Effective Field and Particle Counting in Mean Field Thermal Bose Gases

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Outline

- The model and its Gibbs state
- Random field representation
- Distributions of particle numbers

Literature

- Essential dependence on Deuchert, Nam, Napiórkowski, *The Gibbs state of the mean-field Bose gas*, huge literature justifying Bogoliubov theory in various scaling regimes
- Random fields from Bose gases: Fröhlich, Knowles, Schlein, Sohinger and Lewin, Nam, Rougerie and Nam, Zhu, Zhu, several other even this year

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- Random fields from Bose gases: Fröhlich, Knowles, Schlein, Sohinger and Lewin, Nam, Rougerie and Nam, Zhu, Zhu, several other even this year
- Probabilistic results for interacting Bose gases, e.g. Rademacher, Schlein and Arous, Kirkpatrick, Schlein
- Point processes associated to ideal gas: Shirai, Takahashi and Tamura, Ito and Tamura, Zagrebnov

Mean field bosons

Bosons on a 3-dimensional unit torus Λ, interaction $\sim \frac{1}{N}$:

$$H_N = \bigoplus_{n=0}^{\infty} \Big(\sum_{i=1}^{n} -\Delta_i + \frac{1}{N} \sum_{1 \le i < j \le n} v(x_i - x_j) \Big).$$

Operator on the Fock space $\mathcal{F} = \bigoplus_{n=0}^{\infty} L^2_{\mathrm{sym}}(\Lambda^n)$.

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<u>Assumption</u>: v is nonzero, $v \geq 0$, $\widehat{v} \geq 0$, $\sum_{k \in \Lambda^*} |k| \widehat{v}(k) < \infty$.

Gibbs state

Gibbs state $G_{\beta,N}$ — the Fock space operator minimizing

$$\rho \mapsto \mathrm{Tr}[\rho H_N] + \beta^{-1}\mathrm{Tr}[\rho \log \rho]$$

under the constraints

$$\rho \geq 0, \qquad \mathrm{Tr}[\rho] = 1, \qquad \mathrm{Tr}[\rho \mathcal{N}] = N.$$

Equivalently,

$$G_{\beta,N} = \frac{1}{Z} \exp(-\beta(H_N - \mu_{\beta,N}\mathcal{N})).$$

- $\mathcal{N} = \sum_{p \in \Lambda^*} a_p^* a_p$ — number operator,
- a_p^*, a_p — creator and annihilator of e^{ipx}, satisfying

$$[a_p, a_q^*] = \delta_{p,q}$$

Condensation and BEC transition

Theorem (Deuchert, Nam, Napiórkowski)
Let $\beta = \beta(N)$ be chosen such that

$$\lim_{N\to\infty} \beta(N) N^{\frac{2}{3}} = \frac{\zeta(3/2)^{\frac{2}{3}}}{4\pi}\kappa \in (0,\infty).$$

Then:

$$\lim_{N\to\infty} \frac{1}{N}\mathrm{Tr}[G_{\beta,N} a_p^* a_p] = \begin{cases} (1-\kappa^{-\frac{3}{2}})_+, & p = 0, \\ 0, & p \neq 0. \end{cases}$$

I only discuss the case $\kappa > 1$. A fraction $g(\kappa) = 1 - \kappa^{-\frac{3}{2}}$ of all particles have momentum zero.

Approximate Gibbs state

DNN approximated $G_{\beta,N}$ using a simpler density matrix $\Gamma_{\beta,N}$:

$$\Gamma_{\beta,N} = \int_{\mathbb{C}} |\Omega_z\rangle\langle\Omega_z| \otimes G^{\mathrm{Bog}}(z) g^{\mathrm{BEC}}(z) dz.$$

Random variable z governs the condensate, while $G^{\mathrm{Bog}}(z)$ is the Gibbs state of $H^{\mathrm{Bog}}(z)$ – informally, Hamiltonian expanded up to terms quadratic in a, a^* around $a_0 = z$, $a_0^* = \overline{z}$.

