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Gibbs Scheme

in the Theory of Random Fields

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Mathematical statistical physics

central concept: *Gibbs random field*

<u>physical point of view</u>

DLR–definition based on the notion of potential (Dobrushin-Lanford-Ruelle)

$\Phi \to H^{\Phi} \to Q^{\Phi}$

<u>probabilistic point of view</u>

Pr–*definition* based on the intrinsic properties of a random field

uniform convergence of one-point finite-conditional distributions

first attempt: S. Dachian and B.S. Nahapetian, 2009

Purpose: expand and clarify the concept of the class of **Gibbs random fields** and give its structure the form accepted in the theory of random processes (fields)

- purely probabilistic definition
- theory of Gibbs random fields according to a scheme typical for the theory of random processes
- existence theorem in the framework of DLR–definition as a representation theorem
- all positive random fields admit Gibbsian representation in terms of transition energies
- Gibbsian description of known classes of random fields; validity of limit theorems of probability theory

L.A. Khachatryan and B.S. Nahapetian, *Gibbs scheme in the theory of random fields.* Ann. Henri Poincaré, 2026

Random field

Positive random field $P = \{P_V, V \Subset \mathbb{Z}^d\}$ on $(X^{\mathbb{Z}^d}, \mathscr{B}^{\mathbb{Z}^d})$ with finite state space X

Finite-conditional distributions: for all $V, \Lambda \Subset \mathbb{Z}^d$, $V \cap \Lambda = \varnothing$ and $x \in X^V$, $z \in X^\Lambda$

$$g_V^z(x) = \frac{P_{V\cup\Lambda}(xz)}{P_\Lambda(z)}$$

Let $\{\Lambda_n\}_{n\geq 1}$: $\Lambda_n \subset \Lambda_{n+1} \Subset \mathbb{Z}^d$, $\bigcup \Lambda_n = \mathbb{Z}^d$
Filtration $\mathscr{F} = \{\mathscr{B}^{\Lambda_n}\}_{n\geq 1}$: $\mathscr{B}^{\Lambda_n} \subset \mathscr{B}^{\Lambda_{n+1}} \subset \mathscr{B}^{\mathbb{Z}^d}$ (in what follows, we will consider only such filtrations, usually without explicitly specifying the corresponding increasing sequence.)

Conditional distributions of a random field

Conditional distributions: for all $V \Subset \mathbb{Z}^d$, $x \in X^V$ and P-a.e. $\bar{x} \in X^{V^c}$

$$g_V^{\bar{x}}(x) = \lim_{n\to\infty} g_V^{\bar{x}_{\Lambda_n\setminus V}}(x) = \lim_{n\to\infty} \frac{P_{\Lambda_n}(x\bar{x}_{\Lambda_n\setminus V})}{P_{\Lambda_n\setminus V}(\bar{x}_{\Lambda_n\setminus V})} \qquad (1)$$

Set of admissible boundary conditions:
$\mathscr{X}_V(\mathscr{F}) = \{\bar{x} \in X^{V^c}$: limits in (1) exist for all $x \in X^V\}$, $V \Subset \mathbb{Z}^d$

$G(P,\mathscr{F}) = \{g_V^{\bar{x}}, \bar{x} \in \mathscr{X}_V(\mathscr{F}), V \Subset \mathbb{Z}^d\}$ — *system of conditional distributions of the random field P with respect to $\mathscr{F}$.*

$\star$ Any positive system $G(P,\mathscr{F})$ can be restored by the system of one-point conditional distributions

$$G_1(P,\mathscr{F}) = \{g_t^{\bar{x}}, \bar{x} \in \mathscr{X}_t(\mathscr{F}), t \in \mathbb{Z}^d\}.$$

