The phase transition for random loop models on trees

Volker Betz

TU Darmstadt

Yerevan, 6 September 2019

Joint work with Johannes Ehlert, Benjamin Lees, Lukas Roth

Prelude: my connection to Robert Minlos

My very first paper to be published was entitled
Ground state properties of the Nelson Hamiiltonian - a Gibbs measure based approach
and was jointly with Fumio Hiroshima, Jozsef Lörinczi, Robert Minlos and Herbert Spohn.
Here we investigate the Gibbs measure

$$\frac{1}{Z_T}\,\mathrm{e}^{-\int_{-T}^{T} V(X_s)\,\mathrm{d}s - \int_{-T}^{T}\mathrm{d}s\int_{-T}^{T}\mathrm{d}t\,W(X_t - X_s, t-s)}\,\mathcal{W}_{[-T,T]}(\mathrm{d}X_s)$$

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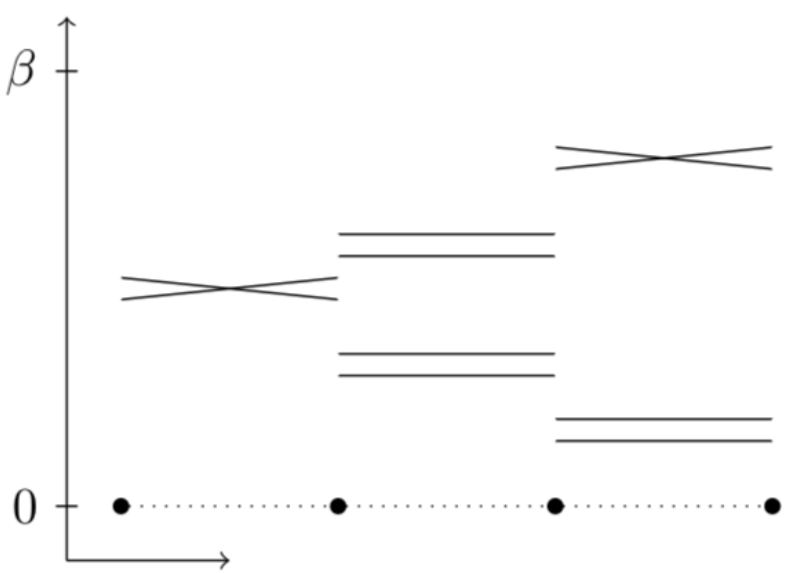
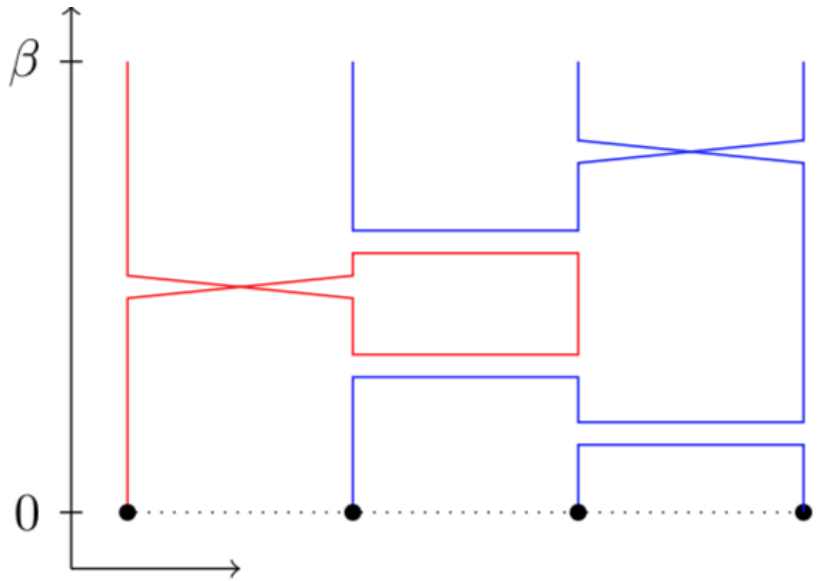
The model is still interesting today - see e.g. Mukherjee / Varadhan 2019 on the polaron.

Interesting unsolved problem: assume W is (very) attractive in the first coordinate, but decays in the second. Assume that V is such that without W the infinite volume Gibbs measure would not exist. Show that the infinite volume Gibbs measure exists if the attraction in W is strong enough.

The random loop model: intuition





The random loop model: definition

- $G = (V, E)$ a graph.
  $\mathbb{T}_\beta$ interval $[0, \beta)$ with periodic b.c.
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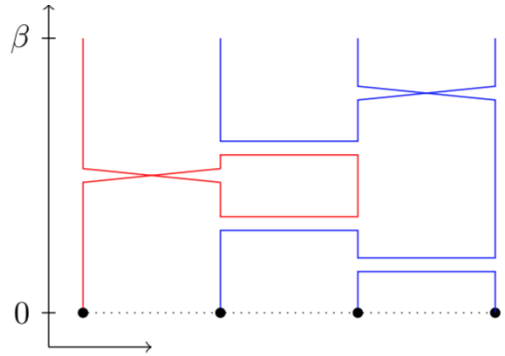

- $(X_e^{\diagdown\!\diagup})_{e\in E}$ and $(X_e^{||})_{e\in E}$ are **iid Poisson point processes** on $\mathbb{T}_\beta$ intensities u and $1-u$, respectively.
- The points of $X_e^{\diagdown\!\diagup,||}$ induce an equivalence relation **('connectedness')** on $\mathcal{X} = \{(v,t) : v \in V, t \in \mathbb{T}_\beta\}$.