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Condensate fluctuations governed by classical probability

$$g^{\mathrm{BEC}}(z) = \frac{1}{Z^{\mathrm{BEC}}} \exp\left(-\frac{\beta\widehat{v}(0)}{2N} |z|^4 + \beta\mu^{\mathrm{BEC}} |z|^2 \right).$$

We have $|z|^2 \approx g(\kappa)N$ with fluctuations of order $N^{\frac{5}{6}} \ll N$.

Approximate Gibbs state

Random field construction

Theorem

There exists a Gaussian random function Φ_N on Λ, satisfying $\widehat{\Phi}_N(0) = 0$ and almost surely smooth, such that:

$$\left\| G_{\beta,N} - \widetilde{\Gamma}_{\beta,N} \right\|_1 \lesssim N^{-\frac{1}{48}},$$

$$\widetilde{\Gamma}_{\beta,N} = \int |\Omega_{z+\frac{z}{|z|}\phi}\rangle\langle\Omega_{z+\frac{z}{|z|}\phi}| g^{\mathrm{BEC}}(z) d\mathbb{P}_{\Phi_N}(\phi) dz.$$

Here $|\Omega_f\rangle = e^{a^*(f)-a(f)}|\Omega\rangle$, *with* $|\Omega\rangle$ *the Fock vacuum.*

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The measure $g^{\mathrm{BEC}}(z) d\mathbb{P}_{\Phi_N}(\phi) dz$ is the *upper symbol* of $\widetilde{\Gamma}_{\beta,N}$. Other semiclassical symbols are less nice: the corresponding random distributions are as singular as white noise.

Asymptotic random field

Theorem
There exists a Gaussian random distribution Ψ *such that we have the convergence*

$$\beta^{\frac{1}{2}}\Phi_N \to \Psi.$$

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- *In Wasserstein distance of probabilities on $H^{-s}(\Lambda)$, $s > \frac{1}{2}$,*
- *of linear observables $\int \beta^{\frac{1}{2}}\Phi_N(x)f(x)dx$ for $f \in H^{-1}(\Lambda)$.*

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Example (ideal gas) To find Ψ, we approximate

$$\frac{1}{e^{\beta(p^2-\mu)}-1} \approx \frac{1}{\beta}\frac{1}{p^2-\mu}$$

Asymptotic random field

$$\Psi(x) = \sum_{p \neq 0} \widehat{\Psi}(p) e^{ipx}.$$

$\widehat{\Psi}(p)$ are jointly Gaussian with the covariance

$$\mathbb{E}[\overline{\widehat{\Psi}(p)}\widehat{\Psi}(q)] = \delta_{p,q} \frac{p^2 + g(\kappa)\widehat{v}(p)}{p^2(p^2 + 2g(\kappa)\widehat{v}(p))},$$
$$\mathbb{E}[\widehat{\Psi}(p)\widehat{\Psi}(q)] = -\delta_{p,-q} \frac{g(\kappa)\widehat{v}(p)}{p^2(p^2 + 2g(\kappa)\widehat{v}(p))}.$$

Recall: $g(\kappa) = (1 - \kappa^{-\frac{3}{2}})_+$ is the limiting *condensate fraction*.

Counting momentum-localized particles

Number operators $\mathcal{N}_p = a_p^* a_p$ are self-adjoint and commute, so they have a joint probability distribution.

Equivalently, there exist random variables N_p such that

$$\mathbb{E}[f(N_{p_1}, \dots, N_{p_k})] = \mathrm{Tr}[G_{\beta,N}\, f(\mathcal{N}_{p_1}, \dots, \mathcal{N}_{p_k})]$$

holds for $f \in C_b(\mathbb{R}^k)$.

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holds for $f \in C_b(\mathbb{R}^k)$.

Theorem

For any nonzero $p_1, \ldots, p_k \in \Lambda^*$ we have the convergence

$$(\beta N_{p_1}, \ldots, \beta N_{p_k}) \to (|\widehat{\Psi}(p_1)|^2, \ldots, |\widehat{\Psi}(p_k)|^2)$$

in Wasserstein 2-distance.

