The peak model for the triplet extensions and their transformations to the reference Hilbert space

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Outline

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- Triplet adjoint in the reference Hilbert space. Transformations preserving the Weyl function

Classical theory for supersingular perturbations

Definitions

- $\cdots \subset \mathfrak{H}_2 \subset \mathfrak{H}_1 \subset \mathfrak{H}_0 \subset \mathfrak{H}_{-1} \subset \mathfrak{H}_{-2} \subset \cdots$ the scale of Hilbert spaces $(\mathfrak{H}_n, \langle\cdot,\cdot\rangle_n)_{n\in\mathbb{Z}}$ associated with a (semibounded) self-adjoint operator $L\colon \mathfrak{H}_2 \to \mathfrak{H}_0$.
- $L_n := L\,|_{\,\mathfrak{H}_{n+2}}$ (self-adjoint in $\mathfrak{H}_n$).
- $\varphi_\sigma \in \mathfrak{H}_{-m-2} \setminus \mathfrak{H}_{-m-1}$ $(m \in \mathbb{N}_0)$; $\sigma \in \mathcal{S}$; $\#\mathcal{S} = d \in \mathbb{N}$.
- $L_{\min} := L\,|_{\,\{f\in\mathfrak{H}_{m+2}\,|\,\langle\varphi,f\rangle=0\}}$ densely def., closed, symm. in $\mathfrak{H}_m$, d.i. (d,d).

It follows that $L_{\min}$ is *esa* in $\mathfrak{H}_0$ (!). Adjoint $L_{\min}^* \supseteq L_m$ with $\operatorname{dom} L_{\min}^* = \mathfrak{H}_{m+2} \dotplus \mathfrak{N}_z(L_{\min}^*)$. Eigenspace $\mathfrak{N}_z(L_{\min}^*) = \operatorname{span}\{G_\sigma(z)\} =: G_z(\mathbb{C}^d)$, $z \in \operatorname{res} L$, $G_\sigma(z) := P(L)^{-1} g_\sigma(z)$, $g_\sigma(z) := (L-z)^{-1}\varphi_\sigma \in \mathfrak{H}_{-m} \setminus \mathfrak{H}_{-m+1}$; *e.g.* $P(L) := \prod_j (L - z_j)$ for $z_j \in \operatorname{res} L \cap \mathbb{R}$, $j \in J := \{1,\dots,m\}$.

Theorem

Let

$$\Gamma_0(f^\# + G_z(c)) := c\,, \quad \Gamma_1(f^\# + G_z(c)) := \langle\varphi, f^\#\rangle + R(z)c$$

for $f^\# \in \mathfrak{H}_{m+2}$, $c \in \mathbb{C}^d$, some Nevanlinna R. The triple $(\mathbb{C}^d, \Gamma_0, \Gamma_1)$ is an OBT for $L_{\min}^*$. The associated γ-field and the Weyl function are

$$\gamma(z) = G_z(\cdot)\,, \quad M(z) = R(z)\,, \quad z \in \operatorname{res} L\,.$$

Triplet adjoint in scale spaces. A general definition

Definition

Consider the Hilbert triple $\mathfrak{h} \subseteq \mathfrak{k} \subseteq \mathfrak{h}'$, with both inclusions being dense. Here $\mathfrak{h}'$ is the dual of $\mathfrak{h}$, *i.e.* an element of $\mathfrak{h}'$ is a continuous linear functional on $\mathfrak{h}$ whose action is defined via the duality pairing $\langle\cdot,\cdot\rangle : \mathfrak{h}' \times \mathfrak{h} \to \mathbb{C}$. The duality pairing is defined by extending the scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ in $\mathfrak{k}$ so that $\langle g, f\rangle$ is well-defined $\forall g \in \mathfrak{h}'$ $\forall f \in \mathfrak{h}$. Let T be a densely defined operator in $\mathfrak{h}$. Then there exists the unique operator $T^\dagger$ in $\mathfrak{h}'$, called the *triplet adjoint* of T, defined by

$$\mathrm{dom}\, T^\dagger := \{g \in \mathfrak{h}' \,|\, (\forall f \in \mathrm{dom}\, T)(\exists g' \in \mathfrak{h}')\, \langle g, Tf\rangle = \langle g', f\rangle\}\,.$$

When such a $g' \in \mathfrak{h}'$ exists, it is unique and denoted by $T^\dagger g$.