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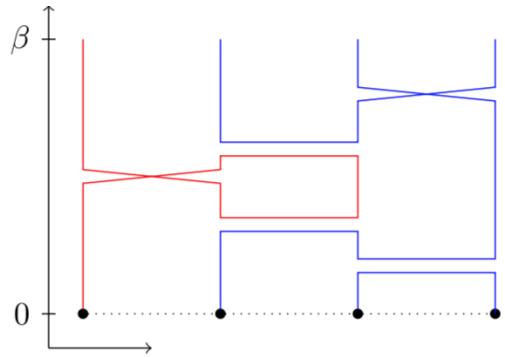

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- (v,t) is connected to (v',t') if
  - $v = v'$ and there is no link between t and t', or
  - $e := \{v, v'\} \in E$, and t and t' connect to the same 'cross' from different sides, or
  - $e := \{v, v'\} \in E$, and t and t' connect to the same 'double bar' from the same side
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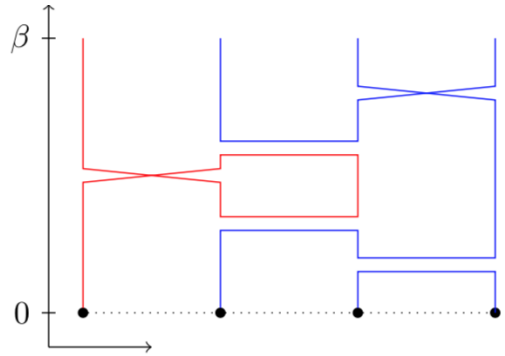

- $(X_e^{\times})_{e\in E}$ and $(X_e^{||})_{e\in E}$ are **iid Poisson point processes** on $\mathbb{T}_\beta$ intensities u and $1-u$, respectively.
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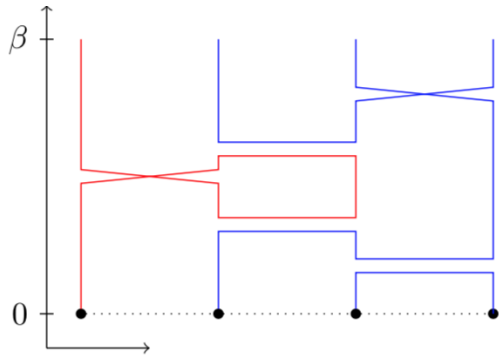
  - Extend by transitivity.
- We thus have a percolation type model.
- **Objective:** prove existence of β_c such that macroscopic loops exist for $\beta > \beta_c$.

The lack of monotonicity

The main difficulty: adding connections can decrease the size of a connected component. Two mechanisms:

1. More than one connection per edge.



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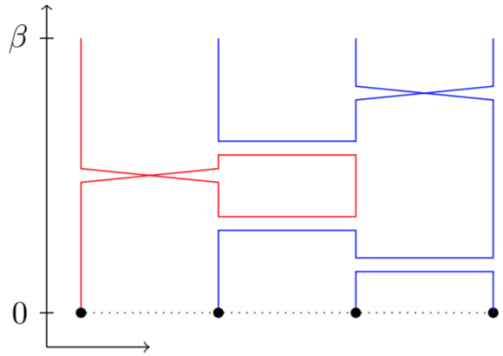

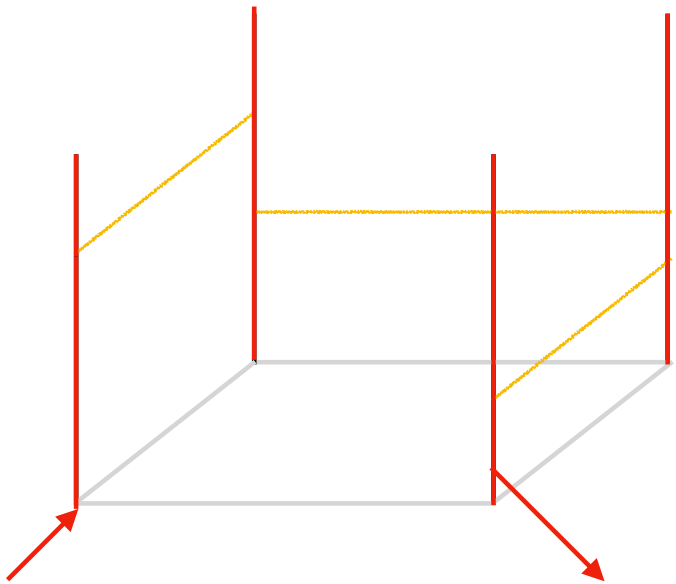

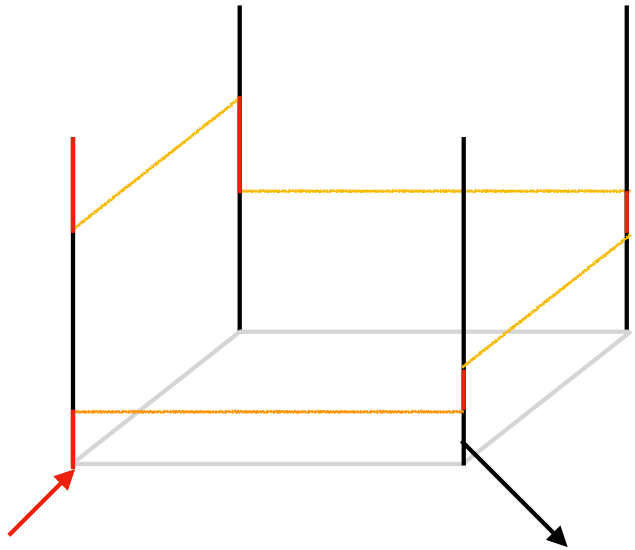

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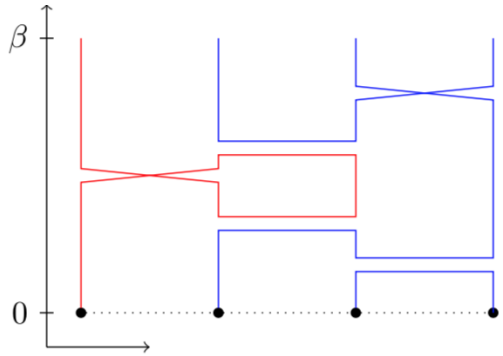

