

Adiabatic charge pumps and Galilei covariance

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Stochastic and Analytic Methods in Mathematical Physics
Yerevan State University
September 7-12, 2026

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Based on joint work with **Tilman Esslinger, Filippo Santi**

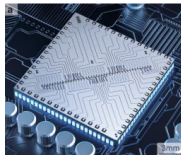
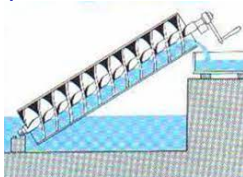
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Galilei covariance

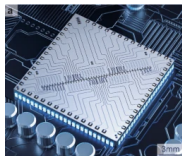
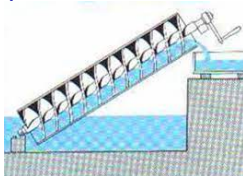
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Quantum pumps: What are they?



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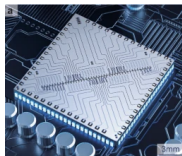
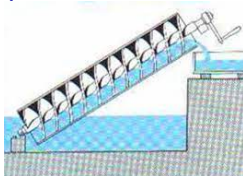
Infinitely extended 1-dimensional system

$$H(t) = p^2 + V(t, x), \quad (p = -i\hbar/dx)$$

depending on time t . Potential $V = V(t, x)$ doubly periodic

$$V(t, x + L) = V(t, x), \quad V(t + T, x) = V(t, x)$$

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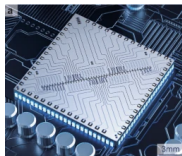
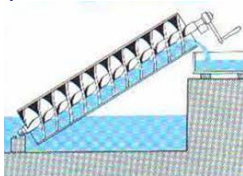
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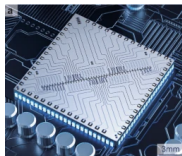
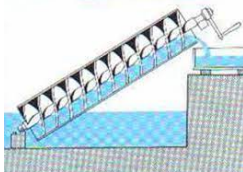
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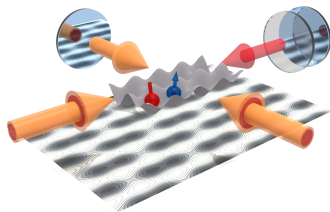
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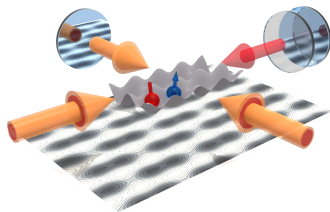
- ▶ Q arises as the Chern number of some vector bundle
- ▶ Its base space is a 2D torus, somehow reflecting space and time periodicity ($1+1=2$)

Cold gases pumped by lasers (experiment by Esslinger et al.)



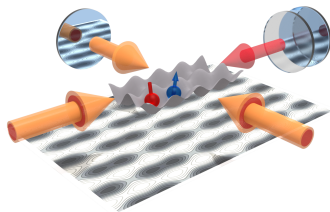
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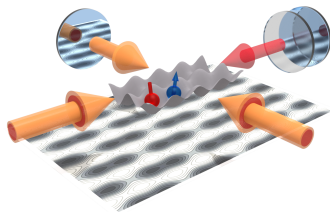
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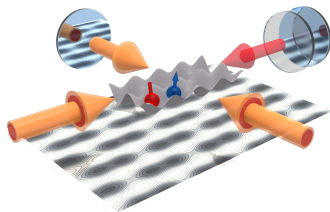
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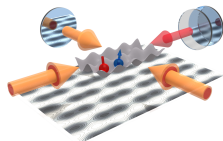
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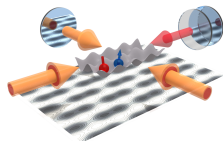


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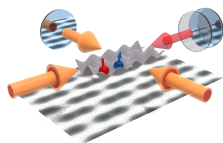
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- ▶ Example (close to experiment): Two-beams landscape

$$V(t, x) = \sum_{i=1,2} V_i \sin^2(k_i x - \omega_i t)$$

For $\omega_1 = 0$ or $\omega_2 = 0$ (i.e. one beam stationary), time periodicity holds true.