The duality pairing $\langle\cdot,\cdot\rangle : \mathfrak{H}_{-m} \times \mathfrak{H}_m \to \mathbb{C}$ $(m \in \mathbb{N}_0)$ for the Hilbert triple $\mathfrak{H}_m \subseteq \mathfrak{H}_0 \subseteq \mathfrak{H}_{-m}$ is defined by

$$\langle g, f\rangle := \langle P(L)^{-1/2} g, P(L)^{1/2} f\rangle_0$$

$\forall f \in \mathfrak{H}_m$ $\forall g \in \mathfrak{H}_{-m}$.

Triplet adjoint in scale spaces

Proposition

The triplet adjoint $L_{\max} := L^{\dagger}_{\min}$ *for the Hilbert triple* $\mathfrak{H}_m \subseteq \mathfrak{H}_0 \subseteq \mathfrak{H}_{-m}$ *is a densely defined, closed, non-symmetric operator in* $\mathfrak{H}_{-m}$ *given by*

$$L_{\max} = P(L_{-m}) L^{*}_{\min} P(L_{-m})^{-1} \supseteq L^{*}_{\min} \,.$$

(Note: $P(L_{n-m})\mathfrak{H}_{n+m} = \mathfrak{H}_{n-m}$)

Corollary (For $d = 1$, see Dijksma et al '05; Kurasov '03, '09)

The operator $L_{\max} \supseteq L_{-m}$ *extends* L_{-m} *to the domain*

$$\operatorname{dom} L_{\max} = \mathfrak{H}_{-m+2} \dotplus \mathfrak{N}_z(L_{\max}) \,, \quad z \in \operatorname{res} L$$

with the eigenspace

$$\mathfrak{N}_z(L_{\max}) = \operatorname{span}\{g_\sigma(z) := (L_{-m-2} - z)^{-1}\varphi_\sigma\} =: g_z(\mathbb{C}^d) \,.$$

Corollary

$L_{\max} = P(L_{-m})^{1/2} \widehat{L}^{*}_0 P(L_{-m})^{-1/2}$ *where* $\widehat{L}^{*}_0$ *is the adjoint in* $\mathfrak{H}_0$ *of* $\widehat{L}_0 := L\,|_{\{u \in \mathfrak{H}_2 \,|\, \langle \widehat{\varphi}, u \rangle = 0\}}$; $\widehat{\varphi}_\sigma := P(L_{-m-2})^{-1/2}\varphi_\sigma \in \mathfrak{H}_{-2} \setminus \mathfrak{H}_{-1}$.

(Note: $P(L_n)^{1/2}\mathfrak{H}_{n+m} = \mathfrak{H}_n$)

Intermediate Hilbert space. Singular elements

Definition (Peak model)

Linear space $\mathfrak{K} := \mathrm{span}\{g_\alpha := g_\sigma(z_j) \in \mathfrak{H}_{-m} \setminus \mathfrak{H}_{-m+1} \,|\, \alpha = (\sigma, j) \in \mathcal{S} \times J\}$.
An element $k \in \mathfrak{K}$ is of the form $k = \sum_\alpha d_\alpha(k) g_\alpha$ with $\mathfrak{K} \ni k \leftrightarrow d(k) \in \mathbb{C}^{md}$ because the Gram matrix $\mathcal{G} = (\langle g_\alpha, g_{\alpha'} \rangle_{-m}) > 0$.

Lemma

$\mathfrak{K}_{\mathrm{min}} \subseteq \mathfrak{K} \subseteq \mathfrak{H}_{-m}$ *where*

$$\mathfrak{K}_{\mathrm{min}} := (\mathfrak{K} \cap \mathfrak{H}_{m-2}) \setminus \mathfrak{H}_{m-1} = \mathrm{span}\{P(L_{-m-2})^{-1} \varphi_\sigma\}\,.$$

An element $k \in \mathfrak{K}_{\mathrm{min}}$ is of the form $k = k_{\mathrm{min}}(c) := \sum_\sigma c_\sigma P(L_{-m-2})^{-1} \varphi_\sigma$ for $c = (c_\sigma) \in \mathbb{C}^d$.

Definition (Cascade (A) model, see Dijksma et al '05 for $d = 1$)

Linear space
$\mathfrak{K}_{\mathrm{A}} := \mathrm{span}\{h_\alpha := [(L - z_1) \cdots (L - z_j)]^{-1} \varphi_\sigma \in \mathfrak{H}_{-m-2+2j} \setminus \mathfrak{H}_{-m-1+2j} \,|\, j \in J\}$.

