# The Quartic Interaction of Spinless Relativistic and Non-Relativistic Bosons

## Why pairwise zero-range forces are not enough in three dimensions

Rodolfo Figari

GSSI, L'Aquila

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# The question

The number-conserving quartic interaction is pairwise:

$$\mathbb{H}_4 = \frac{1}{2} \int V(x-y)\, a_x^\dagger a_y^\dagger a_y a_x \; dx\, dy \qquad \Longleftrightarrow \qquad \sum_{i<j} V(x_i - x_j).$$

In a zero-range limit one would like to write formally

$$\sum_{i<j} \nu\, \delta(x_i - x_j).$$

But for three-dimensional bosons this is precisely where ultraviolet instability appears.

$$\boxed{\text{Can a stable zero-range theory remain purely two-body?}}$$

# The relativistic alternative: point-local vertices

For a neutral spinless scalar field the basic Lorentz-scalar local densities include

$$\phi(x)^2, \qquad \phi(x)^4, \qquad (\partial_\mu\phi)(\partial^\mu\phi), \dots$$

The quartic interaction is therefore written as a spacetime-local vertex,

$$S_{\rm int}[\phi] = -\frac{\lambda}{4!}\int \phi(x)^4 \; d^D x,$$

or, in a Hamiltonian picture at fixed time,

$$H_{\rm int} \sim \int :\phi(t,\mathbf{x})^4: \; d^d x.$$

Relativistic locality pushes the fundamental interaction to coincidence in spacetime.

# Bose–Hubbard as a clean lattice laboratory

For bosons on a lattice $\Lambda$,

$$H_{\mathrm{BH}} = -J \sum_{\langle x,y \rangle} (a_x^\dagger a_y + a_y^\dagger a_x) + \frac{U}{2} \sum_{x \in \Lambda} n_x(n_x - 1), \qquad n_x = a_x^\dagger a_x.$$

**Hopping**

$$-J \sum_{\langle x,y \rangle} a_x^\dagger a_y$$

is a discrete kinetic energy.

**On-site repulsion**

$$\frac{U}{2} n_x(n_x - 1) = \frac{U}{2} (a_x^\dagger)^2 a_x^2$$

is local and quartic.

$\boxed{\text{The lattice separates locality from ultraviolet singularity.}}$

# Quartic means pairwise – higher powers mean genuine many-body

At one site,

$$\frac{U}{2} n_x (n_x - 1)$$

counts unordered *pairs* of particles occupying $x$.

A genuine three-body on-site term is instead

$$\frac{W_3}{3!} n_x (n_x - 1)(n_x - 2) = \frac{W_3}{3!} (a_x^\dagger)^3 a_x^3.$$

More generally,

$$\frac{W_k}{k!} (a_x^\dagger)^k a_x^k$$

acts only when $k$ particles meet at the same site.

$$\boxed{\phi^4 \text{ or } (a^\dagger)^2 a^2 \text{ does not exhaust local bosonic interactions.}}$$

# Selected rigorous landmarks for Bose–Hubbard-type lattice bosons

**Hard-core optical lattice: BEC versus Mott behaviour.** For a hard-core lattice gas at half filling with a periodic optical potential, BEC is proved at small potential and low temperature; for sufficiently large potential the low-temperature phase has no BEC and exhibits a Mott gap.

[1] M. Aizenman, E.H. Lieb, R. Seiringer, J.P. Solovej, J. Yngvason, *Bose–Einstein quantum phase transition in an optical lattice model*, Phys. Rev. A **70** (2004), 023612.

**Hard-core Bose–Hubbard: perturbative Mott boundary.** Rigorous perturbation theory gives, to lowest order in the tunnelling amplitude, the critical line bordering the Mott region.

[2] R. Fernández, J. Fröhlich, D. Ueltschi, *Mott transition in lattice boson models*, Commun. Math. Phys. **266** (2006), 777–795.

**Dilute 3D repulsive Bose–Hubbard gas.** On arbitrary three-dimensional Bravais lattices,

$$e_0(\rho) = 4\pi a_{\mathrm{latt}} \rho^2 \big(1 + O(\rho^{1/6})\big), \qquad \rho \to 0,$$

so the lattice scattering length contains the microscopic information at leading order.