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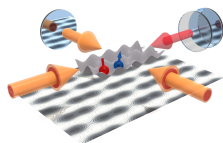
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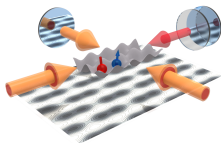
- ▶ By changing frame of reference, frequencies change (Doppler). If beams nearly stationary, one can be made stationary by passing to a moving frame of reference.

A same landscape, experienced by different observers

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(One color for each of them)

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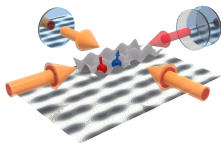


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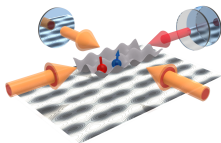


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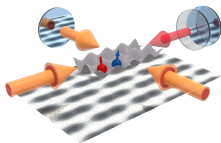


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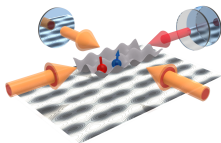


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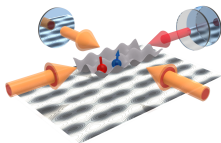


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- ▶ Theory has got to be Galilei covariant, not Q invariant

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Galilei covariance

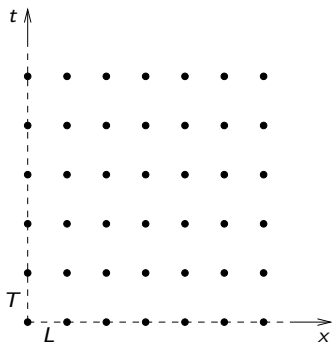
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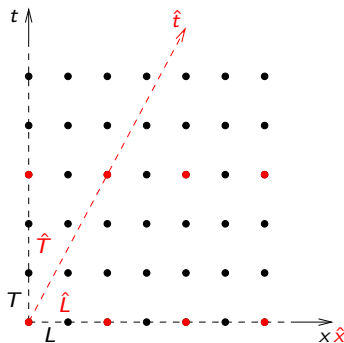
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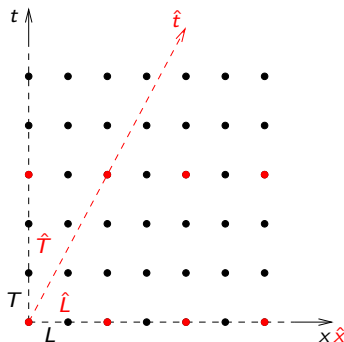


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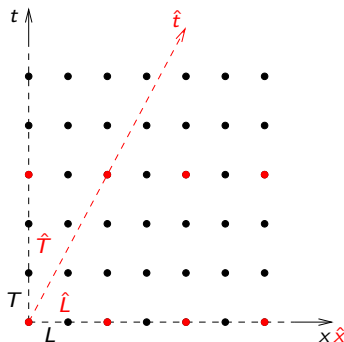
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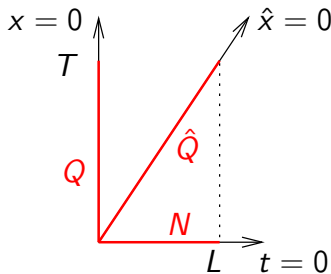
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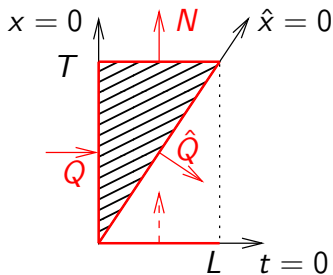
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2-current (ρ, j)
 $\text{div}(\rho, j) = 0$