An element $k \in \mathfrak{K}_{\mathrm{A}}$ is of the form $k = \sum_\alpha d_\alpha(k) h_\alpha$ with
$\mathfrak{K}_{\mathrm{A}} \ni k \leftrightarrow d(k) \in \mathbb{C}^{md}$ because the Gram matrix $\widetilde{\mathcal{G}}_{\mathrm{A}} = (\langle h_\alpha, h_{\alpha'} \rangle_{-m}) > 0$.

Intermediate Hilbert space

Definition (Peak model)

Define the vector space $\mathcal{H} := \mathfrak{H}_m \dot{+} \mathfrak{K}$. The space $\mathcal{H}$ is made into the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ by completing $\mathcal{H}$ with respect to the norm $\|\cdot\|_{\mathcal{H}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathcal{H}}}$, where the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ in $\mathcal{H}$ is defined by

$$\langle f + k, f' + k'\rangle_{\mathcal{H}} := \langle f, f'\rangle_m + \langle k, k'\rangle_{-m}$$

for $f, f' \in \mathfrak{H}_m$ and $k, k' \in \mathfrak{K}$.

The Hilbert space $\mathcal{H}$ isometrically isomorphic to $\mathcal{H}' := (\mathfrak{H}_m \oplus \mathbb{C}^{md}, \langle\cdot,\cdot\rangle_{\mathcal{H}'})$ where

$$\langle (f,\xi), (f',\xi')\rangle_{\mathcal{H}'} := \langle f, f'\rangle_m + \langle \xi, \mathcal{G}\xi'\rangle_{\mathbb{C}^{md}}$$

for $(f,\xi), (f',\xi') \in \mathfrak{H}_m \oplus \mathbb{C}^{md}$. To see this, take $\xi = d(k)$ and $\xi' = d(k')$.

Definition (Cascade (A) model)

Define a linear space $\mathcal{H}_{\mathrm{A}} := \mathfrak{H}_m \dot{+} \mathfrak{K}_{\mathrm{A}}$ with an indefinite metric

$$[f + k, f' + k']_{\mathrm{A}} := \langle f, f'\rangle_m + \langle d(k), \mathcal{G}_{\mathrm{A}} d(k')\rangle_{\mathbb{C}^{md}}$$

for $f, f' \in \mathfrak{H}_m$ and $k, k' \in \mathfrak{K}_{\mathrm{A}}$. An (undefined) Hermitian matrix $\mathcal{G}_{\mathrm{A}}$ is the Gram matrix of the A-model. Thus $\mathcal{H}_{\mathrm{A}}$ is a Hilbert space if $\mathcal{G}_{\mathrm{A}} \geq 0$ and a Pontryagin space otherwise.

Triplet adjoint in an intermediate Hilbert space. Closed restriction

Theorem

Let A_0 be (the graph of) the operator in $\mathcal{H}$ defined by

$$A_0 := \{(f^{\#} + k, L_m f^{\#} + \sum_{\alpha} [Z_d d(k)]_{\alpha} g_{\alpha}) \mid f^{\#} \in \mathfrak{H}_{m+2}\,;\ k \in \mathfrak{K}\}$$

where $Z_d := Z \oplus \cdots \oplus Z$ (d times) is the matrix direct sum of d diagonal matrices $Z := \mathrm{diag}\{z_j; j \in J\}$. Then:

- *A_0 is densely defined, closed, and, in general, non-symmetric operator in $\mathcal{H}$, whose adjoint in $\mathcal{H}$ is the operator given by*

  $$A_0^* = \{(f^{\#} + k, L_m f^{\#} + \sum_{\alpha} [\mathcal{G}^{-1} \mathcal{G}_Z^* d(k)]_{\alpha} g_{\alpha}) \mid f^{\#} \in \mathfrak{H}_{m+2}\,;\ k \in \mathfrak{K}\}$$

  with the matrix $\mathcal{G}_Z := \mathcal{G} Z_d$. It is symmetric, and hence self-adjoint, iff $\mathcal{G}_Z$ is Hermitian.

- *The resolvent*

  $$(A_0 - z)^{-1} = U^* [(L_m - z)^{-1} \oplus (Z_d - z)^{-1}] U$$

  for $z \in \operatorname{res} A_0 = \operatorname{res} L \setminus \{z_j \mid j \in J\}$.