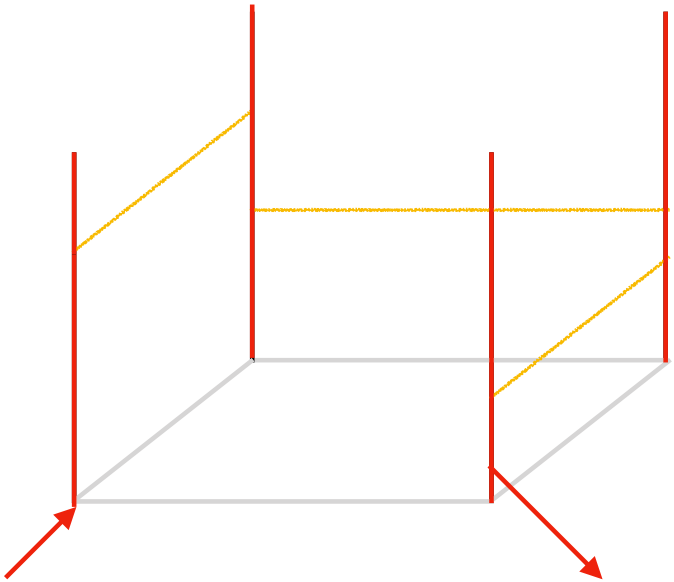

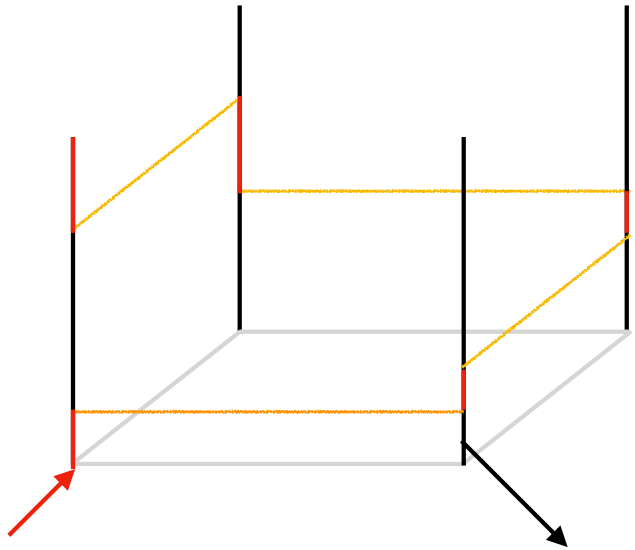

We are only able to address problem 1.

Connections to quantum spin systems

Loop weighted measures: Let $\mathbb{P}_{u,\beta}$ be the measure of the loop model, $\theta > 0$. For a finite Graph G, let

$$\mathbb{P}_{\theta,u,\beta}(A) = \frac{1}{\mathbb{E}_{u,\beta}(\theta^L)} \mathbb{E}_{u,\beta}(\theta^L 1_A)$$

where L is the total number of loops.

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The connection to quantum spin systems occurs for $\theta = 2$: the probability $\mathbb{P}_{2,u,\beta}(x \leftrightarrow y)$ that x and y are in the same cycle is precisely the correlation function between the spins at x and y.

- $H = -2 \sum_{\{x,y\} \in E} S_x^{(1)} S_y^{(1)} + S_x^{(2)} S_y^{(2)} + (2u-1) S_x^{(3)} S_y^{(3)}$.
- $\langle S_x^{(1)} S_y^{(1)} \rangle_\beta \equiv \frac{\mathrm{tr}(\mathrm{S_x^{(1)} S_y^{(1)}\, e^{-\beta H}})}{\mathrm{tr}\, \mathrm{e}^{-\beta \mathrm{H}}} = \mathbb{P}_{2,u,\beta}(x \leftrightarrow y)$.
- Heisenberg ferromagnet ($u = 1$), anti-ferromagnet ($u = 0$) or XY-model ($u = 1/2$).

History of the random loop model

Quantum case ($\theta = 2$):

- Feynman 1953: basic idea to treat thermal states using functional integrals.
- Conlon and Solovej (1991), Toth (1993), Aizenman and Nachtergaele (1994), Ueltschi (2013) refine and use the model in various settings.
- Question of existence of macroscopic loops open in most graphs. Exception: $\mathbb{Z}^d$, with $u \leqslant 1/2$, using reflection positivity (Dyson, Lieb, Simon 1978).

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Classical case ($\theta = 1$), results for diverging degrees:

- Harris 1972: random stirring model.
- Schramm 2005: infinite cycles for the complete graph.
- Berestycki (2011), Berestycki and Kozma (2015): extensions and more slick proofs of Schramms results.
- Kotecky, Milos, Uelschi (2016), results on the hypercube.

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- Björnberg, Ueltschi 2018, 2019: Asymptotics of β_c for large d, and for all θ, u:

$$\frac{\beta_c}{\theta} = \frac{1}{d} + \frac{1 - \theta u(1-u) - \theta^2(1-u)^2/6}{d^2} + o(d^{-2}).$$

No statement for small d.

Main Theorem

Theorem (B., Ehlert, Lees, Roth): Let T be the infinite d-regular tree, r its root, γ_T the loop containing $(r, 0)$.

