Homogenization of parabolic convolution type operators with periodic coefficients randomly evolving in time. Higher order approximation

Andrey Piatnitski

The Arctic University of Norway, campus Narvik

and

Higher School of Modern Math., MIPT, Moscow

The talk is based on joint works with Elena Zhizhina (Narvik and Moscow), Alexandre Popier (Le Mans, France) and Marina Kleptsyna (Le Mans, France)

XII International Conference "Stochastic and Analytic Methods in Mathematical Physics", SAMMP

Yerevan, 10 September 2026

Outline

- Convolution type operatorss with periodic and random coefficients. Assumptions.

Outline

- Convolution type operatorss with periodic and random coefficients. Assumptions.
- Scaled Cauchy problem.

Outline

- Convolution type operatorss with periodic and random coefficients. Assumptions.
- Scaled Cauchy problem.
- Evolution cell problems.

Outline

- Convolution type operatorss with periodic and random coefficients. Assumptions.
- Scaled Cauchy problem.
- Evolution cell problems.
- Auxiliary results. Properties of solutions of cell problems.

Outline

- Convolution type operatorss with periodic and random coefficients. Assumptions.
- Scaled Cauchy problem.
- Evolution cell problems.
- Auxiliary results. Properties of solutions of cell problems.
- Homogenization in randomly moving coordinates.

Outline

- Convolution type operatorss with periodic and random coefficients. Assumptions.
- Scaled Cauchy problem.
- Evolution cell problems.
- Auxiliary results. Properties of solutions of cell problems.
- Homogenization in randomly moving coordinates.
- Mixing properties. Effective SPDE.

Outline

- Convolution type operatorss with periodic and random coefficients. Assumptions.
- Scaled Cauchy problem.
- Evolution cell problems.
- Auxiliary results. Properties of solutions of cell problems.
- Homogenization in randomly moving coordinates.
- Mixing properties. Effective SPDE.
- Symmetric coefficients. Higher order approximations.

Convolution type operators

We consider a zero order convolution type non-local operator A in $L^2(\mathbb{R}^d)$, $d \geq 1$, with periodic coefficients. This operator is defined by

$$Au(x) = \int_{\mathbb{R}^d} \Lambda(x, y) a(x - y)\big(u(y) - u(x)\big)\, dy.$$

Convolution type operators

We consider a zero order convolution type non-local operator A in $L^2(\mathbb{R}^d)$, $d \geq 1$, with periodic coefficients. This operator is defined by

$$Au(x) = \int_{\mathbb{R}^d} \Lambda(x, y)a(x - y)\big(u(y) - u(x)\big)\, dy.$$

The corresponding quadratic form reads

$$-(Au, u) = \frac{1}{2}\int_{\mathbb{R}^d} \Lambda(x, y)a(x - y)\big(u(y) - u(x)\big)^2 dy,$$

Convolution type operators

We consider a zero order convolution type non-local operator A in $L^2(\mathbb{R}^d)$, $d \geq 1$, with periodic coefficients. This operator is defined by

$$Au(x) = \int_{\mathbb{R}^d} \Lambda(x,y)a(x-y)\big(u(y)-u(x)\big)\,dy.$$

The corresponding quadratic form reads

$$-(Au,u) = \frac{1}{2}\int_{\mathbb{R}^d} \Lambda(x,y)a(x-y)\big(u(y)-u(x)\big)^2 dy,$$

here the convolution kernel $a = a(z)$ is a non-negative integrable function, $a : \mathbb{R}^d \mapsto \mathbb{R}^+$, it is responsible for the localization properties of the operator,

Convolution type operators

We consider a zero order convolution type non-local operator A in $L^2(\mathbb{R}^d)$, $d \geq 1$, with periodic coefficients. This operator is defined by

$$Au(x) = \int_{\mathbb{R}^d} \Lambda(x,y)a(x-y)\big(u(y)-u(x)\big)\,dy.$$

The corresponding quadratic form reads

$$-(Au,u) = \frac{1}{2}\int_{\mathbb{R}^d} \Lambda(x,y)a(x-y)\big(u(y)-u(x)\big)^2 dy,$$

here the convolution kernel $a = a(z)$ is a non-negative integrable function, $a : \mathbb{R}^d \mapsto \mathbb{R}^+$, it is responsible for the localization properties of the operator,

$\Lambda(x,y)$ is a periodic function that satisfies the uniform ellipticity conditions. This function Λ represents the local characteristics of the environment.

Evolution problem

Under the above assumptions A is a bounded linear operator in $L^2(\mathbb{R}^d)$. The corresponding evolution equation describes the dynamics of a continuous time jump Markov process in a periodic or random stationary medium.

Under the above assumptions A is a bounded linear operator in $L^2(\mathbb{R}^d)$. The corresponding evolution equation describes the dynamics of a continuous time jump Markov process in a periodic or random stationary medium.

The Cauchy problem for this evolution equation reads as

$$\partial_t u = Au \text{ in } \mathbb{R}^d \times (0, +\infty), \qquad u(x, 0) = \varphi(x),$$

with $\varphi \in L^2(\mathbb{R}^d)$.

When studying the large time behaviour of the said Markov evolution it is natural to make the diffusive scaling of spatial and temporal variables: $x \longrightarrow \varepsilon x, \qquad t \longrightarrow \varepsilon^2 t,$
where ε is a small positive parameter.

When studying the large time behaviour of the said Markov evolution it is natural to make the diffusive scaling of spatial and temporal variables: $x \longrightarrow \varepsilon x, \qquad t \longrightarrow \varepsilon^2 t,$
where ε is a small positive parameter. The generator of the scaled dynamics takes the form

$$(A^\varepsilon u)(x) \ = \ \frac{1}{\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\Lambda\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)\big(u(y)-u(x)\big)\,dy. \qquad (1)$$

Time dependent operator

We also consider a family of operators $A(s)$ that depend on the temporal variable s, $s \in \mathbb{R}$:

$$A(s)u(x) = \int_{\mathbb{R}^d} \Lambda(x, y, s)a(x - y)\big(u(y) - u(x)\big)dy.$$

Time dependent operator

We also consider a family of operators $A(s)$ that depend on the temporal variable s, $s \in \mathbb{R}$:

$$A(s)u(x) = \int_{\mathbb{R}^d} \Lambda(x, y, s)a(x-y)\big(u(y) - u(x)\big)dy.$$

It is assumed that for each $s \in \mathbb{R}$ the operator $A(s)$ satisfies the above formulated conditions.