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- $|\widehat{\Psi}(p)|^2$, $|\widehat{\Psi}(q)|^2$ are independent unless $q = \pm p$.
- The joint distribution of $(|\widehat{\Psi}(p)|^2, |\widehat{\Psi}(-p)|^2)$ has density:

$$f(x,y) = c\, e^{-(p^2+g(\kappa)\widehat{v}(p))(x+y)} I_0(2g(\kappa)\widehat{v}(p)\sqrt{xy}),$$

  where I_0 is a Bessel function.

Counting momentum-localized particles

Proof sketch (one mode, convergence in distribution).

1. State approximation:

$$\mathbb{E}[e^{it\beta N_p}] = \mathrm{Tr}[G_{\beta,N} e^{it\beta \mathcal{N}_p}] = \mathrm{Tr}[\widetilde{\Gamma}_{\beta,N} e^{it\beta \mathcal{N}_p}] + o(1).$$

2. Coherent state representation:

$$\mathrm{Tr}[\widetilde{\Gamma}_{\beta,N} e^{it\beta \mathcal{N}_p}] = \mathbb{E}[\langle \Omega_{z+\frac{z}{|z|}\Phi} | e^{it\beta \mathcal{N}_p} \Omega_{z+\frac{z}{|z|}\Phi} \rangle] = \mathbb{E}[e^{\frac{e^{i\beta t}-1}{\beta} |\beta^{\frac{1}{2}} \widehat{\Phi}_N(p)|^2}].$$

We have $\frac{e^{i\beta t}-1}{\beta} \to it$ and $|\beta^{\frac{1}{2}} \widehat{\Phi}_N(p)|^2 \to |\widehat{\Psi}(p)|^2$. The limit

$$\mathbb{E}[e^{it|\widehat{\Psi}(p)|^2}],$$

is the characteristic function of $|\widehat{\Psi}(p)|^2$. □

Counting particles out of the condensate

Number operator for excited particles: $\mathcal{N}_+ = \sum_{p \neq 0} \mathcal{N}_p$.

In the study of $\mathcal{N}_+$, we encounter the Wick square of Ψ,

$$:|\Psi(x)|^2: = \lim_{q \to \infty} \left(|\Psi^{\leq q}(x)|^2 - \mathbb{E}[|\Psi^{\leq q}(x)|^2] \right),$$

where $\Psi^{\leq q}$ is the projection of Ψ onto momenta $|p| \leq q$:

$$\Psi^{\leq q}(x) = \sum_{|p| \leq q} \widehat{\Psi}(p) e^{ipx}.$$

Counting particles out of the condensate

N_+ — random variable associated to $\mathcal{N}_+$ in the state $G_{\beta,N}$:

$$\mathbb{E}[f(N_+)] = \mathrm{Tr}[G_{\beta,N} f(\mathcal{N}_+)].$$

$\mathbb{E}[N_+] \approx \kappa^{-\frac{3}{2}} N$, fluctuations of order $\beta^{-1} \sim N^{\frac{2}{3}} \ll N$.

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$\mathbb{E}[N_+] \approx \kappa^{-\frac{3}{2}} N$, fluctuations of order $\beta^{-1} \sim N^{\frac{2}{3}} \ll N$.

Theorem
We have the convergence

$$\beta(N_+ - \mathbb{E}[N_+]) \to \int_\Lambda :|\Psi(x)|^2: dx$$

in Wasserstein 2*-distance.*

Counting space-localized particles

$\mathcal{N}_D = \int_D a_x^* a_x dx$ — number operator for particles in $D \subset \Lambda$.

$$\mathbb{E}[f(N_{D_1}, \ldots, N_{D_k})] = \mathrm{Tr}[G_{\beta,N}\, f(\mathcal{N}_{D_1}, \ldots, \mathcal{N}_{D_k})].$$

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Random variables N_D can be realized by a *point process*:

$$N_D = \nu(D), \qquad \nu = \sum_{i=1}^{M} \delta_{X_i},$$

with M a random number and $X_1, \ldots, X_M \in \Lambda$ random points.