Then for all $d \geqslant 3$, $u \in [0, 1]$ there exists $\beta_c > 0$ and $\beta^+ > \beta_c$ such that

1. γ_T is finite almost surely for all $\beta \leqslant \beta_c$,
2. γ_T is infinite with positive probability for all $\beta \in (\beta_c, \beta^+)$.

Moreover, $\beta^+ \geqslant \frac{1}{\sqrt{d}}$ for all $d \geqslant 3$ and $\beta^+ = \infty$ for $d \geqslant 16$.

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We have the expansion

$$\beta_c = \sum_{k=0}^{n} \frac{\alpha_k(u)}{d^{k+1}} + \mathcal{O}(d^{-n-2}),$$

where the α_k are polynomials of order $2k$ in u with recursively computable coefficients.

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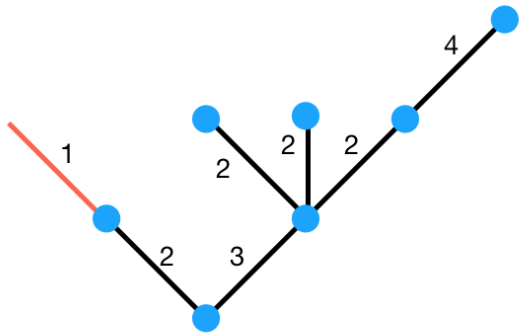
Main progress of our results:
sharp phase transition for much smaller d than known before,
much simpler proof!

Proof: step 1, Galton Watson trees

- Let C_1 be the (random) maximal subtree of T containing the root and where each edge has at least two links.
  Percolation on trees: C is finite almost surely if $\beta^2 \leqslant 1/d$.



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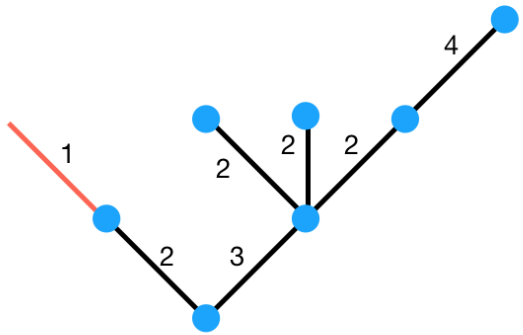

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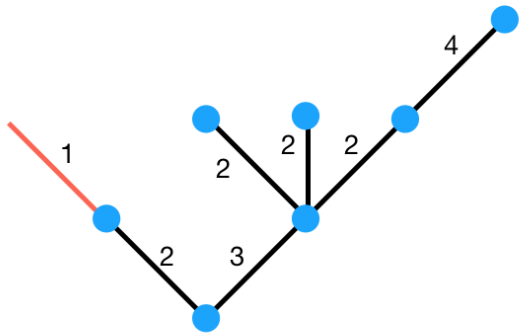

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- So, e serves as a renewal edge, separating future and past.
- Let M_i be the 'living' renewal edges in the i-th generation. $(|M_i|)_{i\in\mathbb{N}}$ is a Galton-Watson process.

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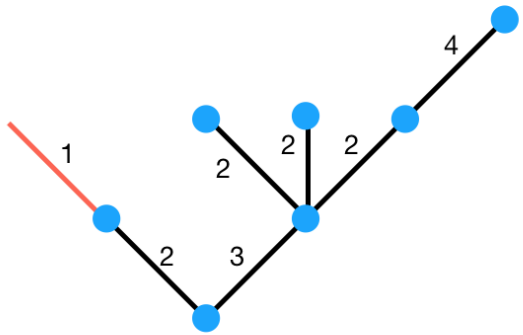

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- So, e serves as a renewal edge, separating future and past.
- Let M_i be the 'living' renewal edges in the i-th generation. $(|M_i|)_{i\in\mathbb{N}}$ is a Galton-Watson process.
- Therefore: γ_T is finite almost surely if and only if $\mathbb{E}_{u,\beta}(M_1) \leqslant 1$.

Proof, step 2: computing $\mathbb{E}_{u,\beta}(M_1)$

Let $\mathcal{C}$ be the set of finite rooted subtrees of T, with edges e labelled by $n_e \geqslant 2$. In the case $\beta < 1/\sqrt{d}$ we find

$$\mathbb{E}(|M_1|) = \sum_{C \in \mathcal{C}} \mathbb{E}\big(|M_1| \,\big|C_1 = C\big)\mathbb{P}(C_1 = C).$$

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By independence, we have

$$\mathbb{P}(C_1 = C) = \prod_{e \in E(C)} \mathbb{P}(|X_e| = n_e) \prod_{e \in \partial^+ C} \mathbb{P}(|X_e| \leqslant 1)$$

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On the other hand,

$$\mathbb{E}\big(|M_1| \,\big|C_1 = C\big) = \frac{\beta}{1+\beta} \sum_{x \in V(C)} (d - d_x) p(u, d)$$

where $p(u, d)$ is a polynomial in u.

Uniqueness and sharpness of the phase transitions

Setting $\alpha = \beta d$, $h = 1/d$, we obtain

$$\mathbb{E}_\beta(|M_1|) = \sum_{C \in \mathcal{C}} f_C(\alpha, h) g_C(h, u), \quad \textit{with}$$

$$f_C(\alpha, h) = \left(\mathrm{e}^{-\alpha} (1 + \alpha h)^{1/h - 1} \right)^{|V(C)|} \frac{\alpha^{N(C)+1}}{(N(C)+1)!} h^{N(C) - E(C)}.$$

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Now a direct computations shows that

$$\partial_\alpha f_C(\alpha, h) = \Big(-|V(C)| + \frac{(1/h - 1)|V(C)|h}{1 + \alpha h} + \frac{N(C) + 1}{\alpha} \Big) f_C(\alpha, h)$$

and since $N(C) + 1 \geqslant 2|E(C)| + 1 \geqslant |V(C)|$, we find that

$$\partial_\alpha f_C(\alpha, h) \geqslant f_C(\alpha, h) |V(C)| \frac{1 - \alpha^2 h}{(1 + \alpha h)\alpha} > 0.$$

for $\alpha^2 h = \beta^2 d < 1$, thus $\beta \mapsto \mathbb{E}_\beta(|M_1|)$ is strictly monotone.