Time dependent operator

We also consider a family of operators $A(s)$ that depend on the temporal variable s, $s \in \mathbb{R}$:

$$A(s)u(x) = \int_{\mathbb{R}^d} \Lambda(x, y, s)a(x - y)\big(u(y) - u(x)\big)dy.$$

It is assumed that for each $s \in \mathbb{R}$ the operator $A(s)$ satisfies the above formulated conditions. Moreover, as a function of time the coefficient $\Lambda(x, y, s)$ is a stationary ergodic process.

Time dependent operator

We also consider a family of operators $A(s)$ that depend on the temporal variable s, $s \in \mathbb{R}$:

$$A(s)u(x) = \int_{\mathbb{R}^d} \Lambda(x, y, s)a(x - y)\big(u(y) - u(x)\big)dy.$$

It is assumed that for each $s \in \mathbb{R}$ the operator $A(s)$ satisfies the above formulated conditions. Moreover, as a function of time the coefficient $\Lambda(x, y, s)$ is a stationary ergodic process.

In other words, we consider a periodic environment that evolves in time, and the evolution is random, stationary and ergodic.

Dynamical system.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space.
We say that a family of measurable maps $\tau_s : \Omega \mapsto \Omega$, $s \in \mathbb{R}$, is a dynamical system, if

- $\tau_{s_1+s_2} = \tau_{s_2} \circ \tau_{s_1}, \quad \tau_0 = Id,$

Dynamical system.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space.
We say that a family of measurable maps $\tau_s : \Omega \mapsto \Omega$, $s \in \mathbb{R}$, is a dynamical system, if

- $\tau_{s_1+s_2} = \tau_{s_2} \circ \tau_{s_1}, \quad \tau_0 = Id,$
- $\mathbf{P}(\tau_s \mathcal{A}) = \mathbf{P}(\mathcal{A})$ for all $s \in \mathbb{R}$ and $\mathcal{A} \in \mathcal{F}$,

Dynamical system.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space.
We say that a family of measurable maps $\tau_s : \Omega \mapsto \Omega$, $s \in \mathbb{R}$, is a dynamical system, if

- $\tau_{s_1+s_2} = \tau_{s_2} \circ \tau_{s_1}, \quad \tau_0 = Id,$
- $\mathbf{P}(\tau_s \mathcal{A}) = \mathbf{P}(\mathcal{A})$ for all $s \in \mathbb{R}$ and $\mathcal{A} \in \mathcal{F}$,
- $\tau.$ is a measurable mapping from $\Omega \times \mathbb{R}$ to Ω; $\Omega \times \mathbb{R}$ being equipped with the product σ-algebra $\mathcal{F} \times \mathcal{B}$.

Dynamical system.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space.
We say that a family of measurable maps $\tau_s : \Omega \mapsto \Omega$, $s \in \mathbb{R}$, is a dynamical system, if

- $\tau_{s_1+s_2} = \tau_{s_2} \circ \tau_{s_1}, \quad \tau_0 = Id$,
- $\mathbf{P}(\tau_s \mathcal{A}) = \mathbf{P}(\mathcal{A})$ for all $s \in \mathbb{R}$ and $\mathcal{A} \in \mathcal{F}$,
- $\tau_\cdot$ is a measurable mapping from $\Omega \times \mathbb{R}$ to Ω; $\Omega \times \mathbb{R}$ being equipped with the product σ-algebra $\mathcal{F} \times \mathcal{B}$.

We say that $\Lambda(x, y, s)$ is stationary, if there exists a random function $\mathbf{\Lambda}(x, y, \omega)$ such that $\Lambda(x, y, s) = \mathbf{\Lambda}(x, y, \tau_s \omega)$.

Dynamical system.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space.
We say that a family of measurable maps $\tau_s : \Omega \mapsto \Omega$, $s \in \mathbb{R}$, is a dynamical system, if

- $\tau_{s_1+s_2} = \tau_{s_2} \circ \tau_{s_1}, \quad \tau_0 = Id$,
- $\mathbf{P}(\tau_s \mathcal{A}) = \mathbf{P}(\mathcal{A})$ for all $s \in \mathbb{R}$ and $\mathcal{A} \in \mathcal{F}$,
- $\tau.$ is a measurable mapping from $\Omega \times \mathbb{R}$ to Ω; $\Omega \times \mathbb{R}$ being equipped with the product σ-algebra $\mathcal{F} \times \mathcal{B}$.

We say that $\Lambda(x, y, s)$ is stationary, if there exists a random function $\mathbf{\Lambda}(x, y, \omega)$ such that $\Lambda(x, y, s) = \mathbf{\Lambda}(x, y, \tau_s \omega)$.

We say that $\tau.$ is ergodic, if for any invariant set $\mathcal{A} \in \mathcal{F}$ we have $\mathbf{P}(\mathcal{A})(1 - \mathbf{P}(\mathcal{A})) = 0$.

Jump Markov process.

We consider a jump Markov process whose Kolmogorov equation reads

$$\begin{gathered}\partial_s u(\tilde{x},s) - A(s)u(\tilde{x},s)\\ = \partial_s u(\tilde{x},s) - \int_{\mathbb{R}^d} a(\tilde{x}-\tilde{y})\Lambda(\tilde{x},\tilde{y},s)\big(u(\tilde{y},s)-u(\tilde{x},s)\big)\,dy = 0.\end{gathered}$$

Jump Markov process.

We consider a jump Markov process whose Kolmogorov equation reads

$$
\begin{aligned}
&\partial_s u(\tilde{x},s) - A(s)u(\tilde{x},s) \\
&= \partial_s u(\tilde{x},s) - \int_{\mathbb{R}^d} a(\tilde{x}-\tilde{y})\Lambda(\tilde{x},\tilde{y},s)\big(u(\tilde{y},s)-u(\tilde{x},s)\big)\,dy = 0.
\end{aligned}
$$

To study the large time behaviour of solutions we make the diffusive scaling $x = \varepsilon\tilde{x}, \quad t = \varepsilon^2 s$, and consider the Cauchy problem for the rescaled equation:

Jump Markov process.

