On the Energy-Probability Relationship in Statistical Physics of Infinite-Volume Systems

S. Dachian
(joint work with B.S. Nahapetian)

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- We omit braces for one-point sets (we write a instead of $\{a\}$).

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- Kolmogorov's description: $\mathbf{P} \longleftrightarrow \boldsymbol{P}$.

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- For all $V \Subset \mathbb{Z}^d$, the probability distributions $\mathrm{Q}_V^{\overline{x}}$ defined by

$$\mathrm{Q}_V^{\overline{x}}(x) = \lim_{\Lambda \uparrow \mathbb{Z}^d \setminus V} \frac{\mathrm{P}_{V \cup \Lambda}(x\overline{x}_\Lambda)}{\mathrm{P}_\Lambda(\overline{x}_\Lambda)}, \quad x \in X^V,$$

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- $\boldsymbol{Q} = \left\{q_V^{\overline{x}},\ V \Subset \mathbb{Z}^d,\ \overline{x} \in X^{\mathbb{Z}^d \setminus V}\right\}$ — *(version of) conditional distribution* of $\mathbf{P}$ if

$$q_V^{\overline{x}} = \mathrm{Q}_V^{\overline{x}}$$

  for all $V \Subset \mathbb{Z}^d$ and almost all $\overline{x} \in X^{\mathbb{Z}^d \setminus V}$.

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[Dobrushin 68]

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Definition. A family $\boldsymbol{Q} = \left\{ q_V^{\overline{x}},\ V \Subset \mathbb{Z}^d,\ \overline{x} \in X^{\mathbb{Z}^d \setminus V} \right\}$ is called *specification* if its elements are consistent in the Dobrushin sense

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$$q_V^{\overline{x}}(xy) = \left(q_V^{\overline{x}}\right)_{V\setminus I}(x)\, q_I^{\overline{x}x}(y)$$

for all $I \subset V \Subset \mathbb{Z}^d$, $x \in X^{V\setminus I}$, $y \in X^I$ and $\overline{x} \in X^{\mathbb{Z}^d \setminus V}$.

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Equivalent form of Dobrushin's consistency conditions:

$$q_V^{\overline{x}}(xy)\, q_{V\setminus I}^{\overline{x}y}(u) = q_V^{\overline{x}}(uy)\, q_{V\setminus I}^{\overline{x}y}(x)$$

for all $I \subset V \Subset \mathbb{Z}^d$, $x, u \in X^{V\setminus I}$, $y \in X^I$ and $\overline{x} \in X^{\mathbb{Z}^d \setminus V}$.

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$\boldsymbol{Q}$ is a *Gibbsian specification* if $q_V^{\overline{x}}$ is defined by the Gibbs formula

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with the *Hamiltonian*

$$H_V^{\overline{x}}(x) = \lim_{\Lambda \uparrow \mathbb{Z}^d \setminus V} \sum_{\emptyset \neq I \subset V} \sum_{J \subset \Lambda} \Phi_{I \cup J}(x_I \overline{x}_J), \quad x \in X^V,$$

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Ruelle's problem: how broad the class of Gibbsian specifications is ? [Ruelle 78, Appendix B (Open problems), B.1]

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2) If, moreover, $\boldsymbol{Q}$ satisfies *Dobrushin's uniqueness condition*

$$\frac{1}{2} \sup_{t \in \mathbb{Z}^d} \sum_{s \in \mathbb{Z}^d \setminus t} \sup_{\overline{x}, \overline{y}\,:\,\overline{x}_{\mathbb{Z}^d \setminus t} = \overline{y}_{\mathbb{Z}^d \setminus t}} \sum_{x \in X^t} \left| q_t^{\overline{x}}(x) - q_t^{\overline{y}}(x) \right| < 1,$$

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Dobrushin's problem: $\boldsymbol{Q} \overset{?}{\longleftrightarrow} \boldsymbol{Q}^{(1)} = \left\{q_t^{\overline{x}},\ t \in \mathbb{Z}^d,\ \overline{x} \in X^{\mathbb{Z}^d \setminus t}\right\}$.

