It is joint work with S. Pirogov and A. Yambartsev.

Publications1

1. E Pechersky, S Pirogov, G M Schuetz, A Vladimirov and A Yambartsev
Large fluctuations of radiation in stochastically activated two-level systems J. Phys. A: Math. Theor. 50, 45 (2017) 455203

2. E Pechersky, S Pirogov, G M Schuetz, A Vladimirov and A Yambartsev
Large Fluctuations in two-level systems with stimulated emission, Theoretical and Mathematical Physics, 198,1, (2019) 133-144, (in russian)

Publications2

3. E. Pechersky, S. Pirogov, G. M. Schuetz, A. Vladimirov, and A. Yambartsev, Large Emission Regime in Mean Field Luminescence, Moscow Mathematical Journal, 19, 1, (2019), 107 -120

4. E. Pechersky, S. Pirogov and A. Yambartsev, Hawking-Penrose Black Hole Model. Large Emission Regime, arxiv.org

About

What are topics:

1. Markov process
2. Functional of the process.
3. Interpretation. Black Hole.
4. Large deviations.
5. The rate function.
8. Hamiltonian and Hamiltonian system.
9. Results and hypothesis.

Markov process 1

Markov process $\xi(t)$ defined on the interval $[0, T]$.
The state space of the process is

$$\mathcal{N} = \{1, ..., N\}$$

The paths of the process are piece-wise constants taking non-negative integer values.
The jumps of the process are $+1$ or -1.
The minimal value of the process is 1, the maximal value is N.

Paths

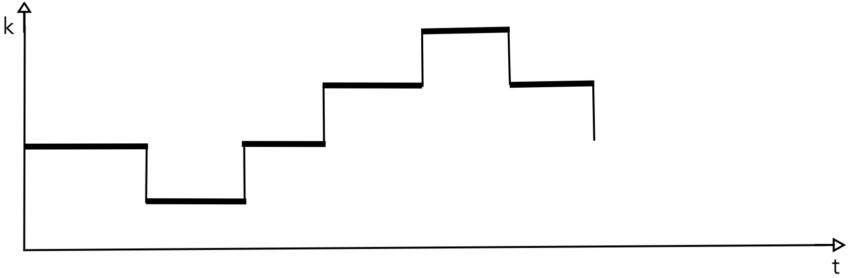

Figure: Path

Markov process 2

Intensities:

$$k \to k+1: \; r_{\uparrow} = \lambda k^2 (N - k) = \lambda N k^2 \left(1 - \frac{k}{N}\right)$$

$$k \to k-1: \; r_{\downarrow} = \mu N \left(\frac{N}{k}\right)^2 (1 - \delta(k-1))$$

Markov process 3

Generator:

$$
\begin{aligned}
Lf(k) = \\
& \lambda N \frac{k^2}{N^2}(1 - \frac{k}{N})[f(k+1) - f(k)] + \\
& \mu N \frac{N^2}{k^2}(1 - \delta(k-1))[f(k-1) - f(k)].
\end{aligned}
$$

Functional of the process

Let $1_+(\gamma) = \begin{cases} 1, & \text{if } \gamma = -1 \\ 0, & \text{otherwise} \end{cases}$.

The emission of the $\xi(\mathrm{t})$ on the interval $[0, \mathcal{T}]$ is

$$\eta(\mathrm{t}) = \sum_{\mathrm{u} \in [0,\mathrm{t}]} 1_+(\xi(\mathrm{u}+0) - \xi(\mathrm{u}))$$

It is the number of the negative jumps (-1) on $[0, \mathrm{t}]$ of ξ.