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$N(C)$ is the number of links on C, and $g_C(h, u)$ a polynomial which is non-negative for all u and all $h = 1/d$, $d \in \mathbb{N}$.
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$$\partial_\alpha f_C(\alpha, h) = \Big(-|V(C)| + \frac{(1/h-1)|V(C)|h}{1+\alpha h} + \frac{N(C)+1}{\alpha}\Big) f_C(\alpha, h)$$

and since $N(C)+1 \geqslant 2|E(C)|+1 \geqslant |V(C)|$, we find that

$$\partial_\alpha f_C(\alpha, h) \geqslant f_C(\alpha, h)|V(C)| \frac{1-\alpha^2 h}{(1+\alpha h)\alpha} > 0.$$

for $\alpha^2 h = \beta^2 d < 1$, thus $\beta \mapsto \mathbb{E}_\beta(|M_1|)$ is strictly monotone.
This proves uniqueness of the phase transition.

Existence of the phase transition

$$\mathbb{E}_\beta(|M_1|) = \sum_{C \in \mathcal{C}} f_C(\beta d, d^{-1}) g_C(d^{-1}, u)$$

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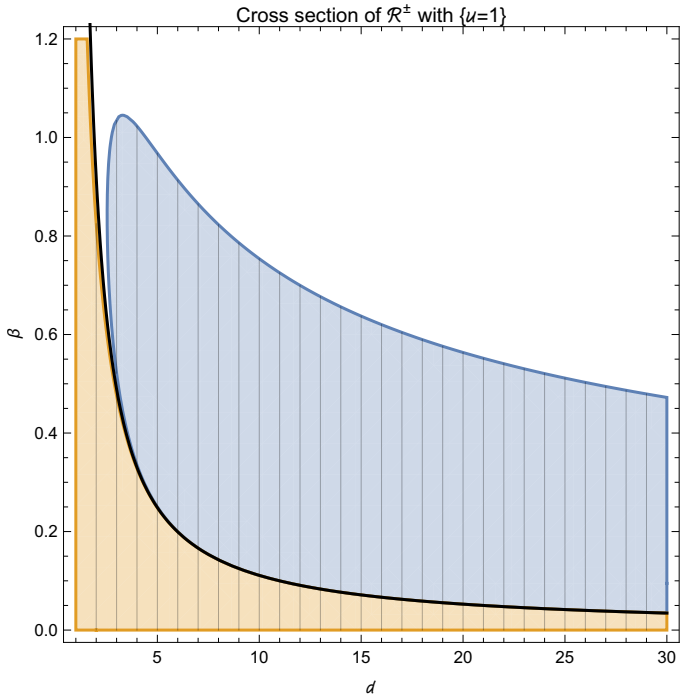

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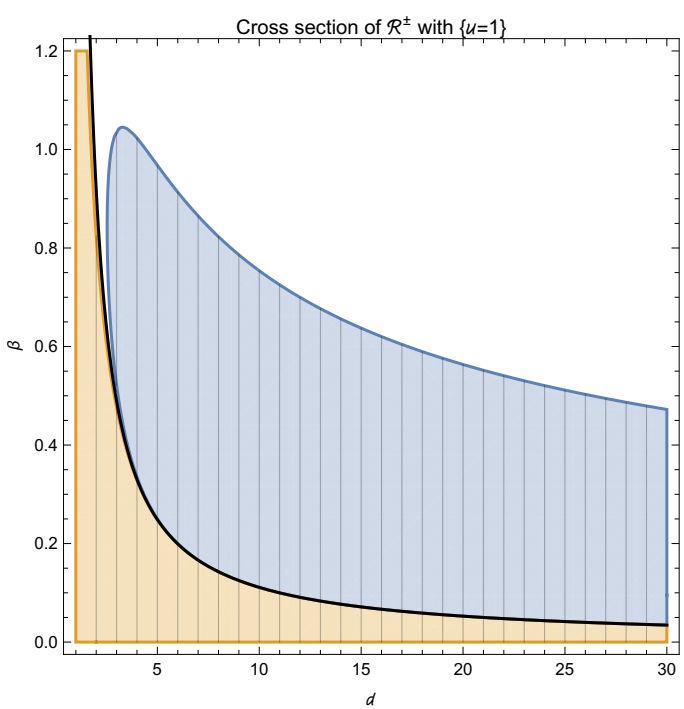