We consider a jump Markov process whose Kolmogorov equation reads

$$
\begin{gathered}
\partial_s u(\tilde{x}, s) - A(s)u(\tilde{x}, s) \\
= \partial_s u(\tilde{x}, s) - \int_{\mathbb{R}^d} a(\tilde{x} - \tilde{y})\Lambda(\tilde{x}, \tilde{y}, s)\big(u(\tilde{y}, s) - u(\tilde{x}, s)\big)\, dy = 0.
\end{gathered}
$$

To study the large time behaviour of solutions we make the diffusive scaling $x = \varepsilon\tilde{x}, \quad t = \varepsilon^2 s$, and consider the Cauchy problem for the rescaled equation:

$$
\begin{gathered}
\partial_t u - \frac{1}{\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\Lambda\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{t}{\varepsilon^2}\Big)\big(u(y,t) - u(x,t)\big)\, dx = 0, \\
u(x, 0) = u_0(x), \quad u_0 \in L^2(\mathbb{R}^d).
\end{gathered}
$$

Zero order convolution type operators appear in many applications, such as models of population dynamics and the continuous contact model, where they describe the evolution of the density of a population.

Motivations and possible applications.

Zero order convolution type operators appear in many applications, such as models of population dynamics and the continuous contact model, where they describe the evolution of the density of a population.

Several models of porous media are also described in terms of convolution type operators with integrable kernels.

Motivations and possible applications.

Zero order convolution type operators appear in many applications, such as models of population dynamics and the continuous contact model, where they describe the evolution of the density of a population.

Several models of porous media are also described in terms of convolution type operators with integrable kernels.

Studying the large time behaviour of such systems naturally leads to homogenization problems for the corresponding non-local functionals and operators.

Existing results for nonlocal operators

- Homogenization a class of integro-differential equations with L´evy operators - M. Arisawa ('09).

Existing results for nonlocal operators

- Homogenization a class of integro-differential equations with L´evy operators - M. Arisawa ('09).
- Scaling limits for symmetric Itˆo–Lévy processes – R. Rhodes and V. Vargas ('09).

Existing results for nonlocal operators

- Homogenization a class of integro-differential equations with L ́evy operators - M. Arisawa ('09).
- Scaling limits for symmetric Itˆo–Lévy processes – R. Rhodes and V. Vargas ('09).
- Homogenization of periodic diffusion with small jumps. – N. Sandric ('16).

Existing results for nonlocal operators

- Homogenization a class of integro-differential equations with L´evy operators - M. Arisawa ('09).
- Scaling limits for symmetric Itˆo–Lévy processes – R. Rhodes and V. Vargas ('09).
- Homogenization of periodic diffusion with small jumps. – N. Sandric ('16).
- Periodic homogenization for nonlinear integro-differential equations. – R. Schwab ('10).

Existing results for nonlocal operators

- Homogenization a class of integro-differential equations with L ´evy operators - M. Arisawa ('09).
- Scaling limits for symmetric It ˆo–Lévy processes – R. Rhodes and V. Vargas ('09).
- Homogenization of periodic diffusion with small jumps. – N. Sandric ('16).
- Periodic homogenization for nonlinear integro-differential equations. – R. Schwab ('10).
- Stochastic homogenization of symmetric Lévy type process in non-uniformly elliptic case. – X. Chen, Z.-Q. Chen, T. Kumagai and J. Wang ('21).

Existing results for nonlocal operators

- Homogenization a class of integro-differential equations with L ´evy operators - M. Arisawa ('09).
- Scaling limits for symmetric It ˆo–Lévy processes – R. Rhodes and V. Vargas ('09).
- Homogenization of periodic diffusion with small jumps. – N. Sandric ('16).
- Periodic homogenization for nonlinear integro-differential equations. – R. Schwab ('10).
- Stochastic homogenization of symmetric Lévy type process in non-uniformly elliptic case. – X. Chen, Z.-Q. Chen, T. Kumagai and J. Wang ('21).
- Periodic homogenization of non-symmetric Lévy type processes – X. Chen, Z.-Q. Chen, T. Kumagai and J. Wang ('22).

Assumptions

We assume that the convolution kernel $a(\cdot)$ and the coefficient $\Lambda(\cdot)$ possess the following properties:

Assumptions

We assume that the convolution kernel $a(\cdot)$ and the coefficient $\Lambda(\cdot)$ possess the following properties:

- Boundedness

C1

$$a(z) \geq 0; \qquad a(z) \in L^1(\mathbb{R}^d).$$

Assumptions

We assume that the convolution kernel $a(\cdot)$ and the coefficient $\Lambda(\cdot)$ possess the following properties:

- Boundedness

C1

$$a(z) \geq 0; \qquad a(z) \in L^1(\mathbb{R}^d).$$

- Normalization and second moments

C3

$$\int_{\mathbb{R}^d} a(z)dz = 1, \quad \int_{\mathbb{R}^d} |z|^2 a(z)dz < \infty.$$

Properties of Λ

- Uniform ellipticity.

C4

$$0 < \Lambda^- \le \Lambda(x, y, s) \le \Lambda^+$$

- Uniform ellipticity.

C4

$$0 < \Lambda^- \leq \Lambda(x, y, s) \leq \Lambda^+$$

Periodicity in spatial variables.

- Periodicity.

P1

$\Lambda(x, y, s)$ is a periodic function in x and y with a period $(0, 1]^{2d}$.

Properties of Λ

- Uniform ellipticity.

C4

$$0 < \Lambda^- \leq \Lambda(x, y, s) \leq \Lambda^+$$

Periodicity in spatial variables.

- Periodicity.

P1

$\Lambda(x, y, s)$ is a periodic function in x and y with a period $(0, 1]^{2d}$.

Randomness in time.

- Stationarity.

R1

$\Lambda(x, y, s)$ is a stationary ergodic random field in s.

Homogenization of "elliptic" problems. Periodic symmetric case.

Consider first the operators with coefficients that do not depend on time. In this case we can study the stationary problem: given a function $f \in L^2(\mathbb{R}^d)$ and $\lambda > 0$ find u^ε that satisfies the equation

$$-A^\varepsilon u(x) + \lambda u(x) \; = \; f(x) \qquad \text{in } \mathbb{R}^d, \tag{2}$$

Homogenization of "elliptic" problems. Periodic symmetric case.