[3] N. Mokrzański, M. Napiórkowski, J. Wojtkiewicz, *Ground state energy of the dilute Bose–Hubbard gas on Bravais lattices*, arXiv:2602.16566 (2026).

Scope: [1]–[2] concern hard-core variants; [3] concerns the repulsive dilute Bose–Hubbard gas itself.

# Dilute 3D Bose–Hubbard gas: universality on the lattice

For bosons on an arbitrary three-dimensional Bravais lattice with positive hopping amplitudes and on-site repulsion,

$$\boxed{e_0(\rho) = 4\pi a_{\mathrm{latt}}\, \rho^2 \left(1 + O(\rho^{1/6})\right), \qquad \rho \to 0.}$$

Here $a_{\mathrm{latt}}$ is defined by the corresponding *two-body* scattering problem.

Thus the microscopic lattice geometry and interaction strength enter the leading term only through one low-energy parameter:

$$\boxed{\text{microscopic lattice model} \longrightarrow a_{\mathrm{latt}} \longrightarrow 4\pi a_{\mathrm{latt}}\rho^2.}$$

Mokrzański–Napiórkowski–Wojtkiewicz (2026), analogue of the Dyson/Lieb–Yngvason leading-order theorem.

# The same number appears in two opposite asymptotic directions

For a dilute continuum Bose gas,

$$\frac{E_0}{N} \sim 4\pi a\rho \qquad (\hbar^2/2m = 1).$$

A regular short-range interaction is replaced, at leading order, by its scattering length.

Conversely, many randomly placed fixed zero-range scatterers may generate an effective regular potential:

$$\boxed{\text{many point scatterers} \implies V_{\mathrm{eff}}(x) = 4\pi a\rho(x).}$$

R. Figari, H. Holden, A. Teta, *A law of large numbers and a central limit theorem for the Schrödinger operator with zero-range potentials*, J. Stat. Phys. (1988).

# Rigorous landmarks: dilute Bose gas and the Gross–Pitaevskii regime

**Homogeneous dilute gas in $\mathbb{R}^3$.** $N$ bosons with a nonnegative, spherically symmetric short-range pair potential, scattering length $a$, density $\rho$; dilute limit $\rho a^3 \to 0$. In units $\hbar^2/2m = 1$,

$$\frac{E_0}{N} = 4\pi a\rho\,(1 + o(1)).$$

E.H. Lieb, J. Yngvason, *Ground State Energy of the Low Density Bose Gas*, Phys. Rev. Lett. **80** (1998), 2504–2507.

**Trapped ground state / complete BEC (GP limit).** Confining external trap, repulsive two-body interaction, $N \to \infty$ with $Na_N \to g$ fixed (trap scale fixed; hence the gas is dilute). The many-body ground-state energy and density converge to the Gross–Pitaevskii theory; the ground state exhibits complete condensation into a GP minimizer.

E.H. Lieb, R. Seiringer, J. Yngvason, *Bosons in a Trap: A Rigorous Derivation of the Gross–Pitaevskii Energy Functional*, Phys. Rev. A **61** (2000), 043602.

E.H. Lieb, R. Seiringer, *Proof of Bose–Einstein Condensation for Dilute Trapped Gases*, Phys. Rev. Lett. **88** (2002), 170409.

**Time-dependent GP limit.** In $d = 3$, take $V_N(x) = N^2 V(Nx)$ with $V \geq 0$ short range, so $a_N = a_0/N$. Initial condensation plus an $O(N)$ energy bound implies persistence of condensation; the condensate wave function obeys the time-dependent GP equation, with coupling fixed by $a_0$.

L. Erdős, B. Schlein, H.-T. Yau, *Rigorous Derivation of the Gross–Pitaevskii Equation with a Large Interaction Potential*, J. Amer. Math. Soc. **22** (2009), 1099–1156.

GP regime $\neq$ mean-field regime: the interaction range and scattering length shrink like $N^{-1}$, while $Na_N = O(1)$; short-scale correlations survive through the scattering length.

# But the fixed-scatterer limit already contains a warning

The 1988 convergence is in probability, not uniformly over all configurations. There are sets of good configurations $\mathcal{G}_N$ such that

$$\mathbb{P}(\mathcal{G}_N) \longrightarrow 1,$$

and the effective limit holds there.

What must be avoided?

$$\boxed{\text{configurations in which some scatterers become anomalously close.}}$$

Dilute asymptotics can make dangerous short-distance configurations rare.

