Canonical DLR equations for the Sine-beta process

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For $\beta > 0$, $n \geq 1$ and $\Lambda_n = [-n/2, n/2]$, we define on Λ_n^n the probability measure

$$\mathbb{P}_n^\beta(dx_1, dx_2, \dots, dx_n) = \frac{1}{Z_n} e^{-\beta H(\{x_1, x_2, \dots, x_n\})} dx_1 dx_2 \dots dx_n.$$

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We define P_n^β its associated point process; the distribution of the random configuration

$$\{X_1, X_2, \dots, X_n\},$$

where $(X_1, X_2, \dots, X_n) \sim \mathbb{P}_n^\beta$.

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The following limit exists

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Proved by: [Valko-Virag 2009], [Bourgade, Erdös, Yau 2012]

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- Distribution of eigenvalues for random matrices:
  - ($\beta = 1$) with Gaussian real input
  - ($\beta = 2$) with Gaussian complex input
  - ($\beta = 4$) with Gaussian Quaternion input
- Universal with respect to the boundary conditions (β-ensembles, β-Circular-ensembles).

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- Determinantal structure for $\beta = 2$.
- Explicit pair correlation functions for $\beta \in \mathbb{Q}^+$ (with very complicated formulas [Forrester ch13])
- Hyperuniform: $\mathrm{Var}_{P^\beta}(N_\Lambda) \sim \log(|\Lambda|)$.
- Variational principle and LDP [Leblé-Serfaty 19]: P^β is the unique minimizer of the free excess energy=Entropy+mean Energy

The DLR equations setting

For $\Lambda \subset \mathbb{R}$:

- (as usual) the local distributions are not tractable;

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$$P^\beta\Big(d\gamma_\Lambda\Big|\gamma_{\Lambda^c}\Big)?$$

- the canonical DLR equations provide the local conditional densities of P^β given the number of points;

$$P^\beta\Big(d\gamma_\Lambda\Big|\gamma_{\Lambda^c}, \#\gamma_\Lambda = k\Big)?$$

What is the energy of a point?

For any stationary point process P, for any $x \in \mathbb{R}$ and for P a.e. γ

$$\sum_{y \in \gamma} -\log(|x-y|) = -\infty.$$

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For P^β-almost all γ and $x \in \mathbb{R}$, the following limit exists

$$V(x,\gamma) = \sum_{y \in \gamma} \log(|y|) - \log(|x-y|).$$

It is the Log-energy we need for moving a particle from 0 to x in γ.

Our canonical DLR equations

Theorem (D.-Hardy-Leblé-Maïda, 2019)

For any bounded $\Lambda \subset \mathbb{R}$, *any* $k \geq 0$ *and* P^β*-almost all* γ

$$P^\beta(dx_1, \dots, dx_k | \gamma_{\Lambda^c}, N_\Lambda = k) = \frac{1}{Z_\Lambda(k, \gamma_{\Lambda^c})} e^{-\beta H(\{x_1, x_2, \dots, x_k\})} \prod_{i=1}^{k} e^{-\beta V(x_i, \gamma_{\Lambda^c})} dx_1^\Lambda dx_2^\Lambda \dots dx_k^\Lambda$$

Locally, the Sine-beta process is a β-ensemble with confining potential V.

A canonical GNZ (Georgii-Nguyen-Zessin) equation

Theorem

There exists a probability mesure Q^β such that for any positive bounded function f

$$E_{P^\beta}\left(\sum_{x\in\gamma} f(x,\gamma\backslash x)\right) = E_{Q^\beta}\left(\int_{\mathbb{R}} e^{-\beta V(x,\gamma)} f(x,\gamma)dx\right)$$

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Are Q^β and P^β equivalent? Are they singular?
If $Q^\beta \ll P^\beta$, a standard GNZ equation exists.

Number-Rigidity

Definition

A point process P in $\mathbb{R}^d$ is said Number-Rigid if for any bounded set $\Lambda \subset \mathbb{R}^d$, there exists a function f_Λ such that for P-a.e. γ

$$N_\Lambda(\gamma) = f_\Lambda(\gamma_{\Lambda^c}).$$

The outside configuration determines the number of points inside Λ.

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Other kind of XXXXXX-rigidity can be considered as well.

Sly-Peres Theorem

The Gaussian perturbed lattice is defined by:

$$\Gamma = \bigcup_{z \in \mathbb{Z}^d} \{z + X_z\},$$

where $(X_z)_{z \in \mathbb{Z}^d}$ are i.i.d $\mathcal{N}(0, \sigma^2)$.