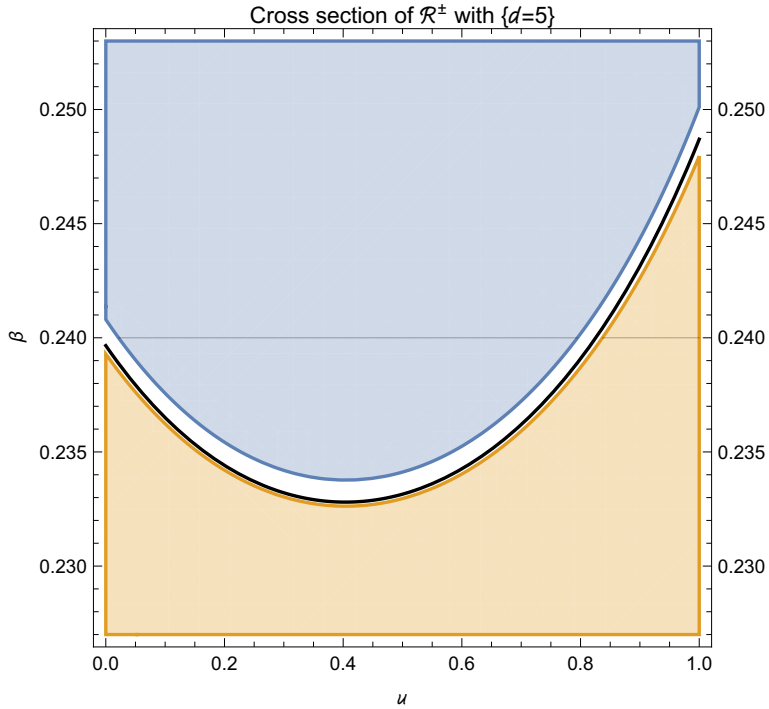

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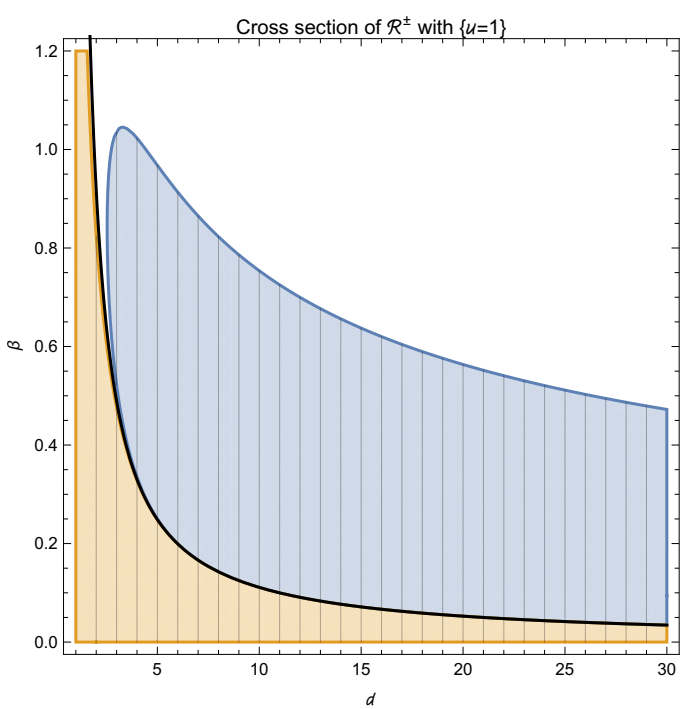
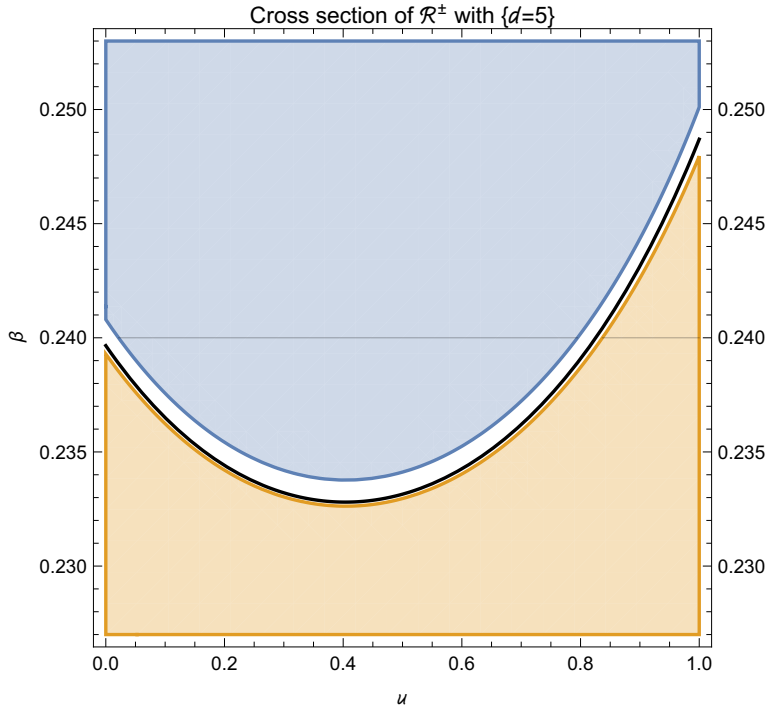
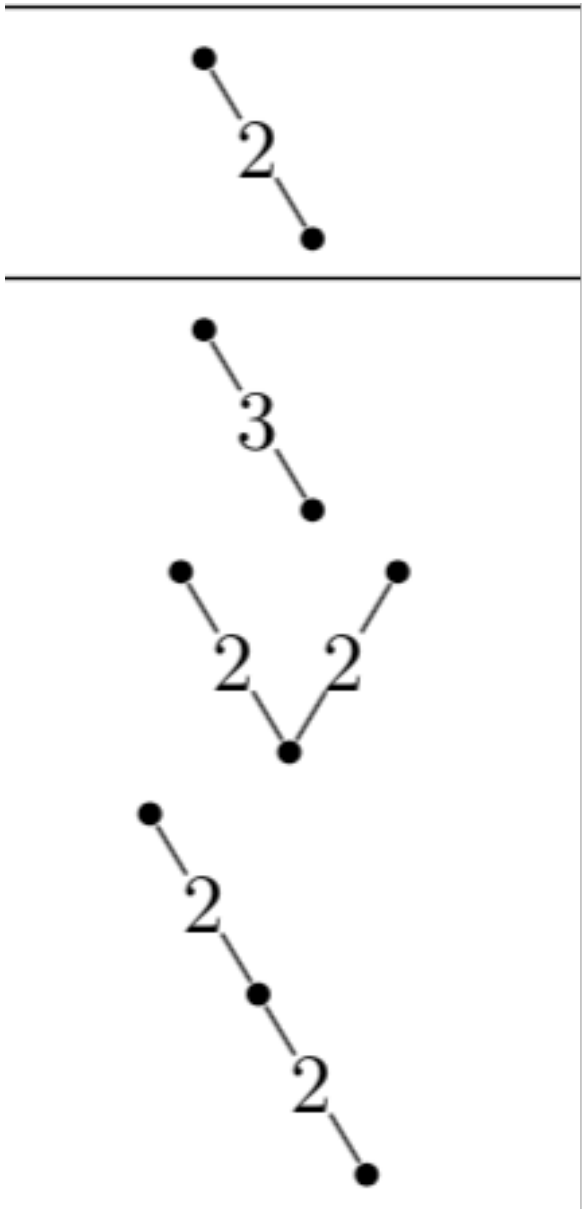