An exact local Hamiltonian must still be defined *on them*.

# Leaving the lattice: quartic locality becomes contact

The number-conserving part of a local relativistic quartic interaction has the schematic form

$$\int (a_x^\dagger)^2 a_x^2 \, dx.$$

On the $N$-particle sector this becomes formally

$$\boxed{\sum_{1 \le i < j \le N} \delta(x_i - x_j).}$$

This is still a *sum of two-body terms*.

The problem is no longer whether contact is a useful low-energy approximation. The problem is:

$$\boxed{\text{Does an exact three-dimensional bosonic contact Hamiltonian exist and remain stable?}}$$

# Contact interaction in 3D is a boundary condition

Let

$$\pi_{ij} = \{x_i = x_j\} \subset \mathbb{R}^{3N}.$$

A two-body Bethe–Peierls / TMS-type condition has the local behaviour

$$\psi(X) = \frac{\xi_{ij}}{|x_i - x_j|} + \alpha\, \xi_{ij} + o(1), \qquad |x_i - x_j| \to 0,$$

with

$$a = -\frac{1}{\alpha}.$$

The crucial point is that $\alpha$ is a constant: the pair $(i, j)$ does not know where the other particles are.

$$\boxed{\text{This is a genuine two-body contact law.}}$$

# The many-center case

A standard "local" family of extensions is obtained for fixed centers

$$\underline{y} = \{y_1, \dots, y_N\} \subset \mathbb{R}^3, \qquad \underline{\alpha} = \{\alpha_1, \dots, \alpha_N\}, \quad \alpha_i \in \mathbb{R}.$$

For $\lambda > 0$, write

$$\Psi = \phi_\lambda + \sum_{k=1}^{N} q_k G_\lambda(\cdot - y_k), \qquad \phi_\lambda \in H^2(\mathbb{R}^3) \qquad G_\lambda(x-y) = \frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|x-y|}$$

with boundary equations

$$\phi_\lambda(y_j) = \sum_{k=1}^{N} [\Gamma_{\underline{\alpha},\underline{y}}(\lambda)]_{jk} q_k, \qquad j = 1, \dots, N,$$

and

$$[\Gamma_{\underline{\alpha},\underline{y}}(\lambda)]_{ij} = \left(\alpha_i + \frac{\sqrt{\lambda}}{4\pi}\right)\delta_{ij} - G_\lambda(y_i - y_j)(1 - \delta_{ij}).$$

Then

$$(H_{\underline{\alpha},\underline{y}} + \lambda)^{-1} = G_\lambda + \sum_{j,k=1}^{N} [\Gamma_{\underline{\alpha},\underline{y}}(\lambda)]^{-1}_{jk} \langle G_\lambda(\cdot - y_j), \cdot\rangle\, G_\lambda(\cdot - y_k).$$

# The many-center case: spectrum and scattering length

The operator $H_{\underline{\alpha},\underline{y}}$ is a self-adjoint extension of

$$-\Delta\Big|_{C_0^\infty(\mathbb{R}^3\setminus\{y_1,\ldots,y_N\})}.$$

Its negative spectrum is determined by the finite-dimensional matrix $\Gamma$:

$$\sigma(H_{\underline{\alpha},\underline{y}}) = \{-\lambda : \lambda > 0,\ \det\Gamma_{\underline{\alpha},\underline{y}}(\lambda) = 0\} \cup [0,\infty),$$

where $[0,\infty)$ is purely absolutely continuous.

At zero energy the scattering length can be expressed through

$$a(\underline{\alpha},\underline{y}) = -\frac{1}{4\pi}\sum_{m,n=1}^{N}[\Gamma_{\underline{\alpha},\underline{y}}(0)]^{-1}_{mn}.$$

At each center $y_i$, functions in the domain satisfy the Bethe–Peierls boundary condition.

# Non-additivity pathology

Consider two identical centers with equal strength $\alpha$:

$$\Gamma_{\alpha,\{y_1,y_2\}}(\lambda) = \begin{pmatrix} \alpha + \frac{\sqrt{\lambda}}{4\pi} & -G_\lambda(y_1 - y_2) \\ -G_\lambda(y_2 - y_1) & \alpha + \frac{\sqrt{\lambda}}{4\pi} \end{pmatrix}.$$

If $|y_1 - y_2| \to 0$, then:

1. the Hamiltonian converges in resolvent sense to the free Laplacian;
2. the scattering length tends to zero;
3. a fixed local boundary parameter $\alpha$ cannot survive the coalescence, unless the charge vanishes.