Probability density of $(X_1, \ldots, X_M)$ = diagonal of the integral kernel of the density matrix on the M-particle subspace.

Counting space-localized particles

Asymptotics of N_D depend on the length-scale we zoom in to:

- **Macro-scale:** D independent of N contains many particles, and $\sqrt{\beta/N}(N_D - \mathbb{E}[N_D])$ converges to a real random variable, not a discrete one.

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- **Micro-scale** — D with volume $\sim \frac{1}{N}$ — N_D is of order 1, can converge to a discrete variable $\longrightarrow$ point process.

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- **Micro-scale** — D with volume $\sim \frac{1}{N}$ — N_D is of order 1, can converge to a discrete variable $\longrightarrow$ point process.
- **Meso-scale** — $\frac{1}{N} \ll |D| \ll 1$ — continuous limit different than for the macro-scale.

Counting space-localized particles

Macro-scale

Theorem

Let $D_1, \ldots, D_k$ have smooth boundary. Then:

$$\sqrt{\beta/N}(N_{D_1} - N|D_1|, \ldots, N_{D_k} - N|D_k|) \xrightarrow{W_2} (V_1, \ldots, V_k),$$

where

$$V_i = \frac{|D_i|}{\widehat{v}(0)} X + 2\sqrt{g(\kappa)} \mathrm{Re}\, \langle \mathbb{1}_D | \Psi \rangle,$$

with $X \sim \mathcal{N}(0,1)$ independent of Ψ.

Counting space-localized particles

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with $X \sim \mathcal{N}(0,1)$ independent of Ψ.

V_i are centered, jointly Gaussian variables with the covariance

$$\mathbb{E}[V_i V_j] = \sum_{p \in \Lambda^*} \frac{2g(\kappa)}{p^2 + 2g(\kappa)\widehat{v}(p)} \overline{\widehat{\mathbb{1}}_{D_i}(p)} \widehat{\mathbb{1}}_{D_j}(p).$$

Theorem (Imprecise formulation)
In the limit $N \to \infty$, the point process associated with $G_{\beta,N}$, zoomed in to the micro-scale $N^{-\frac{1}{3}}$, converges to the point process of ideal Bose gas on $\mathbb{R}^3$ with inverse temperature $\beta_\infty(\kappa) = \frac{\zeta(3/2)^{\frac{2}{3}}}{4\pi}\kappa$ and condensate density $g(\kappa) = 1 - \kappa^{-\frac{3}{2}}$.

Universality: the limit does not depend on the potential v.
It differs from the result for grand canonical ideal gas.

Cox processes

Definition

Let I be a locally finite measure on $\mathbb{R}^3$. Poisson point process with intensity I has its random measure ν characterized by:

- For $B \subset \mathbb{R}^3$ bounded, $\nu(B) \sim \mathrm{Poisson}(I(B))$,
- $(\nu(B_1), \ldots, \nu(B_k))$ are independent for disjoint B_i.

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Definition
Cox processes arise by allowing I itself to be random:

$$\nu(B) \sim \mathrm{Poisson}(I(B)) \text{ conditionally on } I.$$

Counting space-localized particles

Micro-scale

Theorem

Let $D_i = \beta^{\frac{1}{2}} B_i$ *with* $B_i \subset \mathbb{R}^3$ *bounded. Then*

$$(N_{D_1}, \ldots, N_{D_k}) \xrightarrow{d} (\nu(B_1), \ldots, \nu(B_k)),$$

with ν *the random measure of a Cox process*

Counting space-localized particles

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with ν the random measure of a Cox process with the intensity

$$I(B) = \int_B |\sqrt{g(\kappa)} + H(x)|^2 dx,$$

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$$I(B) = \int_B |\sqrt{g(\kappa)} + H(x)|^2 dx,$$

and H — complex Gaussian field on $\mathbb{R}^3$ with the covariance

$$\mathbb{E}[\overline{H(x)} H(y)] = \int_{\mathbb{R}^3} \frac{e^{ip(x-y)}}{e^{\beta_\infty(\kappa) p^2} - 1} \frac{dp}{(2\pi)^3}.$$