Energy $\longleftrightarrow$ probability in finite volume

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Definition. A function $\Delta_V = \left\{\Delta_V(x,u),\ x,u \in X^V\right\}$ is called *transition energy in volume V* if

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$$\Delta_V(x,u) = \ln \frac{\mathrm{P}_V(x)}{\mathrm{P}_V(u)}, \quad x,u \in X^V.$$

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But we do not have a one-to-one relationship $\mathrm{P}_V \longleftrightarrow H_V$ (as H_V is defined up to an arbitrary additive constant).

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Moreover: $\Delta_{V\cup I}^{\overline{x}}(xy,uy) = \Delta(xy\overline{x}, uy\overline{x}) = \Delta_V^{\overline{x}y}(x,u).$

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$$\Delta_{V\cup I}^{\overline{x}}(xy, uy) = \Delta_V^{\overline{x}y}(x, u).$$

for all $V, I \Subset \mathbb{Z}^d$ such that $V \cap I = \emptyset$ and all $x, u \in X^V$, $y \in X^I$ and $\overline{x} \in X^{\mathbb{Z}^d \setminus (V\cup I)}$.

Proposition. We have a one-to-one relationship $\Delta \longleftrightarrow \mathbf{\Delta}$.

Equivalent form of consistency conditions:

$$\Delta_{V\cup I}^{\overline{x}}(xy, uv) = \Delta_V^{\overline{x}y}(x, u) + \Delta_I^{\overline{x}u}(y, v)$$

for all $V, I \Subset \mathbb{Z}^d$ such that $V \cap I = \emptyset$ and all $x, u \in X^V$, $y, v \in X^I$ and $\overline{x} \in X^{\mathbb{Z}^d \setminus (V\cup I)}$.

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a family $\boldsymbol{Q} = \left\{ q_V^{\overline{x}},\ V \Subset \mathbb{Z}^d,\, \overline{x} \in X^{\mathbb{Z}^d \setminus V} \right\}$ is a specification iff

$$q_V^{\overline{x}}(x) = \frac{\exp\{\Delta_V^{\overline{x}}(x,u)\}}{\sum\limits_{z \in X^V} \exp\{\Delta_V^{\overline{x}}(z,u)\}}, \quad x \in X^V,$$

where $u \in X^V$ is arbitrary and $\boldsymbol{\Delta} = \left\{ \Delta_V^{\overline{x}},\ V \Subset \mathbb{Z}^d,\, \overline{x} \in X^{\mathbb{Z}^d \setminus V} \right\}$ is some transition energy field

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$$\Delta_V^{\overline{x}}(x,u) = \ln \frac{q_V^{\overline{x}}(x)}{q_V^{\overline{x}}(u)}, \quad x, u \in X^V.$$

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Proof. We have

$$\Delta_{V\cup I}^{\overline{x}}(xy,uy) = \Delta_V^{\overline{x}y}(x,u) \Longleftrightarrow \frac{q_{V\cup I}^{\overline{x}}(xy)}{q_{V\cup I}^{\overline{x}}(uy)} = \frac{q_V^{\overline{x}y}(x)}{q_V^{\overline{x}y}(u)}.$$

One-point transition energy fields

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Recall that $\Delta^{\overline{x}}_{V\cup I}(xy,uv) = \Delta^{\overline{x}y}_{V}(x,u) + \Delta^{\overline{x}u}_{I}(y,v)$.

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Hence, for $V = \{t_1, t_2, \dots, t_n\}$ we get

$$\Delta^{\overline{x}}_{V}(x,u) = \Delta^{\overline{x}x_{t_2}x_{t_3}\cdots x_{t_n}}_{t_1}(x_{t_1},u_{t_1}) + \Delta^{\overline{x}u_{t_1}x_{t_3}\cdots x_{t_n}}_{t_2}(x_{t_2},u_{t_2})$$

$$+ \Delta^{\overline{x}u_{t_1}u_{t_2}x_{t_4}\cdots x_{t_n}}_{t_3}(x_{t_3},u_{t_3}) + \cdots + \Delta^{\overline{x}u_{t_1}u_{t_2}\cdots u_{t_{n-1}}}_{t_n}(x_{t_n},u_{t_n}),$$

