \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus V}\}$  satisfies

$$H_{V \cup I}^{\bar{x}}(xy) + H_V^{\bar{x}y}(u) = H_{V \cup I}^{\bar{x}}(uy) + H_V^{\bar{x}y}(x)$$

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or, equivalently,

$$H_{V \cup I}^{\bar{x}}(xy) + H_V^{\bar{x}y}(u) + H_I^{\bar{x}u}(v) = H_{V \cup I}^{\bar{x}}(uv) + H_V^{\bar{x}y}(x) + H_I^{\bar{x}u}(y).$$

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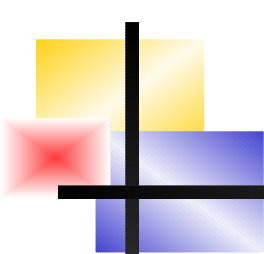
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Then

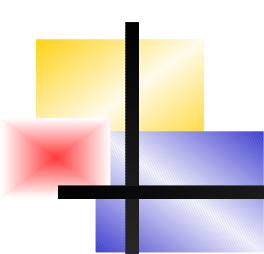
$$q_V^{\bar{x}}(x) = \frac{\exp\{\Delta_V^{\bar{x}}(x, u)\}}{\sum_{z \in X^V} \exp\{\Delta_V^{\bar{x}}(z, u)\}} = \frac{\exp\{-H_V^{\bar{x}}(x)\}}{\sum_{z \in X^V} \exp\{-H_V^{\bar{x}}(z)\}}.$$



# Further consequences

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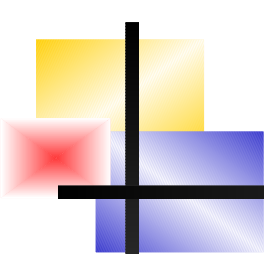
- ◆ Axiomatic definition of the *Hamiltonian*  $H$  (with two equivalent versions of consistency conditions).



# Further consequences

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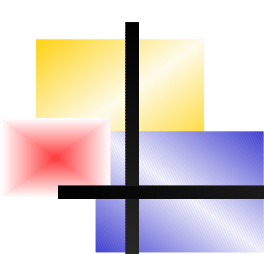
- ◆ Axiomatic definition of the *Hamiltonian*  $H$  (with two equivalent versions of consistency conditions).
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  - ◆ A justification of the Gibbs formula (for infinite volume systems).
  - ◆ An answer to Ruelle's problem.
- ◆ Everything can be reduced to one-point systems, including *one-point Hamiltonian*  $\mathbf{H}^{(1)} = \{H_t^{\bar{x}}, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus t}\}$ :

$$H_t^{\bar{x}y}(x) + H_s^{\bar{x}x}(v) + H_t^{\bar{x}v}(u) + H_s^{\bar{x}u}(y) = H_s^{\bar{x}x}(y) + H_t^{\bar{x}y}(u) + H_s^{\bar{x}u}(v) + H_t^{\bar{x}v}(x).$$