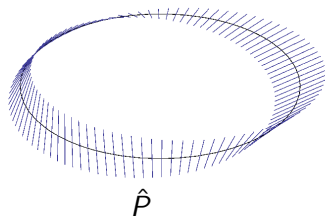
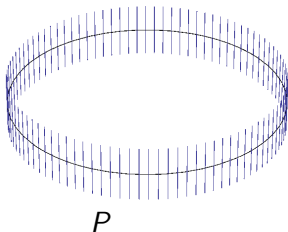
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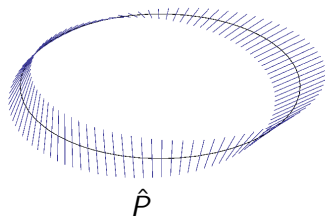
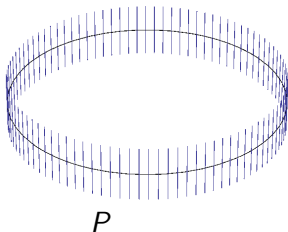
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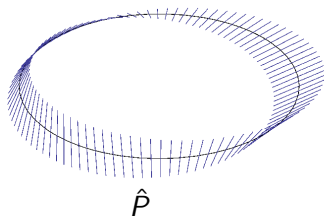
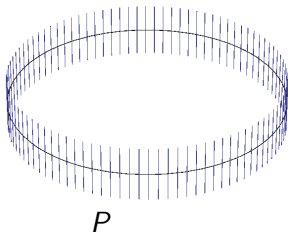


How related?

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- For now: Thouless theory

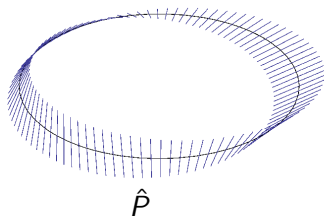
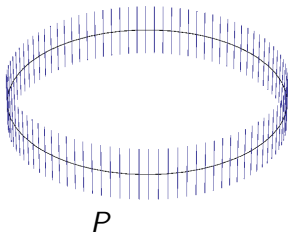
$$N = \text{rk } P, \quad Q = \text{ch } P$$

where $\text{rk } P$ and $\text{ch } P$ are the rank and the Chern number of the Fermi bundle, respectively

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How related?

- For now: Thouless theory

$$N = \text{rk } P, \quad Q = \text{ch } P$$

where $\text{rk } P$ and $\text{ch } P$ are the rank and the Chern number of the Fermi bundle, respectively

- Leads to **Conjecture**

$$\text{rk } \hat{P} = \text{rk } P, \quad \text{ch } \hat{P} = \text{ch } P - \text{rk } P$$

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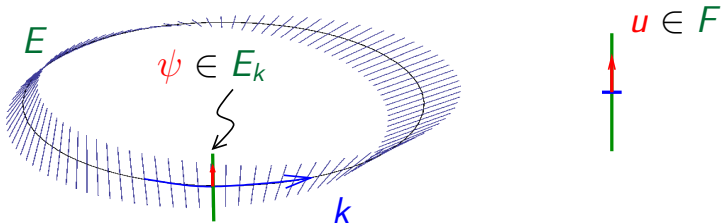
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Concepts reviewed (continued)

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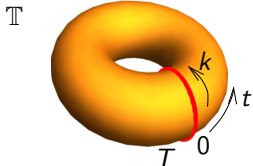
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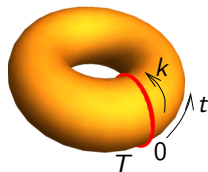
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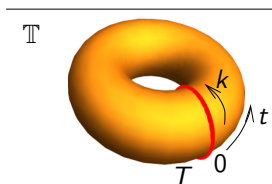
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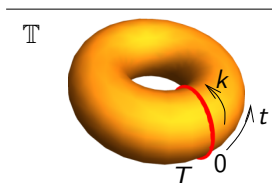
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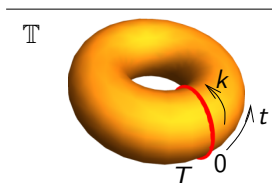
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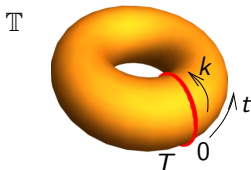
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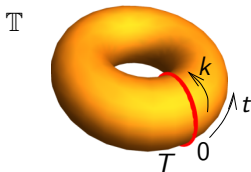
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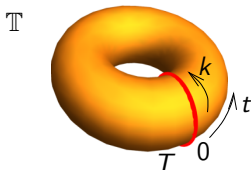
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Thank you all for your attention!

Remark: Strong and weak index

Theorem ($n = m = 1$, i.e. $vL = T$)

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Keep the Bloch bundle, but forgo trivializations.

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Difference between frames $\text{ch } \hat{P} - \text{ch } P$ evaluated not by CS form, but by the **Gauss-Codazzi** (GC) equation.