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Theorem (Sly-Peres 2014)

In dimension $d = 1, 2$ the Gaussian perturbed lattice is Number-Rigid. For any $d \geq 3$, there exists $\sigma_d > 0$ such that

- *for $\sigma < \sigma_d$, the Gaussian perturbed lattice is number-rigid.*
- *for $\sigma > \sigma_d$, the Gaussian perturbed lattice is not number-rigid.*

Number-Rigidity criterion

Theorem (Ghosh-Peres, 2012)

Let P be a point process. Assume that for any bounded $\Lambda \subset \mathbb{R}^d$ and any $\varepsilon > 0$, there exists a function $f : \mathbb{R}^d \to \mathbb{R}$ with compact support such that

- $f(x) = 1$ *for all* $x \in \Lambda$
- $Var_P(\sum_{x \in \Gamma} f(x)) \leq \varepsilon$.

Then P is Number-Rigid.

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Applications: Theorem by Ghosh-Peres, the determinantal Ginibre process is Number-Rigid, etc...

Number-Rigidity of the Sine-beta Process

Theorem (D.-Hardy-Maïda-Leblé 2019)

For any $\beta > 0$, the Sine-beta process P^{β} is Number-Rigid and any other rigidity does not occur.

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-Chaïbi and Najnudel proved simultaneously the Number-Rigidity of the Sine-beta process with a completely different proof.
-Our proof does not use the Ghosh-Peres Criterion. It is a first step toward the general Coulomb case.
- Our proof is based on the canonical GNZ equation. We show that Q^β and P^β are singular.

DLR equations for the Sine-beta Process

Theorem (D.-Hardy-Maïda-Leblé 2019)

For any bounded $\Lambda \subset \mathbb{R}$ and P^β-almost all γ

$$\begin{aligned} P^\beta(d\gamma_\Lambda|\gamma_{\Lambda^c}) &= P^\beta(d\gamma_\Lambda|\gamma_{\Lambda^c}, N_\Lambda = f_\Lambda(\gamma_{\Lambda^c})) \\ &= \frac{1}{Z_\Lambda(f_\Lambda(\gamma_{\Lambda^c}), \gamma_{\Lambda^c})} e^{-\beta H(\{x_1, x_2, \ldots, x_{f_\Lambda(\gamma_{\Lambda^c})}\})} \\ & \quad \prod_{i=1}^{f_\Lambda(\gamma_{\Lambda^c})} e^{-\beta V(x_i, \gamma_{\Lambda^c})} dx_1^\Lambda dx_2^\Lambda \ldots dx_{f_\Lambda(\gamma_{\Lambda^c})}^\Lambda \end{aligned}$$

Unfortunately, the function $f_\Lambda(\gamma_{\Lambda^c})$ is not tractable.

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The centred linear statistic of φ at the scale $n \geq 1$ is defined by

$$\begin{aligned} \mathcal{L}^n(\varphi) &= \sum_{x\in\gamma} \varphi(x/n) - E_P\left(\sum_{x\in\gamma} \varphi(x/n)\right) \\ &= \sum_{x\in\gamma} \varphi(x/n) - \int_{\mathbb{R}} \varphi(x/n)\, dx \end{aligned}$$

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Typical CLT: there exists a sequence $(\alpha_n)_{n\geq 1}$ and $\sigma(\varphi) > 0$ such that

$$\alpha_n \mathcal{L}^n(\varphi) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0, \sigma(\varphi)^2).$$

Poisson case: $\alpha_n = n^{-1/2}$ and $\sigma(\varphi)^2 = \int_{\mathbb{R}} \varphi^2(x)\,dx$.

CLT for the Sine-beta process

Theorem (Leblé 2019)

The Sine-beta process P^β satisfied the CLT for any $\varphi \in \mathcal{C}_c^4(\mathbb{R},\mathbb{R})$

$$\mathcal{L}^n(\varphi) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,\sigma(\varphi)^2),$$

with

$$\sigma(\varphi)^2 = \frac{2}{\beta}\int\int\left(\frac{\varphi(x)-\varphi(y)}{x-y}\right)^2 dxdy.$$

Recall: $\mathcal{L}^n(\varphi) = \sum_{x\in\gamma}\varphi(x/n) - \int_{\mathbb{R}}\varphi(x/n)\,dx$.
The DLR equations furnish a central description in the proof, based on a fine analysis of the Laplace transforms.