Fell bundle over a local group

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Local group

Definition

Let $\mathcal{G}$ be a discrete set and $e \in \mathcal{G}$, W a subset of $\mathcal{G} \times \mathcal{G}$, and let $m : W \longrightarrow \mathcal{G}$ and $i : \mathcal{G} \longrightarrow \mathcal{G}$ be the mappings, $m(a, b) = ab$ and $i(a) = a^{-1}$. Then the system $(\mathcal{G}, e, W, m, i)$ is called a local group (denoted by $\mathcal{G}$) if the following conditions are satisfied:

1) *if $(a, b), (b, c) \in W$ and $(ab, c) \in W$, then $(a, bc) \in W$ and $(ab)c = a(bc)$;*
2) *for all $a \in \mathcal{G}$ (a, e) and $(e, a) \in W$ and $ae = ea = a$;*
3) *for all $a \in \mathcal{G}$ (a, a^{-1}) and $(a^{-1}, a) \in W$ и $aa^{-1} = a^{-1}a = e$.*

Obviously, each symmetric subset with a unit is a local group. The local group of this kind we call the local subgroup.

Definition (R. Exel)

Let G be a discrete group and $\mathfrak{A}$ be a unital C^*-algebra. Then the partial $*$-representation of a group G into $\mathfrak{A}$ is a map $\pi : G \longrightarrow \mathfrak{A}$ such that

1) $\pi(e) = I$,
2) $\pi(a^{-1}) = (\pi(a))^*$,
3) $\pi(a)\pi(b)\pi(b^{-1}) = \pi(ab)\pi(b^{-1})$,
4) $\pi(a^{-1})\pi(a)\pi(b) = \pi(a^{-1})\pi(ab)$.

Definition

Let $\mathcal{G}$ be a local group and $\mathfrak{A}$ be a unital C^*-algebra. The map $\pi : \mathcal{G} \longrightarrow \mathfrak{A}$ is a strong $*$-representation of $\mathcal{G}$ into $\mathfrak{A}$ if

- $\pi(e) = I$,
- $\pi(a^{-1}) = (\pi(a))^*$,
- $\pi(a)\pi(b)\pi(b^{-1}) = \begin{cases} \pi(ab)\pi(b^{-1}) & \text{if} \quad (a, b) \in W; \\ 0, \quad \text{if} \quad (a, b) \notin W. \end{cases}$
- $\pi(a^{-1})\pi(a)\pi(b) = \begin{cases} \pi(a^{-1})\pi(ab) & \text{if} \quad (a, b) \in W; \\ 0, \quad \text{if} \quad (a, b) \notin W. \end{cases}$

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Definition

Let $\mathcal{G}$ be a local group. The map $\pi : \mathcal{G} \longrightarrow B(l^2(\mathcal{G}))$ is a left regular representation if $\pi(a)e_b = \begin{cases} e_{ab} & \textit{if} \quad (a, b) \in W; \\ 0, & \textit{if} \quad (a, b) \notin W. \end{cases}$

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Proposition

Let $\pi : G \longrightarrow \mathfrak{A}$ be a partial $*$-representation of a group G into C^*-algebra $\mathfrak{A}$. Then $\mathcal{G} = \{a \in G : \pi(a) \neq 0\}$ is a local group and the restriction π onto $\mathcal{G}$ is a faithful strong $*$-representation of $\mathcal{G}$.

Let $\mathcal{G} \subset \Gamma$ is a local group.

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$C_r^*(\mathcal{G})$ is a local group graded C^*-algebra.

Definition

Let $\mathfrak{A}$ be a C^*-algebra and $\mathcal{G}$ be a local group. Then $\mathfrak{A}$ is $\mathcal{G}$-graded if the direct sum of Banach spaces $\{B_a\}_{a\in\mathcal{G}}$ is dense in $\mathfrak{A}$, i.e. $\mathfrak{A} = \overline{\bigoplus_{a\in G} B_a}$, and

1) $A \in B_a$ *implies* $A^* \in B_{a^{-1}}$,
2) $A \in B_a$ *and* $B \in B_b$ *implies* $AB \in B_{ab}$, $(a, b) \in W$.