GC equation relates the curvatures of (flat) bundle E and subbundle P .

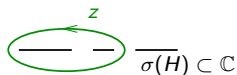
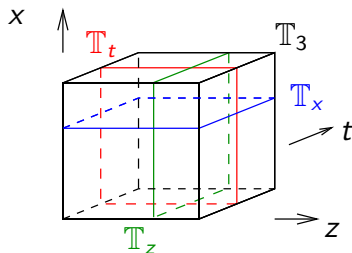
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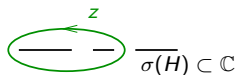
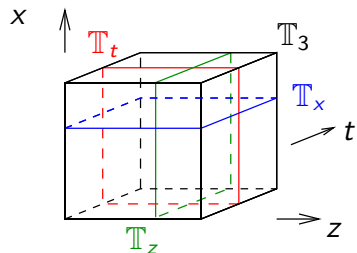
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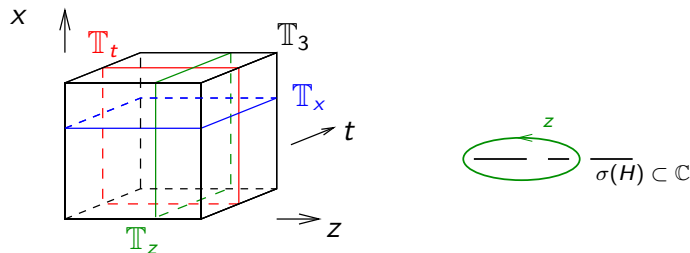


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There is vector bundle P on $\mathbb{T}_3$ by which existing and transported charges (N and Q) are placed on the same footing:

$$\text{ch } P_t = N, \quad \text{ch } P_x = Q, \quad \text{ch } P_z = 0, \quad (\text{any } t, x, z)$$

as Chern numbers of restrictions of bundle P to the different 2-subtori.

Other views III: Scattering approach to pumping (cf. Büttiker)

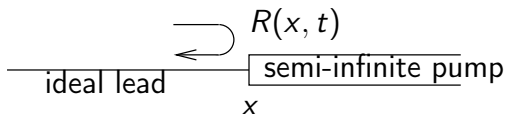
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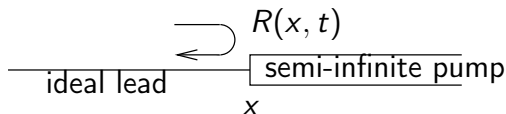


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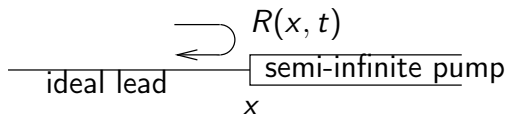


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- ▶ Let $R(x, t)$ be reflection matrix

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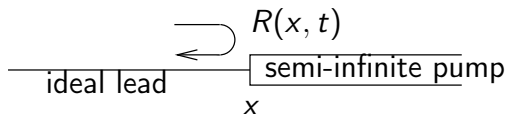


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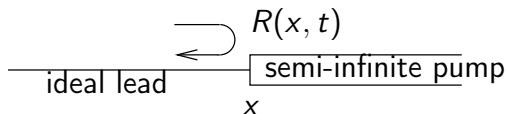
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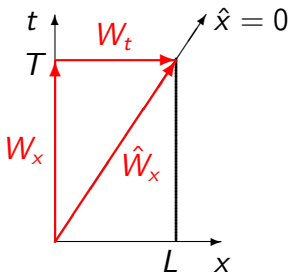
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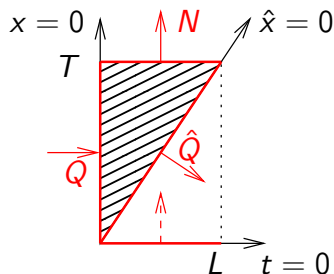
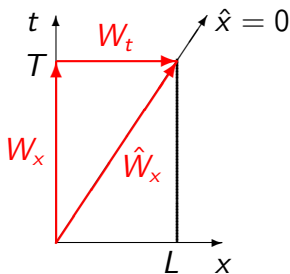


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Do not use any trivialization of the Bloch bundle (forgo use of typical fiber $F = \{u \mid u(x + L) = u(x)\}$)