For $d \geqslant 16$, these regions overlap with the regions where [Hammond 2015] proves infinite loops $\Rightarrow$ unique phase transition.

Expanding β_c in powers of $1/d$.

- Put $\beta = \alpha/d$ and sort the elements of C by their contribution in the limit $d \to \infty$ at fixed $\alpha \approx 1$.



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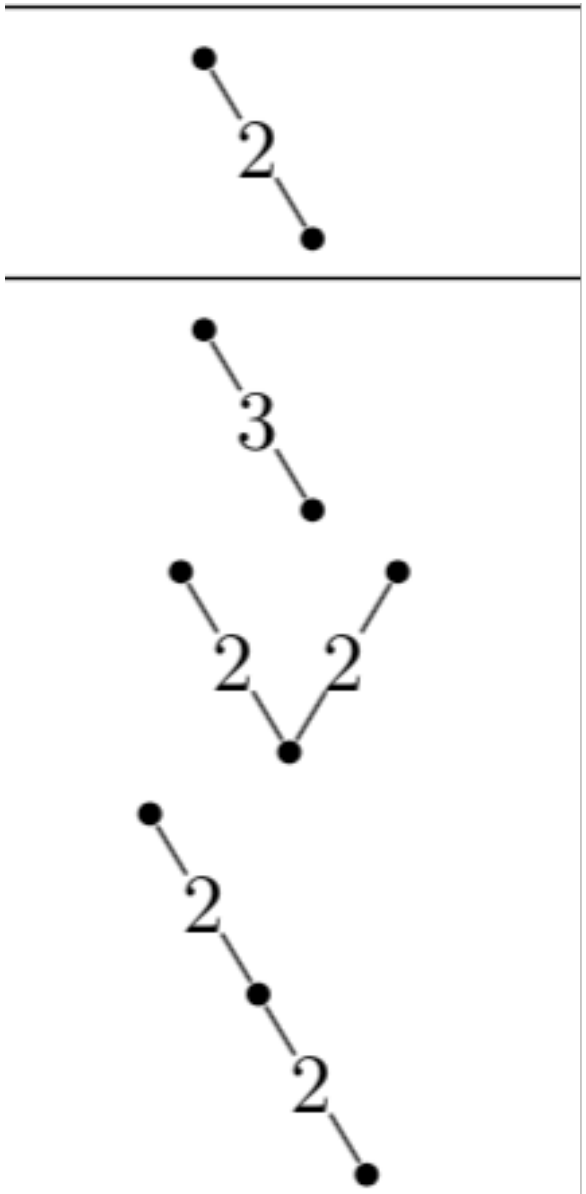
from below: sum only up to order N.



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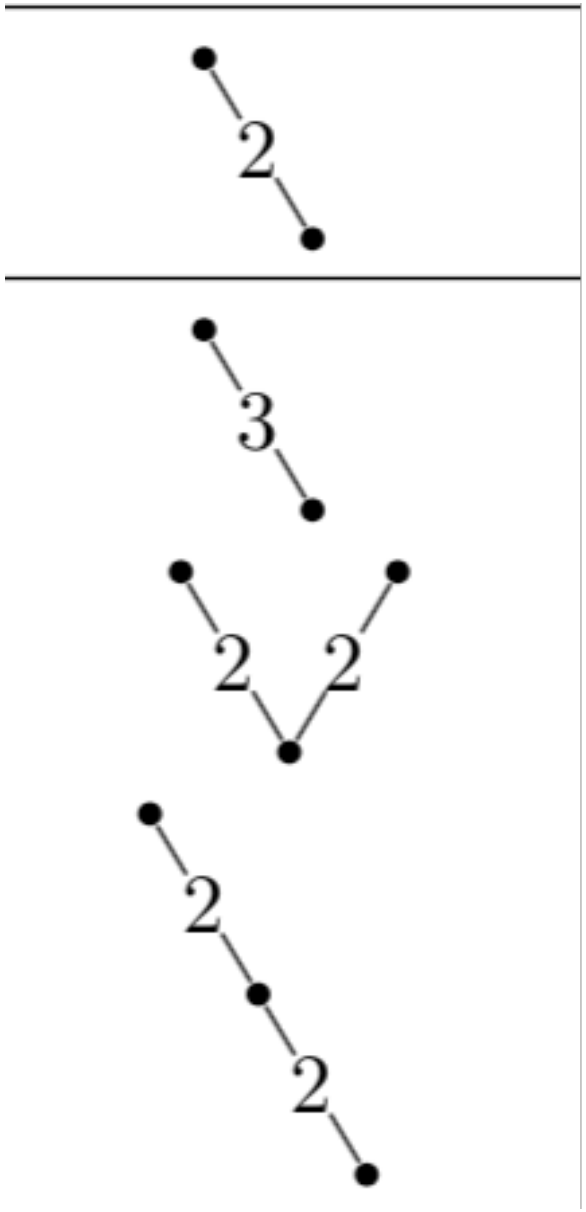