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$$+\Delta_{t_3}^{\overline{x}u_{t_1}u_{t_2}x_{t_4}\cdots x_{t_n}}(x_{t_3},u_{t_3})+\cdots+\Delta_{t_n}^{\overline{x}u_{t_1}u_{t_2}\cdots u_{t_{n-1}}}(x_{t_n},u_{t_n}),$$

So the problem of reconstruction of $\boldsymbol{\Delta}$ by $\boldsymbol{\Delta}^{(1)}=\left\{\Delta_t^{\overline{x}},\ t\in\mathbb{Z}^d,\ \overline{x}\in X^{\mathbb{Z}^d\setminus t}\right\}$ is solved.

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$\boldsymbol{\Delta}^{(1)}=\left\{\Delta_t^{\overline{x}},\ t\in\mathbb{Z}^d,\ \overline{x}\in X^{\mathbb{Z}^d\setminus t}\right\}$ is solved.

For the problem of reconstruction, the natural consistency conditions are:

$$\Delta_t^{\overline{x}y}(x,u)+\Delta_s^{\overline{x}u}(y,v)=\Delta_s^{\overline{x}x}(y,v)+\Delta_t^{\overline{x}v}(x,u).$$

One-point transition energy fields

Definition. A family $\mathbf{\Delta}^{(1)} = \left\{\Delta_t^{\overline{x}},\ t \in \mathbb{Z}^d,\ \overline{x} \in X^{\mathbb{Z}^d \setminus t}\right\}$ is called *one point transition energy field* if its elements are consistent in the following sense:

$$\Delta_t^{\overline{x}y}(x,u) + \Delta_s^{\overline{x}u}(y,v) = \Delta_s^{\overline{x}x}(y,v) + \Delta_t^{\overline{x}v}(x,u)$$

for all $t, s \in \mathbb{Z}^d$ such that $t \neq s$ and all $x, u \in X^t$, $y, v \in X^s$ and $\overline{x} \in X^{\mathbb{Z}^d \setminus \{t,s\}}$.

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Theorem 2. We have a one-to-one relationship $\mathbf{\Delta} \longleftrightarrow \mathbf{\Delta}^{(1)}$.

1–specifications

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$$\Delta_t^{\overline{x}y}(x,u)+\Delta_s^{\overline{x}u}(y,v)=\Delta_s^{\overline{x}x}(y,v)+\Delta_t^{\overline{x}v}(x,u)$$

$$\Updownarrow$$

$$\frac{q_t^{\overline{x}y}(x)}{q_t^{\overline{x}y}(u)}\,\frac{q_s^{\overline{x}u}(y)}{q_s^{\overline{x}u}(v)}=\frac{q_s^{\overline{x}x}(y)}{q_s^{\overline{x}x}(v)}\,\frac{q_t^{\overline{x}v}(x)}{q_t^{\overline{x}v}(u)}$$

$$\Updownarrow$$

$$q_t^{\overline{x}y}(x)\,q_s^{\overline{x}x}(v)\,q_t^{\overline{x}v}(u)\,q_s^{\overline{x}u}(y)=q_s^{\overline{x}x}(y)\,q_t^{\overline{x}y}(u)\,q_s^{\overline{x}u}(v)\,q_t^{\overline{x}v}(x)$$

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$$q_t^{\overline{x}y}(x)\,q_s^{\overline{x}x}(v)\,q_t^{\overline{x}v}(u)\,q_s^{\overline{x}u}(y) = q_s^{\overline{x}x}(y)\,q_t^{\overline{x}y}(u)\,q_s^{\overline{x}u}(v)\,q_t^{\overline{x}v}(x)$$

Definition. A family $\boldsymbol{Q}^{(1)} = \left\{q_t^{\overline{x}},\ t \in \mathbb{Z}^d,\ \overline{x} \in X^{\mathbb{Z}^d \setminus t}\right\}$ is called *1–specification* if its elements are consistent in the following sense:

$$q_t^{\overline{x}y}(x)\,q_s^{\overline{x}x}(v)\,q_t^{\overline{x}v}(u)\,q_s^{\overline{x}u}(y) = q_s^{\overline{x}x}(y)\,q_t^{\overline{x}y}(u)\,q_s^{\overline{x}u}(v)\,q_t^{\overline{x}v}(x)$$

for all $t, s \in \mathbb{Z}^d$ such that $t \neq s$ and all $x, u \in X^t$, $y, v \in X^s$ and $\overline{x} \in X^{\mathbb{Z}^d \setminus \{t,s\}}$.