Gibbs random fields.
Probabilistic definition

A random field P will be called a *Gibbs random field* if there exists a filtration $\mathscr{F} = \{\mathscr{B}^{\Lambda_n}\}_{n\geq 1}$ and sets of admissible boundary conditions $\mathscr{Y}_t(\mathscr{F}) \subset \mathscr{X}_t(\mathscr{F})$, $P(\mathscr{Y}_t(\mathscr{F})) = 1$, $t \in \mathbb{Z}^d$, such that for all $t \in \mathbb{Z}^d$ and any $\bar{x} \in \mathscr{Y}_t(\mathscr{F})$, the limits of finite-conditional distributions

$$\lim_{n\to\infty} \frac{P_{\Lambda_n}(x\bar{x}_{\Lambda_n\setminus t})}{P_{\Lambda_n\setminus t}(\bar{x}_{\Lambda_n\setminus t})} = \lim_{n\to\infty} g_t^{\bar{x}_{\Lambda_n\setminus t}}(x) = g_t^{\bar{x}}(x), \qquad x \in X^t,$$

are positive and convergence with respect to $\bar{x} \in \mathscr{Y}_t(\mathscr{F})$ is uniform.

$\star$ P is Gibbsian if there exists $\mathscr{F}$ and $\mathscr{Y}_t(\mathscr{F}) \subset \mathscr{X}_t(\mathscr{F})$, s.t. $P(\mathscr{Y}_t(\mathscr{F})) = 1$ and $g_t^{\bar{x}}$ are positive and quasilocal.

$\star$ Probabilistic meaning of the class of Gibbs random fields: it is a uniform extension of the class of Markov random fields.

$\star$ Dachian and Nahapetian (2009) used more restrictive conditions: uniform convergence with respect to *all boundary conditions* $x \in X^{\mathbb{Z}^d\setminus t}$

• Uniform convergence with respect to one filtration implies uniform convergence with respect to any filtration.

Consider

$$\mathscr{X}_t = \bigcap_{\mathscr{F}} \mathscr{X}_t(\mathscr{F}), \qquad t \in \mathbb{Z}^d,$$

where the intersection is taken over all possible filtrations $\mathscr{F} = \{\mathscr{B}^{\Lambda_n}\}_{n \geq 1}$.

Theorem. *If P is a Gibbs random field, then for any $t \in \mathbb{Z}^d$, one has $P(\mathscr{X}_t) = 1$.*
If for a random field P, $P(\mathscr{X}_t) \neq 1$ for some $t \in \mathbb{Z}^d$, then P is not a Gibbs random field.

Example: Mixture of Bernoulli random fields

Let $\tau > 0$ and consider the random field with state space $X = \{0, 1\}$ and

$$P_V(x) = \int_0^1 p^{|x|}(1-p)^{|V|-|x|}\tau p^{\tau-1}dp,$$

where $|x| = |\{t \in V : x_t = 1\}|$, $x \in X^V$, $V \Subset \mathbb{Z}^d$.
Let $\mathscr{F} = \{\mathscr{B}^{\Lambda_n}\}_{n\geq 1}$ be a filtration. For all $t \in \mathbb{Z}^d$ and $p \in [0, 1]$, denote

$$U_t^p(\mathscr{F}) = \left\{\bar{x} \in X^{t^c} : \lim_{n\to\infty} \frac{|x_{\Lambda_n}|}{|\Lambda_n|} = p\right\}.$$

The set of admissible boundary conditions is $\mathscr{X}_t(\mathscr{F}) = \bigcup\limits_{p\in[0,1]} U_t^p(\mathscr{F})$.

Then for $\mathscr{X}_t = \bigcap\limits_{\mathscr{F}} \mathscr{X}_t(\mathscr{F})$, we have

$$\mathscr{X}_t = \{\bar{x} \in X^{t^c} : |\{s \in t^c : \bar{x}_s = 1\}| < \infty\} \cup \{\bar{x} \in X^{t^c} : |\{s \in t^c : \bar{x}_s = 0\}| < \infty\}$$

and thus, $P(\mathscr{X}_t) = 0$ for any $t \in \mathbb{Z}^d$.