Consider first the operators with coefficients that do not depend on time. In this case we can study the stationary problem: given a function $f \in L^2(\mathbb{R}^d)$ and $\lambda > 0$ find u^ε that satisfies the equation

$$-A^\varepsilon u(x) + \lambda u(x) \;=\; f(x) \qquad \text{in } \mathbb{R}^d, \tag{2}$$

We first dwell on the symmetric case. In addition to the above formulated conditions we assume that

- Symmetry

Sym

$$\Lambda = \Lambda(x, y), \quad \Lambda(x, y) = \Lambda(y, x), \quad a(-z) = a(z)$$

Homogenization problem

In the symmetric case it is straightforward to show that, for each $\varepsilon > 0$ and any $u \in L^2(\mathbb{R}^d)$,

$$(A^\varepsilon u, u) = -\frac{1}{2\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big) \Lambda\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big) \big(u(y,t) - u(x,t)\big)^2 dxdy.$$

Homogenization problem

In the symmetric case it is straightforward to show that, for each $\varepsilon > 0$ and any $u \in L^2(\mathbb{R}^d)$,

$$(A^\varepsilon u, u) = -\frac{1}{2\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big) \Lambda\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big) \big(u(y,t) - u(x,t)\big)^2 dxdy.$$

This implies that the equation $-A^\varepsilon u + \lambda u = f$ has a unique solution $u^\varepsilon \in L^2(\mathbb{R}^d)$.

In the symmetric case it is straightforward to show that, for each $\varepsilon > 0$ and any $u \in L^2(\mathbb{R}^d)$,

$$(A^\varepsilon u, u) = -\frac{1}{2\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\Lambda\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)\big(u(y,t)-u(x,t)\big)^2 dxdy.$$

This implies that the equation $-A^\varepsilon u + \lambda u = f$ has a unique solution $u^\varepsilon \in L^2(\mathbb{R}^d)$.

Moreover,

$$\|u^\varepsilon\|_{L^2(\mathbb{R}^d)} \leqslant \frac{1}{\lambda}.$$

Homogenization result

Theorem

There exists a positive definite symmetric constant $d \times d$ matrix $a^{\mathrm{eff}} = \{a_{ij}^{\mathrm{eff}}\}$ such that for each $f \in L^2(\mathbb{R}^d)$ the solution u^ε of the problem

$$-A^\varepsilon u + \lambda u = f \qquad \text{in } \mathbb{R}^d$$

converges in $L^2(\mathbb{R}^d)$, as $\varepsilon \to 0$, to a solution u^0 of the homogenized problem

$$-\mathrm{div}\big(a^{\mathrm{eff}} \nabla u\big) + \lambda u = f \quad \text{in } \mathbb{R}^d.$$

Homogenization result

Theorem

There exists a positive definite symmetric constant $d \times d$ matrix $a^{\text{eff}} = \{a_{ij}^{\text{eff}}\}$ such that for each $f \in L^2(\mathbb{R}^d)$ the solution u^ε of the problem

$$-A^\varepsilon u + \lambda u = f \qquad \text{in } \mathbb{R}^d$$

converges in $L^2(\mathbb{R}^d)$, as $\varepsilon \to 0$, to a solution u^0 of the homogenized problem

$$-\mathrm{div}\big(a^{\text{eff}} \nabla u\big) + \lambda u = f \quad \text{in } \mathbb{R}^d.$$

Observe that the limit problem is well posed and has a unique solution.

Homogenization of parabolic problem

In the non-stationary case we consider a Cauchy problem that reads

$$\partial_t u = A^\varepsilon u \qquad \text{in } \mathbb{R}^d \times (0, T], \qquad u(x,0) = u_0 \in L^2(\mathbb{R}^d).$$

Homogenization of parabolic problem

In the non-stationary case we consider a Cauchy problem that reads

$$\partial_t u = A^\varepsilon u \qquad \text{in } \mathbb{R}^d \times (0, T], \qquad u(x,0) = u_0 \in L^2(\mathbb{R}^d).$$

Theorem

For any $u_0 \in L^2(\mathbb{R}^d)$ the solution u^ε of the above Cauchy problem converges in $L^\infty(0, T : L^2(\mathbb{R}^d))$, as $\varepsilon \to 0$, to a solution u^0 of the homogenized Cauchy problem

$$\partial_t u = \mathrm{div}\big(a^{\mathrm{eff}} \nabla u\big) \quad \text{in } \mathbb{R}^d \times (0, T], \qquad u(x,0) = u_0.$$

Non-symmetric autonomous case

In the non-symmetric case the homogenization result holds in moving coordinates.

Non-symmetric autonomous case

In the non-symmetric case the homogenization result holds in moving coordinates.

We assume that $\Lambda(x, y)$ is a periodic function and all the above conditions, except for **Sym**, are fulfilled.

Non-symmetric autonomous case

In the non-symmetric case the homogenization result holds in moving coordinates.

We assume that $\Lambda(x, y)$ is a periodic function and all the above conditions, except for **Sym**, are fulfilled.

We consider the following Cauchy problem:

$$\partial_t u - A^\varepsilon u = 0 \quad \text{in } \mathbb{R}^d \times (0, T], \qquad u(x,0) = u_0(x).$$

Non-symmetric autonomous case

In the non-symmetric case the homogenization result holds in moving coordinates.

We assume that $\Lambda(x, y)$ is a periodic function and all the above conditions, except for **Sym**, are fulfilled.

We consider the following Cauchy problem:

$$\partial_t u - A^\varepsilon u = 0 \quad \text{in } \mathbb{R}^d \times (0, T], \qquad u(x,0) = u_0(x).$$

Since the operator A^ε is bounded for each $\varepsilon > 0$, this problem has a unique solution. Denote it u^ε.

Non-symmetric autonomous case

In the non-symmetric case the homogenization result holds in moving coordinates.

We assume that $\Lambda(x, y)$ is a periodic function and all the above conditions, except for **Sym**, are fulfilled.

We consider the following Cauchy problem:

$$\partial_t u - A^\varepsilon u = 0 \quad \text{in } \mathbb{R}^d \times (0, T], \qquad u(x, 0) = u_0(x).$$

Since the operator A^ε is bounded for each $\varepsilon > 0$, this problem has a unique solution. Denote it u^ε.