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Theorem 3. We have a one-to-one relationship $\boldsymbol{Q}^{(1)} \longleftrightarrow \boldsymbol{\Delta}^{(1)}$

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$$q_t^{\overline{x}}(x) = \frac{\exp\{\Delta_t^{\overline{x}}(x,u)\}}{\sum\limits_{z \in X^t} \exp\{\Delta_t^{\overline{x}}(z,u)\}}, \quad x \in X^t,$$

where $u \in X^t$ is arbitrary and $\boldsymbol{\Delta}^{(1)} = \left\{ \Delta_t^{\overline{x}},\ t \in \mathbb{Z}^d,\ \overline{x} \in X^{\mathbb{Z}^d \setminus t} \right\}$ is some one-point transition energy field

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Answer to Dobrushin's problem

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$$
\begin{tikzcd}
\boldsymbol{Q} & \boldsymbol{Q}^{(1)} \\
\boldsymbol{\Delta} \arrow[u, leftrightarrow, "\text{Th. 1}"] \arrow[r, leftrightarrow, "\text{Th. 2}"] & \boldsymbol{\Delta}^{(1)} \arrow[u, leftrightarrow, "\text{Th. 3}"']
\end{tikzcd}
$$

Answer to Dobrushin's problem

$$\begin{tikzcd} \boldsymbol{Q} \arrow[r, leftrightarrow, red] \arrow[d, leftrightarrow, "\text{Th. 1}"'] & \boldsymbol{Q}^{(1)} \arrow[d, leftrightarrow, "\text{Th. 3}"] \\ \boldsymbol{\Delta} \arrow[r, leftrightarrow, "\text{Th. 2}"] & \boldsymbol{\Delta}^{(1)} \end{tikzcd}$$

Explicit formula

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$$q_V^{\bar{x}}(x) = \frac{\exp\{\Delta_V^{\bar{x}}(x,u)\}}{\sum_{z\in X^V} \exp\{\Delta_V^{\bar{x}}(z,u)\}}$$

$$= \frac{\dfrac{q_{t_1}^{\bar{x}x_{t_2}x_{t_3}\cdots x_{t_n}}(x_{t_1})}{q_{t_1}^{\bar{x}x_{t_2}x_{t_3}\cdots x_{t_n}}(u_{t_1})} \dfrac{q_{t_2}^{\bar{x}u_{t_1}x_{t_3}\cdots x_{t_n}}(x_{t_2})}{q_{t_2}^{\bar{x}u_{t_1}x_{t_3}\cdots x_{t_n}}(u_{t_2})} \cdots \dfrac{q_{t_n}^{\bar{x}u_{t_1}u_{t_2}\cdots u_{t_{n-1}}}(x_{t_n})}{q_{t_n}^{\bar{x}u_{t_1}u_{t_2}\cdots u_{t_{n-1}}}(u_{t_n})}}{\sum_{z\in X^V} \dfrac{q_{t_1}^{\bar{x}x_{t_2}x_{t_3}\cdots x_{t_n}}(x_{t_1})}{q_{t_1}^{\bar{x}x_{t_2}x_{t_3}\cdots x_{t_n}}(u_{t_1})} \dfrac{q_{t_2}^{\bar{x}u_{t_1}x_{t_3}\cdots x_{t_n}}(x_{t_2})}{q_{t_2}^{\bar{x}u_{t_1}x_{t_3}\cdots x_{t_n}}(u_{t_2})} \cdots \dfrac{q_{t_n}^{\bar{x}u_{t_1}u_{t_2}\cdots u_{t_{n-1}}}(x_{t_n})}{q_{t_n}^{\bar{x}u_{t_1}u_{t_2}\cdots u_{t_{n-1}}}(u_{t_n})}}$$

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Hence: $\boldsymbol{Q}$ is quasilocal $\Longleftrightarrow$ $\boldsymbol{Q}^{(1)}$ is quasilocal.