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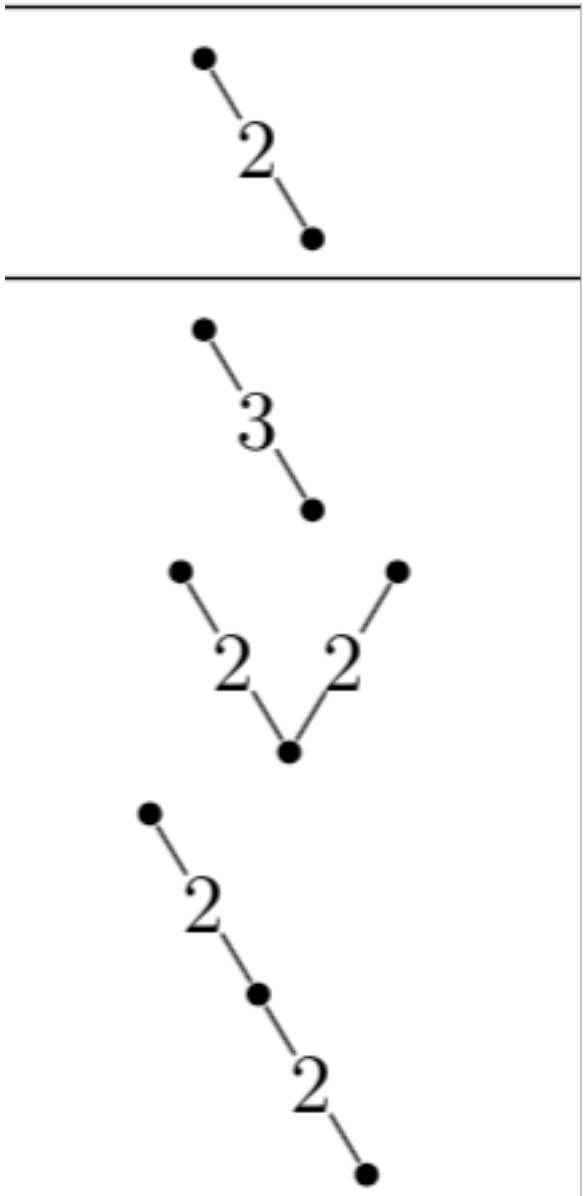

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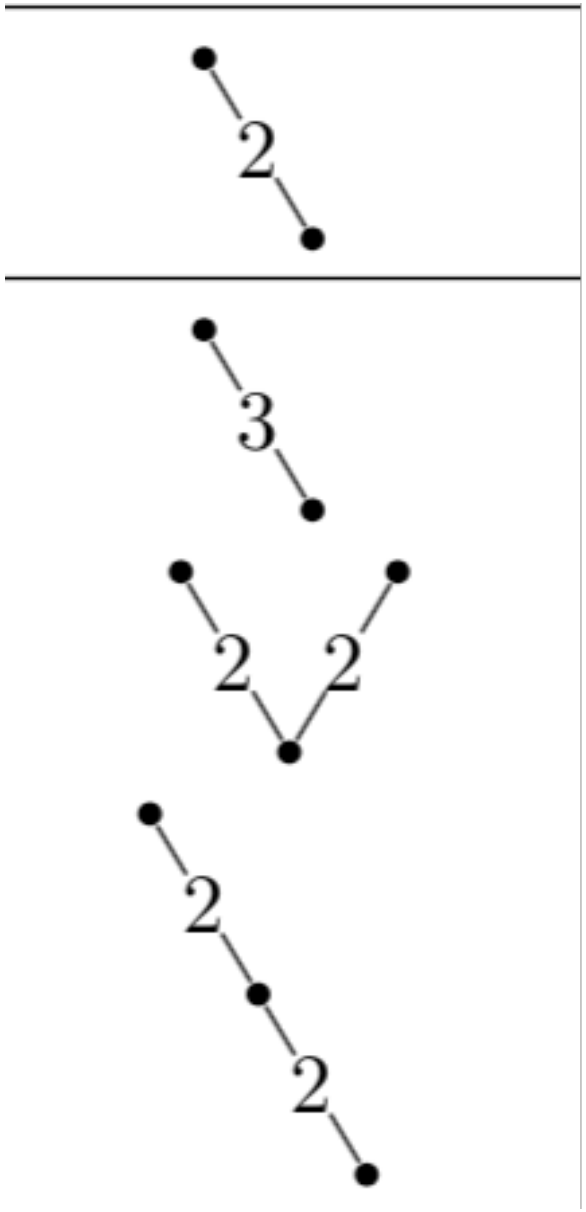

An intriguing observation

We calculated β_c up to sixth order. The result is

$$\beta_c(u,d) = \sum_{k=0}^{5} \frac{\alpha_k(u)}{d^{k+1}} + \mathcal{O}(d^{-7}),$$

with $\alpha_k(u) = \sum_{j=0}^{2k} \alpha_{k,j} \binom{2k}{j} u^j (1-u)^{2k-j}$, and

$\alpha_{k,j}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
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So, in the Bernstein basis, all coefficients seem to be in $[0,1]$. We have no idea why this is so and whether it persists.

Conclusion

- For the d-regular tree, the loop model with $\theta = 1$ is now pretty well understood.
- The case $\theta > 1$ can't use renewal theory directly, but is not hopeless $\rightarrow$ work in progress.
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Thank you for your attention!