DLR–Gibbsian random fields are Gibbsian

DLR–definition: A Gibbs random field corresponding to a given uniformly convergent potential Φ is a random field having a version of its conditional distribution that almost everywhere coincides with the Gibbsian specification constructed by Φ.

Equivalently: A random field $P = \{P_V, V \in W\}$ is a Gibbs random field corresponding to the uniformly convergent potential Φ if for P-a.e. boundary conditions $\bar{x}$,

$$\lim_{n\to\infty} \frac{P_{\Lambda_n}(x\bar{x}_{\Lambda_n\setminus V})}{P_{\Lambda_n\setminus V}(\bar{x}_{\Lambda_n\setminus V})} = \frac{\exp\{-H_V^{\bar{x}}(x)\}}{\sum\limits_{z\in X^V} \exp\{-H_V^{\bar{x}}(z)\}} \qquad (2)$$

where $x \in X^V$, $\bar{x} \in \mathscr{X}_V(\mathscr{F})$, $V \in W$, $\mathscr{F} = \{\mathscr{B}^{\Lambda_n}\}_{n\geq 1}$ is a filtration and $H_V^{\bar{x}}(x) = \sum\limits_{t,s\in V} \Phi_{t,s}(x_t x_s) + \sum\limits_{t\in V}\sum\limits_{s\in V^c} \Phi_{t,s}(x_t\bar{x}_s)$.

Limits in (2) are quasilocal $\Rightarrow$ the convergence is uniform.

Transition energies of a random field

System of transition energies

$$\boldsymbol{\Delta}(P,\mathscr{F}) = \{\Delta_V^{\bar{x}}, \bar{x} \in \mathscr{X}_V(\mathscr{F}), V \in W\}$$

of a random field P with positive system $G(P,\mathscr{F})$

$$\Delta_V^{\bar{x}}(x,u) = \ln \frac{g_V^{\bar{x}}(x)}{g_V^{\bar{x}}(u)} = \lim_{n\to\infty} \ln \frac{P_{\Lambda_n}(x\bar{x}_{\Lambda_n\setminus V})}{P_{\Lambda_n}(u\bar{x}_{\Lambda_n\setminus V})}, \quad x,u \in X^V \quad (3)$$

Functions (3) satisfy

$$\Delta_V^{\bar{x}}(x,u) = \Delta_V^{\bar{x}}(x,y) + \Delta_V^{\bar{x}}(y,u), \qquad x,u,y \in X^V;$$

$$\Delta_{V\cup I}^{\bar{x}}(xy,uv) = \Delta_V^{\bar{x}y}(x,u) + \Delta_I^{\bar{x}x}(y,v), \qquad x,u \in X^V, y,v \in X^I$$

⋆ Autonomous (*a priori*) given analogue — transition energies field — and axiomatic definition of Hamiltonian were introduced by Dachian and Nahapetian in Markov Processes Relat. Fields, 2019

Gibbs form of conditional distributions

Theorem. *Let P be a random field with a positive system $G(P, \mathscr{F})$ of its conditional distributions and let $\mathbf{\Delta}(P, \mathscr{F})$ be its system of transition energies. Then the elements of $G(P, \mathscr{F})$ necessarily have a Gibbs form*

$$g_V^{\bar{x}}(x) = \frac{\exp\{\Delta_V^{\bar{x}}(x, u)\}}{\sum\limits_{\alpha \in X^V} \exp\{\Delta_V^{\bar{x}}(\alpha, u)\}},$$

$x, u \in X^V$, $\bar{x} \in \mathscr{X}_V(\mathscr{F})$, $V \Subset \mathbb{Z}^d$.