Lemma

The family u^ε satisfies the following uniform estimate:

$$\|u^\varepsilon\|^2_{L^\infty(0,T;L^2(\mathbb{R}^d))} - \int_0^T (A^\varepsilon u(\cdot, t), u(\cdot, t)dt \leqslant C\|u_0\|^2_{L^2(\mathbb{R}^d)}.$$

Theorem

In the non-symmetric case there exist a constant vector $b \in \mathbb{R}^d$ and a positive definite matrix a^{eff} such that for any $u_0 \in L^2(\mathbb{R}^d)$ we have

$$\int_0^T \int_{\mathbb{R}^d} \left| u^\varepsilon(x,t) - u^0\Big(x - \frac{b}{\varepsilon}t, t\Big)\right|^2 dxdt \to 0, \quad as\ \varepsilon \to 0,$$

where u^0 is a solution of the homogenized Cauchy problem that reads

$$\partial_t u - \mathrm{div}\big(a^{\mathrm{eff}} \nabla u\big) = 0, \qquad u(x,0) = u_0(x).$$

Time dependent coefficients

We turn to the case of time dependent coefficients and consider the Cauchy problem

$$\partial_t u - \frac{1}{\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\Lambda\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{t}{\varepsilon^2}\Big)\big(u(y,t) - u(x,t)\big)\, dx = 0,$$

$$u(x,0) = u_0(x), \quad u_0 \in L^2(\mathbb{R}^d).$$

Time dependent coefficients

We turn to the case of time dependent coefficients and consider the Cauchy problem

$$\partial_t u - \frac{1}{\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\Lambda\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{t}{\varepsilon^2}\Big)\big(u(y,t) - u(x,t)\big)\, dx = 0,$$

$$u(x,0) = u_0(x), \quad u_0 \in L^2(\mathbb{R}^d).$$

By the Shur test (Shur theorem) the operator

$$A^\varepsilon(t)u(x,t) = \frac{1}{\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\Lambda\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{t}{\varepsilon^2}\Big)\big(u(y,t) - u(x,t)\big)\, dx$$

is bounded in $L^2(\mathbb{R}^d)$,

Time dependent coefficients

We turn to the case of time dependent coefficients and consider the Cauchy problem

$$\partial_t u - \frac{1}{\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\Lambda\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{t}{\varepsilon^2}\Big)\big(u(y,t) - u(x,t)\big)\, dx = 0,$$

$$u(x,0) = u_0(x), \quad u_0 \in L^2(\mathbb{R}^d).$$

By the Shur test (Shur theorem) the operator

$$A^\varepsilon(t)u(x,t) = \frac{1}{\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\Lambda\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{t}{\varepsilon^2}\Big)\big(u(y,t) - u(x,t)\big)\, dx$$

is bounded in $L^2(\mathbb{R}^d)$, and its norm satisfies the estimate

$$\|A^\varepsilon(t)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leqslant \frac{2}{\varepsilon^2}\Lambda^+.$$

If the coefficient $\Lambda(x, y, t)$ is periodic in all the variables, then the following result holds:

Theorem

There exist a constant vector $b \in \mathbb{R}^d$ and a positive definite matrix a^{eff} such that for any $u_0 \in L^2(\mathbb{R}^d)$ we have

$$\int_0^T \int_{\mathbb{R}^d} \left| u^\varepsilon(x,t) - u^0\left(x - \frac{b}{\varepsilon}t, t\right) \right|^2 dxdt \to 0, \quad \text{as } \varepsilon \to 0,$$

where u^0 is a solution of the homogenized Cauchy problem that reads

$$\partial_t u - \mathrm{div}\left(a^{\mathrm{eff}} \nabla u\right) = 0, \qquad u(x,0) = u_0(x).$$

Therefore, for each $\varepsilon > 0$ the studied Cauchy problem is well posed and has a unique solution.

Therefore, for each $\varepsilon > 0$ the studied Cauchy problem is well posed and has a unique solution.

In order to formulate the homogenization result we recall the definition of the strong mixing coefficient.

Therefore, for each $\varepsilon > 0$ the studied Cauchy problem is well posed and has a unique solution.

In order to formulate the homogenization result we recall the definition of the strong mixing coefficient.

Denote by $\mathcal{F}_{\leqslant s}$ the σ-algebra generated by $\{\Lambda(\cdot, \cdot, r)\}$ with $r \leqslant s$. Similarly, $\mathcal{F}_{\geqslant s}$ is the σ-algebra generated by $\{\Lambda(\cdot, \cdot, r)\}$ with $r \geqslant s$.

Therefore, for each $\varepsilon > 0$ the studied Cauchy problem is well posed and has a unique solution.

In order to formulate the homogenization result we recall the definition of the strong mixing coefficient.

Denote by $\mathcal{F}_{\leqslant s}$ the σ-algebra generated by $\{\Lambda(\cdot,\cdot,r)\}$ with $r \leqslant s$. Similarly, $\mathcal{F}_{\geqslant s}$ is the σ-algebra generated by $\{\Lambda(\cdot,\cdot,r)\}$ with $r \geqslant s$.

The function $\alpha(r)$ defined as

$$\alpha(r) = \sup_{\mathcal{S}_1\in\mathcal{F}_{\leqslant 0},\, \mathcal{S}_2\in\mathcal{F}_{\geqslant r}} \big|\mathbf{P}(\mathcal{S}_1\cap\mathcal{S}_2) - \mathbf{P}(\mathcal{S}_1)\,\mathbf{P}(\mathcal{S}_2)\big|$$

is called the strong mixing coefficient of the random function Λ.

Mixing condition

- Mixing

Mix

$$\int_0^\infty \sqrt{\alpha(r)}dr < +\infty.$$

Mixing condition

- Mixing

Mix

$$\int_0^{\infty} \sqrt{\alpha(r)}dr < +\infty.$$

In the next statement we do not impose mixing condition **Mix**.

Theorem

Let conditions **C1** $-$ **C4**, **P1** *and* **R1** *be fulfilled. Then there exists a positive definite symmetric matrix* $\Theta^{\rm eff}$ *and a vector-valued stationary ergodic process* $\beta_\omega(t) \in L^\infty(\mathbb{R}; \mathbb{R}^d)$ *such that for any* $T > 0$ *and for a.e.* $\omega \in \Omega$

$$\|u^\varepsilon_\omega(x,t) - u^0\Big(x - \frac{1}{\varepsilon}\int_0^t \beta_\omega\Big(\frac{s}{\varepsilon^2}\Big)ds, t\Big)\|_{L^\infty(0,T;L^2(\mathbb{R}^d))} \to 0, \ \textit{as } \varepsilon \to 0.$$

where u^0 *is a solution to the homogenized Cauchy problem*

$$\partial_t u - {\rm div}\big(\Theta^{\rm eff}\nabla u\big) = 0, \qquad u(x,0) = u_0(x).$$

Comments

Remark

If the coefficients $a(x-y)$ and $\Lambda(x,y,t)$ are symmetric in the variables x and y, then $\beta_\omega(s)=0$, and the limit function is deterministic.