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Hence: $\boldsymbol{Q}$ is quasilocal $\Longleftrightarrow$ $\boldsymbol{Q}^{(1)}$ is quasilocal.

Moreover: $\mathbf{P}$ is compatible with $\boldsymbol{Q}$ $\Longleftrightarrow$ $\mathbf{P}$ is *compatible* with $\boldsymbol{Q}^{(1)}$.

Reformulation of Dobrushin's theorems

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Theorem. Let $\boldsymbol{Q} = \left\{ q_V^{\overline{x}},\ V \Subset \mathbb{Z}^d,\ \overline{x} \in X^{\mathbb{Z}^d \setminus V} \right\}$ be some specification.

1) If $q_t^{\overline{x}}(x)$ (where $t \in \mathbb{Z}^d$) is quasilocal w.r.t. $\overline{x}$, then there exists a random field P compatible with $\boldsymbol{Q}$.

2) If, moreover,

$$\frac{1}{2} \sup_{t \in \mathbb{Z}^d} \sum_{s \in \mathbb{Z}^d \setminus t} \sup_{\overline{x}, \overline{y} \,:\, \overline{x}_{\mathbb{Z}^d \setminus t} = \overline{y}_{\mathbb{Z}^d \setminus t}} \sum_{x \in X^t} \left| q_t^{\overline{x}}(x) - q_t^{\overline{y}}(x) \right| < 1,$$

then the random field P compatible with $\boldsymbol{Q}$ is unique.

Reformulation of Dobrushin's theorems

Theorem. Let $\boldsymbol{Q}^{(1)} = \left\{q_t^{\overline{x}},\ t \in \mathbb{Z}^d,\ \overline{x} \in X^{\mathbb{Z}^d \setminus t}\right\}$ be some 1–specification.

1) If $\boldsymbol{Q}^{(1)}$ is quasilocal (i.e., if $q_t^{\overline{x}}(x)$ is quasilocal w.r.t. $\overline{x}$), then there exists a random field P compatible with $\boldsymbol{Q}^{(1)}$.

2) If, moreover,

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A family $\boldsymbol{\Delta} = \left\{ \Delta_V^{\overline{x}},\ V \Subset \mathbb{Z}^d,\ \overline{x} \in X^{\mathbb{Z}^d \setminus V} \right\}$ is a transition energy field iff

$$\Delta_V^{\overline{x}}(x,u) = H_V^{\overline{x}}(u) - H_V^{\overline{x}}(x),$$

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Then

$$q_V^{\overline{x}}(x) = \frac{\exp\{\Delta_V^{\overline{x}}(x,u)\}}{\sum\limits_{z\in X^V} \exp\{\Delta_V^{\overline{x}}(z,u)\}} = \frac{\exp\{-H_V^{\overline{x}}(x)\}}{\sum\limits_{z\in X^V} \exp\{-H_V^{\overline{x}}(z)\}}.$$

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- Theorem 1 says that a family $\boldsymbol{Q} = \left\{q_V^{\overline{x}},\ V \Subset \mathbb{Z}^d,\ \overline{x} \in X^{\mathbb{Z}^d \setminus V}\right\}$ is a specification iff it has Gibbsian form (with some Hamiltonian satisfying above axiomatic definition).
  - A justification of the Gibbs formula (for infinite volume systems).
  - An answer to Ruelle's problem.
- Everything can be reduced to one-point systems, including *one-point Hamiltonian* $\boldsymbol{H}^{(1)} = \left\{H_t^{\overline{x}},\ t \in \mathbb{Z}^d,\ \overline{x} \in X^{\mathbb{Z}^d \setminus t}\right\}$:

$$H_t^{\overline{x}y}(x)+H_s^{\overline{x}x}(v)+H_t^{\overline{x}v}(u)+H_s^{\overline{x}u}(y)=H_s^{\overline{x}x}(y)+H_t^{\overline{x}y}(u)+H_s^{\overline{x}u}(v)+H_t^{\overline{x}v}(x).$$