Theorem. *If a random field P has a quasilocal system of transition energies $\mathbf{\Delta}_1(P)$*

$$\lim_{\Lambda \uparrow t^c} \sup_{\bar{x}, \bar{y} \in \mathscr{X}_t(\mathscr{F}) : \bar{x}_\Lambda = \bar{y}_\Lambda} \left| \Delta_t^{\bar{x}}(x, u) - \Delta_t^{\bar{y}}(x, u) \right| = 0.$$

then P is Gibbsian.

Gibbs random fields. Representation theorem

Theorem. *For any quasilocal one-point transition energy field* $\mathbf{\Delta}_1 = \{\delta_t^{\bar{x}}, \bar{x} \in X^{t^c}, t \in \mathbb{Z}^d\}$, *there exists a Gibbs random field* P *corresponding to* $\mathbf{\Delta}_1$*:*

$$\Delta_t^{\bar{x}}(x,u) = \delta_t^{\bar{x}}(x,u), \qquad x,u \in X^t,$$

for all $t \in \mathbb{Z}^d$, $\bar{x} \in X_t(\mathscr{F})$ *and any filtration* $\mathscr{F}$, *where* $\mathbf{\Delta}_1(P,\mathscr{F}) = \{\Delta_t^{\bar{x}}, \bar{x} \in X_t(\mathscr{F}), t \in \mathbb{Z}^d\}$.

- Let $\Phi = \{\Phi_V, V \in W\}$ be a uniformly convergent potential. Then the set $\mathbf{\Delta}_1^{\Phi} = \{\delta_t^{\bar{x}}, \bar{x} \in X^{t^c}, t \in \mathbb{Z}^d\}$ of functions

$$\delta_t^{\bar{x}}(x,u) = \sum_{J \subset W(t^c)} (\Phi_{t\cup J}(u\bar{x}_J) - \Phi_{t\cup J}(x\bar{x}_J)),$$

$x,u \in X^t$, $\bar{x} \in X^{t^c}$, $t \in \mathbb{Z}^d$, forms a quasilocal transition energy field.

- discrete Widom–Rowlinson model (see Dachian, Nahapetian, 2019)

Gibbsian description of Gaussian random fields

Let P be a Gaussian r.f. with mean vector $m = \{m_t, t \in \mathbb{Z}^d\}$ and covariance matrix $C = \{c_{ts}, t, s \in \mathbb{Z}^d\}$. Denote

$$C_\Lambda = \{c_{ts}, t, s \in \Lambda\}, \quad A_\Lambda = \{a^\Lambda_{ts}, t, s \in \Lambda\} = C_\Lambda^{-1}, \quad \Lambda \Subset \mathbb{Z}^d$$

Let $\{\Lambda_n\} \uparrow \mathbb{Z}^d$ be such that for any $t \in \mathbb{Z}^d$, there exists

$$a_{tt} = \lim_{n\to\infty} \frac{\det C_{\Lambda_n \setminus \{t\}}}{\det C_{\Lambda_n}}.$$

Proposition. *The elements of the one-point conditional distribution* $G_1(P,\mathscr{F})=\{g_t^{\bar{x}},\mathscr{X}_t(\mathscr{F}),t\in\mathbb{Z}^d\}$ *have the following form:*

$$g_t^{\bar{x}}(x)=\sqrt{\frac{a_{tt}}{2\pi}}\exp\left\{-\frac{a_{tt}}{2}\Big((x-m_t)+\frac{1}{a_{tt}}\lim_{n\to\infty}\sum_{s\in\Lambda_n\setminus t}a_{ts}^{\Lambda_n}(\bar{x}_s-m_s)\Big)^2\right\},$$

$x\in\mathbb{R}^t$, *and for each* $t\in\mathbb{Z}^d$, *the set of admissible boundary conditions* $\mathscr{X}_t(\mathscr{F})$ *is the set of those configurations* $\bar{x}\in\mathbb{R}^{t^c}$ *for which there exist limits*

$$\lim_{n\to\infty}\sum_{s\in\Lambda_n\setminus t}a_{ts}^{\Lambda_n}(\bar{x}_s-m_s).$$