Comments

Remark

If the coefficients $a(x-y)$ and $\Lambda(x,y,t)$ are symmetric in the variables x and y, then $\beta_\omega(s)=0$, and the limit function is deterministic.

Remark

Denote $b=\mathbf{E}\beta_\cdot(0)$. By the Birkhoff ergodic theorem a.s.

$$\frac{1}{\varepsilon}\int_0^t \beta_\omega\Big(\frac{s}{\varepsilon^2}\Big)ds = \frac{1}{\varepsilon}b(1+o(1)).$$

Therefore, the principal term of the effective convection is deterministic and is equal to $\frac{b}{\varepsilon}$.

Comments

Remark

If the coefficients $a(x-y)$ and $\Lambda(x,y,t)$ are symmetric in the variables x and y, then $\beta_\omega(s)=0$, and the limit function is deterministic.

Remark

Denote $b=\mathbf{E}\beta_\cdot(0)$. By the Birkhoff ergodic theorem a.s.

$$\frac{1}{\varepsilon}\int_0^t \beta_\omega\Big(\frac{s}{\varepsilon^2}\Big)ds=\frac{1}{\varepsilon}b(1+o(1)).$$

Therefore, the principal term of the effective convection is deterministic and is equal to $\frac{b}{\varepsilon}$.

Concerning the lower order term we only know that it is of order $o(\varepsilon^{-1})$.

Comments

Remark

If the coefficients $a(x-y)$ and $\Lambda(x,y,t)$ are symmetric in the variables x and y, then $\beta_\omega(s)=0$, and the limit function is deterministic.

Remark

Denote $b=\mathbf{E}\beta_\cdot(0)$. By the Birkhoff ergodic theorem a.s.

$$\frac{1}{\varepsilon}\int_0^t \beta_\omega\Big(\frac{s}{\varepsilon^2}\Big)ds=\frac{1}{\varepsilon}b(1+o(1)).$$

Therefore, the principal term of the effective convection is deterministic and is equal to $\frac{b}{\varepsilon}$.

Concerning the lower order term we only know that it is of order $o(\varepsilon^{-1})$.

However, under additional mixing conditions, the behaviour of the second term can be understood better.

Denote

$$g_\omega(t) = \beta_\omega(t) - b.$$

Denote

$$g_\omega(t) = \beta_\omega(t) - b.$$

Lemma

If condition **Mix** *holds, then the integral*

$$B = \int_0^\infty \mathbf{E}(g_\omega(r) \otimes g_\omega(0))dr$$

converges.

Denote

$$g_\omega(t) = \beta_\omega(t) - b.$$

Lemma

If condition **Mix** *holds, then the integral*

$$B = \int_0^\infty \mathbf{E}(g_\omega(r) \otimes g_\omega(0))dr$$

converges. The matrix B is non-negative, and there exists a matrix σ such that

$$\sigma\sigma^* = 2B.$$

Denote

$$g_\omega(t) = \beta_\omega(t) - b.$$

Lemma

If condition **Mix** *holds, then the integral*

$$B = \int_0^\infty \mathbf{E}(g_\omega(r) \otimes g_\omega(0))dr$$

converges. The matrix B is non-negative, and there exists a matrix σ such that

$$\sigma\sigma^* = 2B.$$

The process $G_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t g\left(\frac{s}{\varepsilon^2}\right)ds$ *converges in law in the space $C[0,T]^d$ to the process σW_t, where W_t is the standard Brownian motion.*

Theorem

Under condition **Mix** *we have*

$$u^\varepsilon\Big(x+\frac{b}{\varepsilon}t,t\Big)\overset{\mathcal{L}}{\to}u_0(x-\sigma W_t,t)\quad \textit{in law in } L^2(0,T;L^2(\mathbb{R}^d)).$$

Theorem

Under condition **Mix** *we have*

$$u^{\varepsilon}\Big(x+\frac{b}{\varepsilon}t,t\Big)\xrightarrow{\mathcal{L}}u_0(x-\sigma W_t,t)\quad \textit{in law in } L^2(0,T;L^2(\mathbb{R}^d)).$$

Applying Itô's formula, as a consequence we obtain

Theorem

The family $u^{\varepsilon}\Big(x+\frac{c}{\varepsilon}t,t\Big)$ *converges in law, as* $\varepsilon\to 0$, *in the space* $L^2(0,T;L^2(\mathbb{R}^d))$ *to a solution of the effective SPDE that reads*

$$du=\mathrm{div}\big(\Xi^{\mathrm{eff}}\nabla u\big)dt-\nabla u\cdot\sigma dW_t,\quad u(x,0)=u_0(x).$$

with $\Xi^{\mathrm{eff}}=\Theta^{\mathrm{eff}}+\frac{1}{2}\sigma\sigma^*$.

Higher order approximation

If the operator $A_\varepsilon(t)$ is symmetric for each t, then the limit is deterministic and the effective problem reads

$$\partial_t u = \mathrm{div}\big(a^{\mathrm{eff}} \nabla u\big) \ \text{ in } \mathbb{R}^d \times (0, T], \qquad u(x,0) = u_0.$$

Higher order approximation

If the operator $A_\varepsilon(t)$ is symmetric for each t, then the limit is deterministic and the effective problem reads

$$\partial_t u = \mathrm{div}\big(a^{\mathrm{eff}} \nabla u\big) \ \text{ in } \mathbb{R}^d \times (0, T], \qquad u(x,0) = u_0.$$

In this case, it is natural to raise the question of higher order approximations. We study the normalized difference

$$U^\varepsilon := \frac{u^\varepsilon - u^0}{\varepsilon}.$$

Higher order approximation

If the operator $A_\varepsilon(t)$ is symmetric for each t, then the limit is deterministic and the effective problem reads

$$\partial_t u = \mathrm{div}\big(a^{\mathrm{eff}} \nabla u\big) \ \text{ in } \mathbb{R}^d \times (0, T], \qquad u(x,0) = u_0.$$