Markov process with the emission

It is two dimension process

$$(\xi(t), \eta(t))$$

whose generator is

$$Lf(k,m) =$$

$$\lambda N \frac{k^2}{N^2}\left(1 - \frac{k}{N}\right)[f(k+1,m) - f(k,m)] +$$

$$\mu N \frac{N^2}{k^2}(1 - \delta(k-1))\,[f(k-1,m+1) - f(k,m)]\,,$$

Interpretation. Black hole

Hawking-Penrose black hole

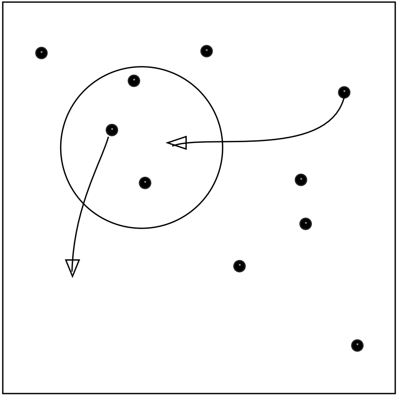

Figure: Black Hole

Hawking, Penrose

Hawking, S.W., Black hole explosions? Nature, 248, 30, 1974

Hawking, S.W., Particle creation by black holes, Comm. Math. Phys., 43, 199, 1975

Hawking,S.W.,Breakdownofpredictabilityingravitationalcollaps, Phys.Rev.D14,10,(1976)246

Penrose, R., Singularity and time-asymmetry, in General relativity: an Einstein centenary survey, eds Israel, Hawking, Cambridge, (1979) 581-638

Universe containing the black hole

The box (in the picture) is a space of Universe inhabited by photons (dots).

The region (the disk) in the box is a horizon surrounding the black hole (the center of the disk)

We consider Schwarzschild black hole, no rotations, no charges.

Dynamics

The black hole attracts the photons from outside. It is by the gravitation.

The black hole emits the photons. It is by Hawking radiation.

Parameters of the black hole

Let R be the radius of the black hole horizon.

The radius R is a function of the number of the photons inside the black hole.
It is proportional to number of the photons in the black hole.

$$R \sim k$$

if k is the photon number in the black hole.

Stochasticity of the black hole

Stochastic regime of the black hole is a Markov jump wise process $\xi(\mathrm{t})$ on a finite interval $[0, \mathcal{T}]$.
The state space is

$$\mathcal{N} = \{1, ..., \mathrm{N}\}$$

N is total number of the photons in Universe.
$\xi(\mathrm{t}) = \mathrm{k} \in \mathcal{N}$ means that the black hole contains k photons.

Stochasticity 2

The intensities of the photon transmissions depend on the black hole radius R.

The absorption of one photon means that $k \to k + 1$ if $k < N$.
The absorption intensity $r_{\uparrow}$ is proportional to R^2

The emission of one photon means that $k \to k - 1$ if $k > 0$.
The emission intensity $r_{\downarrow}$ is inversely proportional to R^2

Generator

$$Lf(k) =$$

$$\lambda N(1 - \frac{k}{N})R^2[f(k+1) - f(k)]+$$

$$\mu N(1 - \delta(k-1))\frac{1}{R^2}[f(k-1) - f(k)].$$

Because $R \sim k$ we have

$$Lf(k) =$$

$$\lambda N(1 - \frac{k}{N})k^2[f(k+1) - f(k)]+$$

$$\mu N(1 - \delta(k-1))\frac{1}{k^2}[f(k-1) - f(k)].$$

Generator of $(\xi(t), \eta(t))$

where $\eta(t)$ is number of the photons emitted on the interval $[0, t]$.

$$Lf(k,m) =$$

$$\lambda N k^2 \left(1 - \frac{k}{N}\right) [f(k+1,m) - f(k,m)] +$$

$$\mu N \frac{1}{k^2} (1 - \delta(k-1)) \, [f(k-1,m+1) - f(k,m)] \, ,$$

Problem

We are looking for the probability

$$\mathsf{Pr}\Big(\eta(\mathcal{T}) > \mathrm{B}\mathcal{T}\Big),$$

when B is very large.

It means that very large emission happens on $[0, \mathcal{T}]$.

Scaling

The problem belongs to the area of the large deviations theory.