If for all $t\in\mathbb{Z}^d$,

$$a_{tt}=\lim_{n\to\infty}\frac{\det C_{\Lambda_n\setminus\{t\}}}{\det C_{\Lambda_n}}>0,$$

then system $G_1(P,\mathscr{F})$ *is positive.*

Proposition. *Let Gaussian random field P be such that $a_{tt} > 0$, $t \in \mathbb{Z}^d$. Then the elements of its system $\mathbf{\Delta}_1(P, \mathscr{F})$ have the following form:*

$$\Delta_t^{\bar{x}}(x,u) = \frac{a_{tt}}{2}\Big((u-m_t)^2 - (x-m_t)^2\Big) +$$

$$+ (u-x) \lim_{n\to\infty} \sum_{s\in\Lambda_n\setminus t} a_{ts}^{\Lambda_n}(\bar{x}_s - m_s),$$

$x, u \in \mathbb{R}^t$, $\bar{x} \in \mathscr{X}_t(\mathscr{F})$, $t \in \mathbb{Z}^d$.

Theorem. *Let P be a Gaussian random field with covariance matrix C and a vector of mean values m. Suppose there is an increasing sequence $\{\Lambda_n\}_{n\geq 1}$ of finite sets converging to $\mathbb{Z}^d$ such that limits*

$$a_{tt} = \lim_{n\to\infty} \frac{\det C_{\Lambda_n}}{\det C_{t\cup\Lambda_n}}, \qquad t \in \mathbb{Z}^d,$$

exist and are positive. If there exist sets of admissible boundary conditions $\mathscr{Y}_t(\mathscr{F}) \subset \mathscr{X}_t(\mathscr{F})$, $P(\mathscr{Y}_t(\mathscr{F})) = 1$, $t \in \mathbb{Z}^d$, such that for all $t \in \mathbb{Z}^d$, the convergence in limits

$$\lim_{n\to\infty} \sum_{s\in\Lambda_n\setminus t} a_{ts}^{\Lambda_n}(\bar{x}_s - m_s),$$

where $a_{ts}^{\Lambda_n} = (C_{\Lambda_n}^{-1})_{ts}$, is uniform with respect to $\bar{x} \in \mathscr{Y}_t(\mathscr{F})$, then P is a Gibbs random field.

Thank you

for your attention!

References

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Hamiltonian of a random field

For $\mathbf{\Delta}(P,\mathscr{F})$, we have

$$\Delta_V^{\bar{x}}(x,u) = H_V^{\bar{x}}(u) - H_V^{\bar{x}}(x), \qquad x,u \in X^V.$$

$H(P) = \{H_V^{\bar{x}}, \bar{x} \in \mathscr{X}_V, V \in W\}$ is a *Hamiltonian corresponding to the random field* P.

Corollary. *Let conditional distributions* $G(P)$ *of the random field* P *be positive. Then they have a Gibbs form*

$$g_V^{\bar{x}}(x) = \frac{\exp\{-H_V^{\bar{x}}(x)\}}{\sum\limits_{\alpha \in X^V} \exp\{-H_V^{\bar{x}}(\alpha)\}},$$

$x \in X^V$, $\bar{x} \in \mathscr{X}_V$, $V \in W$, *where* $H(P) = \{H_V^{\bar{x}}, \bar{x} \in \mathscr{X}_V, V \in W\}$ *is a Hamiltonian corresponding to the random field* P.

The elements of $H(P)$ are consistent in the following sense:

$$H_{V\cup I}^{\bar{x}}(xy) + H_V^{\bar{x}y}(u) = H_{V\cup I}^{\bar{x}}(uy) + H_V^{\bar{x}y}(x)$$

$V, I \in W$, $V \cap I = \varnothing$, $\bar{x} \in \mathscr{X}_{V\cup I}$, $x,u \in X^V$, $y,v \in X^I$.