Fell bundle over a local group

A collection of Banach spaces $\mathcal{B} = \{B_g\}_{a \in \mathcal{G}}$ with operations $\cdot : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B}$ and $* : \mathcal{B} \longrightarrow \mathcal{B}$ is a Fell bundle over a local group $\mathcal{G}$ if for all $a, b \in \mathcal{G}$ and $A, B \in \mathcal{B}$:

1) $B_a B_b \subseteq B_{ab}$ для $(a, b) \in W$,
2) $\cdot : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B}$ is associative and bi-linear for $(a, b) \in W$ and $(ab, c) \in W$,
3) $\|AB\| \leq \|A\|\|B\|$,
4) $(B_a)^* \subseteq B_{a^{-1}}$,
5) $* : B_a \longrightarrow B_{a^{-1}}$ is conjugate-linear,
6) B_e is a C^*-algebra.
7) $A^*A \in B_e$.

A strong $*$-representation $\pi : \mathcal{G} \longrightarrow \mathfrak{A}$ of a local group naturally generates a Fell bundle over a local group.
Denote by $B_a, \;\; a \in \mathcal{G}$, the Banach space
$\mathrm{Span}\{A = \pi(a_1)\pi(a_2)\dots\pi(a_n) : n \in \mathbb{N},\; a_i \in \mathcal{G},\; a_1a_2\dots a_n = a\}$.
Then a collection $\mathcal{B} = \{B_a\}_{a\in\mathcal{G}}$ is a Fell bundle over local group $\mathcal{G}$ with the operations borrowed from $\mathfrak{A}$.

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Denote by $B_a, \ \ a \in \mathcal{G}$, the Banach space
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Then a collection $\mathcal{B} = \{B_a\}_{a\in\mathcal{G}}$ is a Fell bundle over local group $\mathcal{G}$ with the operations borrowed from $\mathfrak{A}$.
Let F_n be a free group with n generators $\{u_i\}_{i=1}^n$ and F_n^+ be a subset generated by the positive powers of u_i.
Let define a local subgroup $\mathcal{F}_n \subset F_n$ by
$\mathcal{F}_n = \{a \in F_n : a = vw^{-1}, v, w \in F_n^+\}$ with a multiplication $\cdot$ with the rules:

1) $u_i \cdot e = e \cdot u_i = u_i$ and $u_i^{-1} \cdot e = e \cdot u_i^{-1} = u_i^{-1}$ for all i,
2) $u_i \cdot u_i^{-1} = u_i^{-1} \cdot u_i = e$ for all i,
3) if a is of the form $a' u_i$ then $a \cdot b = ab, \ \ a, a', b \in \mathcal{F}_n$,
4) if a is of the form $a' u_i^{-1}$ then $a \cdot b = ab$ if b is of the form $u_i b'$, or $u_j^{-1} b'$ for all $i, j, \ \ a, a', b, b' \in \mathcal{F}_n$.

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Let $U_a : l^2(F_n) \longrightarrow l^2(F_n)$ be a unitary defined by $U_a b = ab$, and $J : l^2(F_n^+) \longrightarrow l^2(F_n)$ be a natural inclusion.

Proposition

The map π defined by

$$\pi : \mathcal{F}_n \longrightarrow B(l^2(F_n^+)); \ \pi(a) = J^* U_a J,$$

is a faithful strong $*$-representation of a local group $\mathcal{F}_n$.

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The map π defined by

$$\pi : \mathcal{F}_n \longrightarrow B(l^2(F_n^+)); \; \pi(a) = J^* U_a J,$$

is a faithful strong $*$-representation of a local group $\mathcal{F}_n$.