Indeed, a fixed Bethe–Peierls law would require simultaneously

$$\frac{q}{|x - y_1|} - \frac{q}{a} = \frac{q}{|x - y_2|} - \frac{q}{a}.$$

$\boxed{\text{Point interactions are not additive when the centers coalesce.}}$

# The Born–Oppenheimer approximation

Consider two heavy particles of mass $M$ and one light particle of mass $m$. With

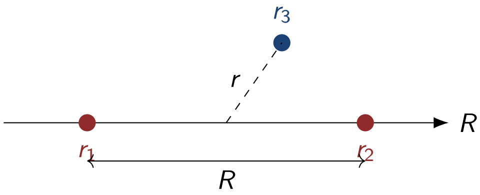
$$R = r_1 - r_2, \qquad r = r_3 - \frac{1}{2}(r_1 + r_2), \qquad \xi = \frac{M}{m},$$

set

$$\mu = \frac{\xi}{2}, \qquad \nu = \frac{2\xi}{2\xi + 1}.$$

Then

$$H = -\frac{1}{\mu}\Delta_R - \frac{1}{\nu}\Delta_r + V(r + R/2) + V(r - R/2).$$



The Born–Oppenheimer ansatz is

$$\Psi(r,R) = \psi(r;R)\,\Phi(R),$$

where the light-particle state solves, for fixed $R$,

$$\left[-\frac{1}{\nu}\Delta_r + V(r+R/2) + V(r-R/2)\right]\psi(r;R) = \epsilon(R)\psi(r;R).$$

The heavy degree of freedom then sees the effective potential $\epsilon(R)$:

$$\left[-\frac{1}{\mu}\Delta_R + \epsilon(R)\right]\Phi(R) = E\Phi(R).$$

Three-body short-distance structure is encoded in the effective potential $\epsilon(R)$.

# The no-go mechanism: three bosons are already enough

For three identical bosons in $\mathbb{R}^3$, the genuine two-body TMS condition is not sufficient.

Minlos–Faddeev: the three-body extension requires extra information at triple coincidence, and the resulting standard family is unbounded from below.

$$\boxed{x_1 \simeq x_2 \simeq x_3 \quad \Longrightarrow \quad \text{interaction becomes too attractive} \quad \Longrightarrow \quad \text{Thomas collapse.}}$$

For $N > 3$, the same obstruction cannot disappear.

A purely pairwise zero-range law is too rigid in three dimensions.

# Replace a constant scattering parameter by many-body information

The stabilizing idea is to replace

$$\alpha \quad \longrightarrow \quad A(z, y_1, \ldots, y_{N-2}),$$

where the effective contact strength depends on the other particles. A model choice is

$$A = \alpha + \gamma \sum_k \frac{\theta(|y_k - z|)}{|y_k - z|} + \frac{\gamma}{2} \sum_{k<\ell} \frac{\theta(|y_k - y_\ell|)}{|y_k - y_\ell|}.$$

Interpretation:

- the first correction weakens the pair interaction when a third particle approaches;
- the second controls simultaneous approach of another pair.

$\boxed{\text{The boundary condition is local, but no longer genuinely two-body.}}$

# A stable $N$-boson contact Hamiltonian

For $N \geq 3$ spinless bosons in $\mathbb{R}^3$, the modified boundary condition can be implemented by a quadratic-form construction.

For sufficiently strong short-distance repulsion ($\gamma > \gamma_c(N)$) one obtains a Hamiltonian

$$\mathcal{H}_A$$

which is

$$\boxed{\text{self-adjoint and bounded from below.}}$$

Away from multiple coincidences, one may choose $\theta$ with arbitrarily small support so that the usual two-body contact condition is restored.

D. Ferretti, A. Teta, *Hamiltonian for a Bose gas with Contact Interactions*, Rev. Math. Phys. 38 (2026), 2550020; arXiv:2403.12594.