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Identities

$$(a) \quad P[[P, x], [P, p]]P = [P_x P, P_p P] - P[x, p]P$$

$$(b) \quad \text{tr}_{E_{t,k}}[P_x P, P_p P] = i \frac{\partial}{\partial k} \text{tr}_{E_{t,k}} P_p P$$

$$(c) \quad \text{tr}_{E_{t,k}} P[x, p]P = i \text{tr}_{E_{t,k}} P = i \text{rk } P$$

- ▶ (b) is total derivative, does not contribute
- ▶ (c) Heisenberg, yields result, by $(\nu/2\pi)T(2\pi/L) = 1$
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- (a) is Gauss-Codazzi equation relating curvatures of (flat) bundle E and subbundle P . **Indeed** Gauss-Codazzi says

$$PRP = R^{(P)} - A^* \wedge A.$$

(R ambient curvature (here = 0), $R^{(P)}$ induced curvature, A 2nd fundamental form, $\nabla_X = \nabla_X^{(P)} + A(X)$). Identity (a) is (literally)

$$A^* \wedge A = R^{(P)}$$

with $X = x$, $Y = p$ and

$$A(X) = [X, P]P, \quad R^{(P)}(X, Y) = [\nabla_X^{(P)}, \nabla_Y^{(P)}] - \nabla_{[X, Y]}^{(P)}$$

Other views II: Improper solutions

Setup

As before,

$$H = -\frac{d^2}{dx^2} + V(x) \quad \text{on } L^2(\mathbb{R}_x, \mathbb{C}^n)$$

with $V(x) = V^*(x) \in M_n(\mathbb{C})$.

Schrödinger equation (as 2nd order ODE) and its “real adjoint”:

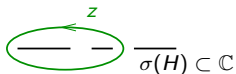
$$(H - z)\psi(x) = 0, \quad \tilde{\psi}(x)(H - z) = 0$$

with $\psi(x)$ (resp. $\tilde{\psi}(x)$) column (row) vectors in $\mathbb{C}^n$

The solutions ψ of the 2nd order ODE are determined by their initial values (ψ_0, ψ'_0) at any point $x = x_0$;

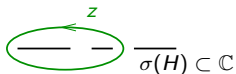
A circle and a bundle

- ▶ Let $\mathbb{C} \ni z$ be the (complex) energy plane. Spectrum $\sigma(H)$ is real subset
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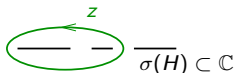


The diagram shows a horizontal line representing the real axis in the complex plane. On this line, there are two short horizontal segments representing parts of the spectrum $\sigma(H)$. A green oval loop, labeled γ above it, encircles these segments. To the right of the loop, the text $\overline{\sigma(H)} \subset \mathbb{C}$ is written.

- ▶ On γ there is a (trivial) vector bundle E with fiber $F = \mathbb{C}^n \oplus \mathbb{C}^n$

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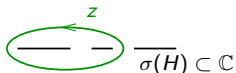
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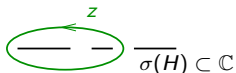
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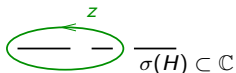
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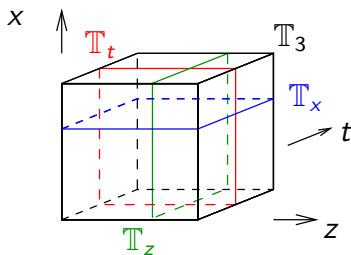
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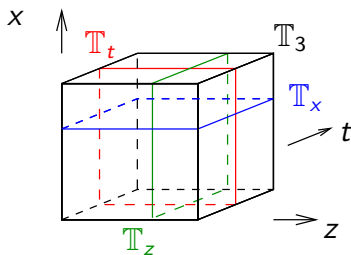
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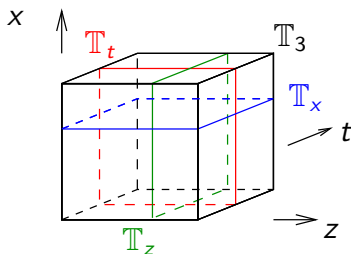
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Theorem [Implementation of Galilei transformations]

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