In this case, it is natural to raise the question of higher order approximations. We study the normalized difference

$$U^\varepsilon := \frac{u^\varepsilon - u^0}{\varepsilon}.$$

From now on we assume that, in addition to the above assumptions, $a(z)$ has a finite third moment:

$$\int_{\mathbb{R}^d} |z|^3 a(z)dz < +\infty.$$

Auxiliary problem

We define $\chi_1(\xi, s)$ as a periodic in ξ and stationary solution of the equation:

$$\partial_s \chi_1(\xi, s) - (L_s \chi_1)(\xi, s) = -\int_{\mathbb{R}^d} a(z) \Lambda(\xi, \xi - z, s)\, z dz,$$

which satisfies the normalization condition

$$\int_{\mathbb{T}^d} \chi_1(\xi, s) d\xi = 0.$$

Auxiliary problem

We define $\chi_1(\xi, s)$ as a periodic in ξ and stationary solution of the equation:

$$\partial_s \chi_1(\xi, s) - (L_s \chi_1)(\xi, s) = -\int_{\mathbb{R}^d} a(z)\Lambda(\xi, \xi - z, s)\, z dz,$$

which satisfies the normalization condition

$$\int_{\mathbb{T}^d} \chi_1(\xi, s) d\xi = 0.$$

The existence and uniqueness of the function χ_1 is granted by the following statement:

Auxiliary function

Lemma

If the coefficients $a(\xi-\eta)$ *and* $\Lambda(\xi,\eta,s)$ *are symmetric in the variables* ξ *and* η*, then the above auxiliary problem has a periodic in* ξ *stationary solution. This solution is unique up to an additive (random) constant.*

Auxiliary function

Lemma

If the coefficients $a(\xi-\eta)$ and $\Lambda(\xi,\eta,s)$ are symmetric in the variables ξ and η, then the above auxiliary problem has a periodic in ξ stationary solution. This solution is unique up to an additive (random) constant.

Later on we assume that mixing condition **Mix** is fulfilled.

Diffusion approximation

Theorem

There exist

- *a symmetric constant positive semi-definite* $d^2 \times d^2$ *matrix* Θ^{eff}
- *a constant tensor* $H^{\mathrm{eff}} = \{H^{\mathrm{eff}}_{jkl}\}_{j,k,l=1}^{d}$

such that the process $\frac{1}{\varepsilon}(u^\varepsilon - u^0) - \chi_1\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\nabla u^0$ *converges in law in* $L^2(\mathbb{R}^d \times (0,T))$ *to the unique solution of the following SPDE:*

$$dU_t - \mathrm{div}(a^{\mathrm{eff}}\nabla U)dt = \Theta^{\mathrm{eff}}\nabla\nabla u^0\, dW_t + H^{\mathrm{eff}}\nabla\nabla\nabla u^0 dt,$$
$$U(x,0) = 0,$$

where u^0 *satisfies the equation*

$$\partial_t u^0 - \mathrm{div}(a^{\mathrm{eff}}\nabla u^0) = 0, \qquad u(x,0) = u_0.$$

We begin by using the asymptotic expansions technique.

Some elements of the proof

We begin by using the asymptotic expansions technique. Assuming that the limit function $u(x,t)$ belongs to $C^\infty([0,T];\mathbb{S}(\mathbb{R}^d))$, where the symbol $\mathbb{S}$ stands for the Schwartz class, we represent u^ε as the asymptotic series

$$u^\varepsilon(x,t) = u(x^\varepsilon,t) + \varepsilon\varkappa_1\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\nabla u(x^\varepsilon,t) + \dots$$

We begin by using the asymptotic expansions technique. Assuming that the limit function $u(x,t)$ belongs to $C^\infty([0,T];\mathbb{S}(\mathbb{R}^d))$, where the symbol $\mathbb{S}$ stands for the Schwartz class, we represent u^ε as the asymptotic series

$$u^\varepsilon(x,t) = u(x^\varepsilon, t) + \varepsilon \varkappa_1\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\Big)\nabla u(x^\varepsilon, t) + \dots$$

with

$$x^\varepsilon = x - \frac{1}{\varepsilon}\gamma^\varepsilon(t), \qquad \gamma^\varepsilon(t) = \int_0^t \beta\Big(\frac{s}{\varepsilon^2}\Big)ds.$$

Elements of the proof

One can rewrite the operator $A^\varepsilon(t)$ as follows:

$$A^\varepsilon(t)u(x,t) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} a(z)\Lambda\Big(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^2}\Big)\big(u(x,t)-u(x-\varepsilon z,t)\big)\,dz.$$

One can rewrite the operator $A^\varepsilon(t)$ as follows:

$$A^\varepsilon(t)u(x,t) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} a(z)\Lambda\Big(\frac{x}{\varepsilon}, \frac{x}{\varepsilon}-z, \frac{t}{\varepsilon^2}\Big)\big(u(x,t)-u(x-\varepsilon z, t)\big)\,dz.$$

We then expand $u(x^\varepsilon - \varepsilon z, t)$ in Taylor series,

$$u(x^\varepsilon - \varepsilon z, t) = u(x^\varepsilon, t) - \varepsilon z \nabla u(x^\varepsilon, t) + \frac{1}{2}\varepsilon^2 z \otimes z \nabla\nabla u(x^\varepsilon, t) + \dots,$$

Elements of the proof

One can rewrite the operator $A^\varepsilon(t)$ as follows:

$$A^\varepsilon(t)u(x,t) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} a(z)\Lambda\Big(\frac{x}{\varepsilon}, \frac{x}{\varepsilon}-z, \frac{t}{\varepsilon^2}\Big)\big(u(x,t)-u(x-\varepsilon z,t)\big)\,dz.$$

We then expand $u(x^\varepsilon - \varepsilon z, t)$ in Taylor series,

$$u(x^\varepsilon-\varepsilon z,t) = u(x^\varepsilon,t)-\varepsilon z\nabla u(x^\varepsilon,t)+\frac{1}{2}\varepsilon^2 z\otimes z\nabla\nabla u(x^\varepsilon,t)+\dots,$$

substitute the right-hand side for $u(x^\varepsilon - \varepsilon z, t)$ and collect power-like terms in the resulting relation.