To apply the the large deviations theory we introduce the scaling versions of the studied process

$$\xi_{\mathrm{N}}(\mathrm{t}) = \frac{\xi(\mathrm{t})}{\mathrm{N}}, \ \eta_{\mathrm{N}}(\mathrm{t}) = \frac{\eta(\mathrm{t})}{\mathrm{N}}.$$

and find an asymptotic

$$\frac{1}{\mathrm{N}} \ln \Pr\Big(\eta_{\mathrm{N}}(\mathcal{T}) > \mathrm{B}\mathcal{T}\Big),$$

as $\mathrm{N} \to \infty$

Rate function

By the large deviations theory this asymptotic is

$$\lim_{N\to\infty} \frac{1}{N} \ln \Pr\Big(\eta_N(\mathcal{T}) > B\mathcal{T}\Big) = -\inf I((x,y) \in \mathcal{E}),$$

where

$$\mathcal{E} = \Big\{(x(\cdot), y(\cdot)) : \ y(\mathcal{T}) \geq B\mathcal{T}, y(0) = 0\Big\}$$

is the set of all paths of the processes $(\xi_N(t), \eta_N(t)), t \in [0, \mathcal{T}]$ conditioned by $(\eta_N(\mathcal{T}) > B\mathcal{T})$, and I is the rate function.

Rate function2

In our case

$$\begin{aligned} I(x,y) &= \int_0^{\mathcal{T}} \mathcal{L}(x,y)dt \\ &= \int_0^{\mathcal{T}} \sup_{\varkappa_1(t),\varkappa_2(t)} \Big\{ \varkappa_1(t)\dot{x}(t) + \varkappa_2(t)\dot{y}(t) - \\ &\qquad \lambda x^2(t)(1-x(t))[e^{\varkappa_1(t)}-1] - \mu\tfrac{1}{x^2(t)}[e^{-\varkappa_1(t)+\varkappa_2(t)}-1] \Big\} dt, \end{aligned}$$

Hamiltonian

where

$$\mathcal{L}(x(t), y(t)) = \sup_{\varkappa_1(t),\varkappa_2(t)} \{\varkappa_1(t)\dot{x}(t) + \varkappa_2(t)\dot{y}(t) - H(x(t), y(t), \varkappa_1(t), \varkappa_2(t))\}$$

is Legendre transform of Hamiltonian

$$H(x, y, \varkappa_1, \varkappa_2) = \lambda x^2(1 - x)[e^{\varkappa_1} - 1] + \mu \frac{1}{x^2}[e^{-\varkappa_1+\varkappa_2} - 1].$$

Infimum

One a way to find

$$\operatorname{inf} \mathrm{I}((\mathrm{x}, \mathrm{y}) \in \mathcal{E})$$

is to solve the Hamiltonian system

Hamiltonian system

$$\begin{cases} \dot{x} &= \lambda x^2(1-x)\exp\{\varkappa_1\} - \mu\frac{1}{x^2}\exp\{-\varkappa_1+\varkappa_2\}, \\ \dot{y} &= \mu\frac{1}{x^2}\exp\{-\varkappa_1+\varkappa_2\}, \\ \dot{\varkappa}_1 &= -\lambda(2x-3x^2)[\exp\{\varkappa_1\}-1] + \mu\frac{2}{x^3}[\exp\{-\varkappa_1+\varkappa_2\}-1], \\ \dot{\varkappa}_2 &= 0, \end{cases} \tag{0.1}$$

with suitable boundary conditions.

Extremal

For any $B > 0$
there exists $q_B \in [0, 1]$ such that

$$x(t) \equiv q_B \text{ and } y(t) = Bt,$$

$$e^{\varkappa_1(t)} = \frac{B}{\lambda q_B^2 (1 - q_B)}$$

and

$$e^{\varkappa_2(t)} = B^2 \frac{1}{\lambda\mu(1 - q_B)}$$

are the solutions of the Hamiltonian system,
where $t \in [0, \mathcal{T}]$

We see that $x(t) \equiv q_B$ is a constant. The relation between q_B and B is

$$q_B \sim \left(\frac{2\mu}{B}\right)^{\frac{1}{3}}$$

Unfortunately this solution does not give $\inf \mathrm{I}((x, y) \in \mathcal{E})$.