# 1977: a stable $N$-boson contact Hamiltonian was already there

Albeverio–Høegh-Krohn–Streit approached the problem through Dirichlet forms. For $m \geq 0$, introduce the positive weight

$$\varphi_m(X) = \frac{1}{4\pi} \sum_{i<j} \frac{e^{-m|x_i - x_j|}}{|x_i - x_j|}.$$

Their form can be written schematically as

$$Q_D[\psi] = \int_{\mathbb{R}^{3N}} \varphi_m^2(X) \left| \nabla \left( \frac{\psi}{\varphi_m} \right)(X) \right|^2 dX - 2m^2 \|\psi\|^2.$$

It defines an $N$-body Hamiltonian $-\Delta_m$ which acts as the free Hamiltonian away from the coincidence set and satisfies

$$\boxed{-\Delta_m \geq -2m^2.}$$

S. Albeverio, R. Høegh-Krohn, L. Streit, *Energy forms, Hamiltonians, and distorted Brownian paths*, J. Math. Phys. 18 (1977), 907–917; many-particle example around p. 911.

# Ferretti–Teta: what was hidden in the 1977 construction?

Ferretti–Teta identify the AHKS Hamiltonian as a special case of their boundary-condition family. The precise choice is

$$\boxed{\alpha_0 = -m, \qquad \gamma = 2, \qquad \theta(r) = e^{-mr}.}$$

Hence the old Dirichlet-form model already contains the short-distance many-body corrections which are needed to avoid collapse.

But in the AHKS model a *single parameter m* fixes simultaneously

two-body scattering length, three-body repulsion, four-body repulsion.

$$\boxed{\text{the 1977 stable model is one distinguished point in a larger stable family.}}$$

# Ferretti–Teta: the broader result

The modern construction separates the parameters which the 1977 model ties together. One prescribes a contact law of the form

$$A = \alpha_0 + \gamma \sum_k \frac{\theta(|y_k - z|)}{|y_k - z|} + \frac{\gamma}{2} \sum_{k<\ell} \frac{\theta(|y_k - y_\ell|)}{|y_k - y_\ell|},$$

with $\gamma > \gamma_c(N)$ and a suitable short-range cutoff $\theta$. Then:

- the associated quadratic form is closed and bounded from below;
- the Hamiltonian is self-adjoint and semibounded;
- the ordinary two-body Bethe–Peierls law is recovered away from multiple coincidence;
- a generalized Dirichlet-form construction with a broader class of weights produces essentially the same family of Hamiltonians.

D. Ferretti, A. Teta, *Hamiltonian for a Bose gas with contact interactions*, Rev. Math. Phys. 38 (2026), 2550020; see also *Hamiltonians for Quantum Systems with Contact Interactions*, Springer (2025).

# A sequence of models: from three bodies to $N$ bodies

TMS / pairwise contact law $\xrightarrow{\text{Thomas instability}}$ regularized three-boson model $\xrightarrow{\text{many-body regularization}}$ stable $N$-boson contact model

- Basti–Cacciapuoti–Finco–Teta: regularized three-body Hamiltonian; effective repulsive three-body force; norm-resolvent approximation by renormalized short-range nonlocal potentials.
- Ferretti–Teta: extension to a stable gas of $N$ bosons with configuration-dependent contact strength.
- Basti–Ferretti–Teta (2026): zero-range Efimov model in the Born–Oppenheimer picture – a current spectral direction built on the same contact framework.

# Take-home message

$$\boxed{\phi^4 \text{ interaction } \implies \text{ pair interactions in fixed-}N\text{ sectors}}$$

Bose–Hubbard $\longrightarrow$ dilute universality $\longrightarrow$ contact limit $\longrightarrow$ multiple coincidence.

In three-dimensional bosonic zero-range models,

$$\boxed{\text{stability requires the contact law to know about more than one pair.}}$$

This may be encoded as effective three-/four-/$N$-body terms or, equivalently, as configuration-dependent boundary conditions at the coincidence manifolds.

**Selected references:** Albeverio–Høegh-Krohn–Streit, J. Math. Phys. 18 (1977); Figari–Holden–Teta, J. Stat. Phys. (1988); Basti–Cacciapuoti–Finco–Teta, Ann. Henri Poincaré 24 (2023); Figari–Saberbaghi–Teta, J. Phys. A 57 (2024); Ferretti–Teta, Rev. Math. Phys. 38 (2026); Basti–Ferretti–Teta (2026).