Auxiliary problem

At the first step this yields the evolution equation on $\mathbb{T}^d \times (-\infty, +\infty)$ that reads

$$\partial_s \varkappa_1(\xi, s) - \int_{\mathbb{R}^d} a(\xi - \eta)\Lambda(\xi, \eta, s)\big(\varkappa_1(\eta, s) - \varkappa_1(\xi, s)\big)\, d\eta = h(\xi, s)$$

Auxiliary problem

At the first step this yields the evolution equation on $\mathbb{T}^d \times (-\infty, +\infty)$ that reads

$$\partial_s \varkappa_1(\xi, s) - \int_{\mathbb{R}^d} a(\xi-\eta)\Lambda(\xi,\eta,s)\big(\varkappa_1(\eta,s) - \varkappa_1(\xi,s)\big)\,d\eta = h(\xi,s)$$

with

$$h(\xi,s) = \int_{\mathbb{R}^d} a(\xi-\eta)\Lambda(\xi,\eta,s)\big(\eta-\xi\big)\,d\eta - \beta(s)$$

Auxiliary problem

At the first step this yields the evolution equation on $\mathbb{T}^d \times (-\infty, +\infty)$ that reads

$$\partial_s \varkappa_1(\xi, s) - \int_{\mathbb{R}^d} a(\xi - \eta)\Lambda(\xi, \eta, s)\big(\varkappa_1(\eta, s) - \varkappa_1(\xi, s)\big)\, d\eta = h(\xi, s)$$

with

$$h(\xi, s) = \int_{\mathbb{R}^d} a(\xi - \eta)\Lambda(\xi, \eta, s)\big(\eta - \xi\big)\, d\eta - \beta(s)$$

Denote

$$A(s)v(\xi) = \int_{\mathbb{R}^d} a(\xi - \eta)\Lambda(\xi, \eta, s)\big(v(\eta) - v(\xi)\big)\, d\eta, \quad v \in L^2(\mathbb{T}^d),$$

Auxiliary problem

At the first step this yields the evolution equation on $\mathbb{T}^d \times (-\infty, +\infty)$ that reads

$$\partial_s \varkappa_1(\xi, s) - \int_{\mathbb{R}^d} a(\xi - \eta)\Lambda(\xi, \eta, s)\big(\varkappa_1(\eta, s) - \varkappa_1(\xi, s)\big)\, d\eta = h(\xi, s)$$

with

$$h(\xi, s) = \int_{\mathbb{R}^d} a(\xi - \eta)\Lambda(\xi, \eta, s)\big(\eta - \xi\big)\, d\eta - \beta(s)$$

Denote

$$A(s)v(\xi) = \int_{\mathbb{R}^d} a(\xi - \eta)\Lambda(\xi, \eta, s)\big(v(\eta) - v(\xi)\big)\, d\eta, \quad v \in L^2(\mathbb{T}^d),$$

$$A^*(s)p(\xi) = \int_{\mathbb{R}^d} a(\eta - \xi)\Lambda(\eta, \xi, s)p(\eta) - \int_{\mathbb{R}^d} a(\xi - \eta)\Lambda(\xi, \eta, s)\, d\eta\, p(\xi).$$

Adjoint problem

Theorem

The equation

$$-\partial p(\xi, s) = A^*(s)p(\xi, s)$$

has a unique up to a multiplicative constant periodic in ξ stationary solution.

Adjoint problem

Theorem

The equation

$$-\partial p(\xi, s) = A^*(s) p(\xi, s)$$

has a unique up to a multiplicative constant periodic in ξ *stationary solution. It satisfies the relation*

$$\int_{\mathbb{T}^d} p(\xi, s)\, d\xi = C.$$

Adjoint problem

Theorem

The equation

$$-\partial p(\xi, s) = A^*(s)p(\xi, s)$$

has a unique up to a multiplicative constant periodic in ξ stationary solution. It satisfies the relation

$$\int_{\mathbb{T}^d} p(\xi, s)\, d\xi = C.$$

Under the normalization condition

$$\int_{\mathbb{T}^d} p(\xi, s)\, d\xi = 1$$

this solution satisfies the estimates

$$0 < p^- \leqslant p(\xi, s) \leqslant p^+ < +\infty.$$

Auxiliary problem

We denote this solution by $p^*(\xi, s)$.

We denote this solution by $p^*(\xi, s)$. Recalling that

$$h(\xi, s) = \int_{\mathbb{R}^d} a(\xi - \eta)\Lambda(\xi, \eta, s)\big(\eta - \xi\big)\,d\eta - \beta(s)$$

Auxiliary problem

We denote this solution by $p^*(\xi, s)$. Recalling that

$$h(\xi, s) = \int_{\mathbb{R}^d} a(\xi - \eta)\Lambda(\xi, \eta, s)\big(\eta - \xi\big)\, d\eta - \beta(s)$$

we choose

$$\beta(s) = \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} a(\xi - \eta)\Lambda(\xi, \eta, s)\big(\eta - \xi\big) p^*(\xi, s) d\eta d\xi.$$

Notice that $h \in L^\infty(\mathbb{T}^d \times (-\infty, +\infty))$.

Auxiliary problem

Under this choice, $\beta(s)$ is a stationary process, and

$$\int_{\mathbb{T}^d} h(\xi, s) p^*(\xi, s)\, d\xi = 0$$

for all ω and s.

Auxiliary problem

Under this choice, $\beta(s)$ is a stationary process, and

$$\int_{\mathbb{T}^d} h(\xi, s) p^*(\xi, s)\, d\xi = 0$$

for all ω and s. Recall that

$$A(s)v(\xi) = \int_{\mathbb{R}^d} a(\xi - \eta)\Lambda(\xi, \eta, s)\big(v(\eta) - v(\xi)\big)\, d\eta, \quad v \in L^2(\mathbb{T}^d).$$

Auxiliary problem

Under this choice, $\beta(s)$ is a stationary process, and

$$\int_{\mathbb{T}^d} h(\xi,s)p^*(\xi,s)\,d\xi = 0$$

for all ω and s. Recall that

$$A(s)v(\xi) = \int_{\mathbb{R}^d} a(\xi-\eta)\Lambda(\xi,\eta,s)\big(v(\eta)-v(\xi)\big)\,d\eta, \quad v \in L^2(\mathbb{T}^d).$$

Theorem

The equation

$$\partial_s \varkappa_1 - A(s)\varkappa_1(\xi,s) = h(\xi,s)$$

has a stationary solution, which is unique up to an additive constant vector. Moreover, $\varkappa_1 \in L^\infty(\mathbb{T}^d \times (-\infty,+\infty))$.

1. Piatnitski, A.; Zhizhina, E.; Homogenization of non-autonomous evolution problems for convolution type operators in randomly evolving media. *J.Math. Pures Appl;*, **194**, (2025), paper 103660.

2. Kleptsyna, M; Piatnitski, A.; Popier A.; Homogenization of nonlocal equations in randomly evolving media. Diffusion approximation. arXiv:2602.09584 (2026)

Thank you for your attention