Triplet adjoint in an intermediate Hilbert space. Closable restriction

Theorem

The restriction $A_{\max}$ to $\mathcal{H}$ of $L_{\max}$ is an extension of A_0 to the domain $\operatorname{dom} A_{\max} = \operatorname{dom} A_0 \dot{+} \mathfrak{N}_z(A_{\max})$ *where the eigenspace* $\mathfrak{N}_z(A_{\max}) = \mathfrak{N}_z(L_{\max})$ *with* $z \in \operatorname{res} A_0$. *The operator $A_{\max}$ is not closed, in general, but it is closable, and the closure is given by* $A_{\max}^{**} = A_{\min}^{*}$, *where* $A_{\min} := A_{\max}^{*} \subseteq A_0^{*}$ *with*

$$\operatorname{dom} A_{\min} = \{f^{\#} + k \in \mathfrak{H}_{m+2} \dot{+} \mathfrak{K} \,|\, \langle \varphi, f^{\#} \rangle = \mathcal{G}_b^{*} d(k)\}$$

and where $A_{\min}^{*} \supseteq A_0$ *with* $\operatorname{dom} A_{\min}^{*} = \operatorname{dom} A_0 \dot{+} \mathfrak{N}_z(A_{\min}^{*})$, $z \in \operatorname{res} A_0$; *the eigenspace*

$$\mathfrak{N}_z(A_{\min}^{*}) = \operatorname{span}\{F_\sigma(z) + \sum_{\alpha'} [\Lambda(z)]_{\alpha'\sigma} g_{\alpha'}\}\,, \quad F_\sigma(z) := P(z)^{-1} g_\sigma(z)\,.$$

The above matrices $\mathcal{G}_b, \Lambda(z) \in [\mathbb{C}^d, \mathbb{C}^{md}]$ are defined by

$$[\mathcal{G}_b]_{\alpha\sigma'} := \sum_j \mathcal{G}_{\alpha,\sigma' j} b_j(z_j)^{-1}\,, \quad b_j(\cdot) := \prod_{j' \in \{1,\dots,j-1,j+1,\dots,m\}} (\cdot - z_{j'})$$

and

$$\Lambda(z) := [(z\mathcal{G} - \mathcal{G}_Z^{*})^{-1} - (z\mathcal{G} - \mathcal{G}_Z)^{-1}]\mathcal{G}_b\,.$$

Triplet adjoint in an intermediate Hilbert space. Boundary triple

Corollary

$\mathrm{dom}\, A_{\max} = \mathrm{dom}\, L^*_{\min} \dot{+} \mathfrak{K}$.

Corollary

$A_{\max}$ is closed in $\mathcal{H}$, i.e. $A^*_{\min} = A_{\max}$, iff $\mathcal{G}_Z$ is Hermitian.

Theorem

Let

$$\widetilde{\Gamma}_0(f^\# + G_z(c) + k) := c\,, \quad \widetilde{\Gamma}_1(f^\# + G_z(c) + k) := \langle \varphi, f^\# \rangle + R(z)c - \mathcal{G}_b^* d(k)$$

for $f^\# \in \mathfrak{H}_{m+2}$, $c = (c_\sigma) \in \mathbb{C}^d$, $k \in \mathfrak{K}$, some Nevanlinna R. For an Hermitian $\mathcal{G}_Z$, the triple $(\mathbb{C}^d, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1)$ is an OBT for the adjoint $A^*_{\min}$ of a densely defined, closed, symmetric operator $A_{\min}$ in $\mathcal{H}$. The associated γ-field and the Weyl function are

$$\widetilde{\gamma}(z)c = \sum_\sigma c_\sigma F_\sigma(z)\,, \quad \widetilde{M}(z) = R(z) + Q_{\mathcal{G}}(z)\,, \quad z \in \mathrm{res}\, A_0\,,$$

$$[Q_{\mathcal{G}}(z)]_{\sigma\sigma'} := \sum_j \frac{[\mathcal{G}_b^*]_{\sigma,\sigma' j}}{(z_j - z) b_j(z_j)} = \sum_j \frac{\mathcal{G}_{\sigma j,\sigma' j}}{(z_j - z) b_j(z_j)^2}\,.$$

Example and Remark

Corollary (cf. Kurasov '09 for Θ real)

Let $d = 1$ and let $\mathcal{G}Z = Z\mathcal{G}$; then

$$(A_\Theta - z)^{-1} = (A_0 - z)^{-1} + \frac{g(z)U^*[\langle G(\overline{z}), \cdot\rangle_m \oplus \langle b, \mathcal{G}(z-Z)^{-1}\cdot\rangle_{\mathbb{C}^m}]U}{P(z)[\Theta - R(z) + \langle b, \mathcal{G}(z-Z)^{-1}b\rangle_{\mathbb{C}^m}]}$$

for $z \in \operatorname{res} A_0 \cap \operatorname{res} A_\Theta$ and $\Theta \in \mathbb{C} \cup \{\infty\}$; $b := (b_j(z_j)^{-1}) \in \mathbb{C}^m$.