Theorem

There exists a Fell bundle over the local group $\mathcal{F}_n$ which is grading for the Cuntz algebra, $\mathcal{O}_n = \overline{\bigoplus_{a \in \mathcal{F}_n} B_a}$.

If $\{V_i\}_{i=1}^n$ are the generators of the Cuntz algebra then the respective subspace B_a is a subspace spanned on the operators of the form $V'^{k_1}_{i_1} V'^{k_2}_{i_2} \dots V'^{k_s}_{i_s}$, here $V'_{i_l} \in \{V_{i_l}, V^*_{i_l}\}$ and $u'^{k_1}_{i_1} u'^{k_2}_{i_2} \dots u'^{k_s}_{i_s} \in \mathcal{F}_n$ and equals a, $u'_{i_l} \in \{u_{i_l}, u^{-1}_{i_l}\}$.

Algebra $C_r^*(P)$

Let Γ be a discrete group and $P \subset \Gamma$ be a subset. Then

$$\Gamma_P = \{a \in \Gamma : aP \cap P \neq \emptyset\} \quad \text{is a local group.}$$

Proposition

There exists a strong $*$-representation $\pi_P : \Gamma_P \longrightarrow B(l^2(P))$, namely the restriction of left regular representation of Γ_P onto $l^2(P)$.

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If $\{U_a\}$, $U_a e_b = e_{ab}$, is the group of unitary shift operators on $l^2(\Gamma)$, then

$$\pi_P(a) = J^* U_a J, \quad J : l^2(P) \hookrightarrow l^2(\Gamma).$$

Definition

The C^*-algebra generated by the set P is a C^*-algebra $C_r^*(P)$ generated by the image of the strong $*$-representation π_P.

Let $\mathrm{Mon}(P)$ be the semigroup generated by $\{\pi_P(a)\}$.

If $W \in \mathrm{Mon}(P)$ and $We_b = e_c$ then $\mathrm{ind}W = cb^{-1}$.

Proposition

The index of W is well defined and
if $W_1 W_2 \neq 0$ then $\mathrm{ind}(W_1 W_2) = \mathrm{ind}(W_1)\mathrm{ind}(W_2)$.

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Proposition

The semigroup $\mathrm{Mon}(P)$ is inverse and the set $\{\mathrm{ind}(\mathrm{Mon}(P))\}$ is a local group and coincides with Γ_P.

Let $\mathrm{Mon}_a(P) = \{W \in \mathrm{Mon}(P) : \mathrm{ind} W = a\}$ and let B_a be an operator space generated by $\mathrm{Mon}_a(P)$.

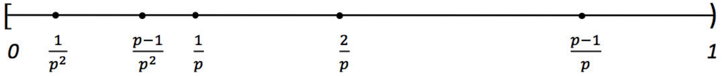
Theorem

- *The collection $\{B_a\}_{a \in \Gamma_P}$ is a Fell bundle over local group Γ_P.*
- *$\{B_a\}_{a \in \Gamma_P}$ is a grading for $C_r^*(P)$, $C_r^*(P) = \overline{\bigoplus_{a \in \Gamma_P} B_a}$.*
- *There exists a conditional expectation $\Phi : C_r^*(P) \longrightarrow B_e$.*
- *B_e is a Cartan subalgebra in $C_r^*(P)$.*

Example 1

Let $\Gamma = \mathbb{Q}_p = \{\frac{m}{p^k}\}_{k\in\mathbb{N}, m\in\mathbb{Z}}$. Define

$$P_0 = [0,1)\cap\mathbb{Q}_p, \quad \text{then} \quad \Gamma_{P_0} = (-1,1)\cap\mathbb{Q}_p.$$



$$l^2(P_0), \quad \{e_{\frac{m}{p^k}}\}_{m<p^k}, \quad T_a e_{\frac{m}{p^k}} = \begin{cases} e_{a+\frac{m}{p^k}} \text{ if } a+\frac{m}{p^k}\in P_0; \\ 0, \text{ if } a+\frac{m}{p^k}\notin P_0. \end{cases}$$

Lemma

For all $k\in\mathbb{N}$ $l^2(P_0)$ can be decomposed as $\bigoplus_{0\le a<\frac{1}{p^k}} H_a, \quad a\in P_0,$

and every H_a is invariant subspace for $T_{\frac{1}{p^k}}$ with dimension p^k.