Remark

The Krein Q-function $\widetilde{Q}(z) := Q(z) + Q_\mathcal{G}(z)$, where $Q(z) := R(z) - R(z_0)$, fixed $z_0 \in \operatorname{res} L$. Both, Q and $Q_\mathcal{G}$, belong to Nevanlinna class. In contrast, with a suitable choice of model parameters, the Q-function in the A-model is $Q_\mathrm{A}(z) = Q(z) + r(z)$, where the generalized Nevanlinna function

$$r(z) := -\sum_{j=1}^{m} \frac{\alpha_{jm}}{(z - z_1)^{m-j+1}}\,, \quad [\alpha_{jm}]_{\sigma\sigma'} := [\mathcal{G}_\mathrm{A}]_{\sigma m, \sigma' j}\,.$$

Now, formally ignore that $\mathcal{G}$ is diagonal in $j \in J$, put $z_j = z_1 - \delta_{j-1}$ for $j \in \{2, 3, \ldots, m\}$ $(m \geq 2)$, and take the limits $\delta_j \to \delta_{j-1}$ and $\delta := \delta_1 \to 0$; then

$$Q_\mathcal{G}(z) \sim \widetilde{r}(z) + O(\delta)\,, \quad \widetilde{r}(z) := -\sum_{j=1}^{m} \frac{\widetilde{\alpha}_{jm}}{(z - z_1)^{m-j+1}}\,, \quad [\widetilde{\alpha}_{jm}]_{\sigma\sigma'} := [\widetilde{\mathcal{G}}_\mathrm{A}]_{\sigma m, \sigma' j}\,.$$

Triplet adjoint in the reference Hilbert space. Principal goals

The goal is to construct the operator in $\mathfrak{H}_0$ whose Weyl function is that of A_Θ in $\mathcal{H}$. By Langer–Textorius: If Q-functions of two simple, closed, densely defined, symmetric operators coincide, then the operators are unitarily equivalent.

Definition

Consider an operator $P_{\mathcal{H}} \in [\mathfrak{H}_{-m}, \mathcal{H}]$ ($m \in \mathbb{N}$) and let $P^*_{\mathcal{H}} \in [\mathcal{H}, \mathfrak{H}_{-m}]$ be its adjoint. Define another bounded operator $\Omega := P_{\mathcal{H}} P(L_{-m})^{1/2} \colon \mathfrak{H}_0 \to \mathcal{H}$, whose adjoint $\Omega^* = P(L_{-m})^{-1/2} P^*_{\mathcal{H}} \colon \mathcal{H} \to \mathfrak{H}_0$.

Definition

Let Θ be a linear relation in $\mathbb{C}^d$ and let $\mathcal{G}_Z$ be Hermitian. Define the operator in $\mathfrak{H}_0$ by $\widehat{A}^\Omega_\Theta := \Omega^* A_\Theta \Omega$, and in particular put $\widehat{A}^\Omega_{\mathrm{min}} := \Omega^* A_{\mathrm{min}} \Omega$ and $\widehat{A}^\Omega_{\mathrm{max}} := \Omega^* A_{\mathrm{max}} \Omega$ and $\widehat{A}^\Omega_0 := \Omega^* A_0 \Omega$.

Theorem

The adjoint in $\mathfrak{H}_0$ is the operator $(\widehat{A}^\Omega_\Theta)^* = \widehat{A}^\Omega_{\Theta^*}$. Moreover, $\widehat{A}^\Omega_{\mathrm{min}} \subseteq \widehat{A}^\Omega_\Theta \subseteq \widehat{A}^\Omega_{\mathrm{max}}$. That is, $\widehat{A}^\Omega_\Theta$ is a proper extension of a densely defined, closed, and symmetric operator $\widehat{A}^\Omega_{\mathrm{min}}$ in $\mathfrak{H}_0$, whose adjoint in $\mathfrak{H}_0$ is the operator $\widehat{A}^\Omega_{\mathrm{max}}$.

We construct conditions, under which Ω is not necessarily unitary, and yet Ω preserves $\widetilde{M}$.

Triplet adjoint in the reference Hilbert space. Eigenspace

Definition

Let $\iota := \Omega\Omega^* = P_{\mathcal{H}} P_{\mathcal{H}}^* \in [\mathcal{H}]$, so that $\iota = |P_{\mathcal{H}}^*|^2 > 0$ is a bounded, positive, self-adjoint operator in $\mathcal{H}$.

Lemma

$\Omega\mathfrak{N}_z(\widehat{A}_{\Theta}^{\Omega}) \subseteq \mathfrak{N}_z(\iota A_{\Theta})$. If in particular, $P_{\mathcal{H}}$ is invertible and leaves $\operatorname{dom} A_{\Theta}$ *invariant, for some Θ, then $P_{\mathcal{H}}$ (and hence Ω) is unitary, and the inclusion is the equality.*

Lemma

Consider an element $f = f^{\#} + \gamma(z)c + k \in \operatorname{dom} A_{\max}$; $f^{\#} \in \mathfrak{H}_{m+2}$, $k \in \mathfrak{K}$, $c \in \mathbb{C}^d$. Then $f \in \mathfrak{N}_z(\iota A_{\max})$ for $z \in \Sigma_\iota := \operatorname{res} A_0 \cap \operatorname{res}(\iota A_0)$ iff $f = H_z(c)$, where

$$H_z(c) := [I - z(\iota A_0 - z)^{-1}(\iota - I)]\gamma(z)c - (\iota A_0 - z)^{-1}\iota k_{\min}(c)\,.$$

In particular, $f^{\#} = 0$ iff $(\forall z \in \Sigma_\iota)(\exists n \in \mathbb{Z}_{\leq m+2})\, H_z(\mathbb{C}^d) \cap \mathfrak{H}_n = \{0\}$.

(For $\iota = I$, $H_z(c) = \widetilde{\gamma}(z)c$ and $f^{\#} = 0$)

Triplet adjoint in the reference Hilbert space. Boundary triple

Theorem

Assume that $P_{\mathcal{H}}$ leaves $\operatorname{dom} A_{\max}$ *invariant. Then the triple* $(\mathbb{C}^d, \widehat{\Gamma}_0^{\Omega}, \widehat{\Gamma}_1^{\Omega})$ *is an OBT for* $\widehat{A}_{\max}^{\Omega}$*; here* $\widehat{\Gamma}_0^{\Omega} := \widetilde{\Gamma}_0 \Omega$ *and* $\widehat{\Gamma}_1^{\Omega} := \widetilde{\Gamma}_1 \Omega$ *are single-valued surjective operators from* $\operatorname{dom} \widehat{A}_{\max}^{\Omega}$ *onto* $\mathbb{C}^d$*. The (graph of the) associated γ-field is given by*

$$\widehat{\gamma}^{\Omega}(z) = \{(c, u) \in \mathbb{C}^d \times \mathfrak{N}_z(\widehat{A}_{\max}^{\Omega}) \,|\, \Omega u = H_z(c)\}$$

and the Weyl function is

$$\widehat{M}^{\Omega}(z) = R(z) - \widetilde{\Gamma}_1[(\iota A_0 - z)^{-1}((\iota - I)z\gamma(z) + \iota k_{\min}(\cdot))]$$

on $\mathbb{C}^d$*, with $z \in \Sigma_\iota$. Moreover, the operator* $\widehat{A}_{\Theta}^{\Omega}$ *in $\mathfrak{H}_0$ corresponds to the operator ιA_{Θ} in $\mathcal{H}$ in the sense that*

$$\Omega(\widehat{A}_{\Theta}^{\Omega} - z)^{-1} = (\iota A_{\Theta} - z)^{-1}\Omega\,, \quad z \in \operatorname{res} \widehat{A}_{\Theta}^{\Omega} = \operatorname{res}(\iota A_{\Theta})\,.$$

Corollary

$\widehat{M}^{\Omega} = \widetilde{M}$ *on* $\operatorname{res} A_0$ *iff* $(\forall c \in \mathbb{C}^d)$ $(\forall z \in \Sigma_\iota)$

$$[(A_0 - z)^{-1} - (\iota A_0 - z)^{-1}\iota]k_{\min}(c) - (\iota A_0 - z)^{-1}(\iota - I)z\gamma(z)c \in \operatorname{dom} A_{\min}\,.$$