Example 1

Let $\mathfrak{A}_k$ be the C^*-algebra generated by $T_{\frac{1}{p^k}}$.

Corollary

$$\mathfrak{A}_k \simeq M_{p^k}(\mathbb{C}).$$

Note that $T_{\frac{1}{p}} = T^p_{\frac{1}{p^2}}$, hence

$$\mathfrak{A}_1 \hookrightarrow \mathfrak{A}_2 \hookrightarrow \mathfrak{A}_3 \ldots, \quad C^*_r(P_0) = \overline{\bigcup_k \mathfrak{A}_k}.$$

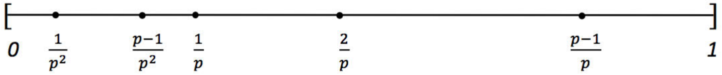
Theorem

$C^*_r(P_0)$ is a UHF-algebra.

Example 2

Again $\Gamma = \mathbb{Q}_p = \{\frac{m}{p^k}\}_{k\in\mathbb{N}, m\in\mathbb{Z}}$,

$$P_1 = [0,1] \cap \mathbb{Q}_p, \quad \text{then} \quad \Gamma_{P_1} = [-1,1] \cap \mathbb{Q}_p.$$



$$l^2(P_1) = l^2(P_0) \oplus \mathbb{C}\{e_1\},\ \{e_{\frac{m}{p^k}}\}_{m\le p^k},\ T_a e_{\frac{m}{p^k}} = \begin{cases} e_{a+\frac{m}{p^k}} \text{ if } a+\frac{m}{p^k} \in P_1; \\ 0, \text{ if } a+\frac{m}{p^k} \notin P_1. \end{cases}$$

Lemma

For all $k \in \mathbb{N}$ $l^2(P_1)$ can be decomposed as $H_0 \oplus (\bigoplus\limits_{0\le a<\frac{1}{p^k}} H_a)$, $a \in P_1$, and every H_a is invariant subspace for $T_{\frac{1}{p^k}}$ with dimension p^k, and H_0 is invariant subspace with dimension p^k+1.

Example 2

Let $\mathfrak{A}_k$ be the C^*-algebra generated by $T_{\frac{1}{p^k}}$.

Corollary

$$\mathfrak{A}_k \simeq M_{p^k+1}(\mathbb{C}) \oplus M_{p^k}.$$

Again $T_{\frac{1}{p}} = T_{\frac{1}{p^2}}^p$, hence

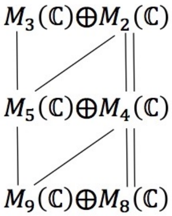
$$\mathfrak{A}_1 \hookrightarrow \mathfrak{A}_2 \hookrightarrow \mathfrak{A}_3 \ldots, \quad C_r^*(P_1) = \overline{\bigcup_k \mathfrak{A}_k}.$$

Theorem

$C_r^*(P_1)$ is an AF-algebra. $C_r^*(P_1)$ contains the algebra of compact operators K and the following short sequence is exact

$$0 \longrightarrow K \longrightarrow C_r^*(P_1) \longrightarrow C_r^*(P_0) \longrightarrow 0.$$

In case $p = 2$ we have the following diagram



In both algebras the semigroup

$$\mathrm{Mon}(P) = E_0 \cup E_1,$$

where E_0 is semilattice of idempotents and E_1 is set of nilpotent operators.

Thank you!