Integration by parts for pinned point processes

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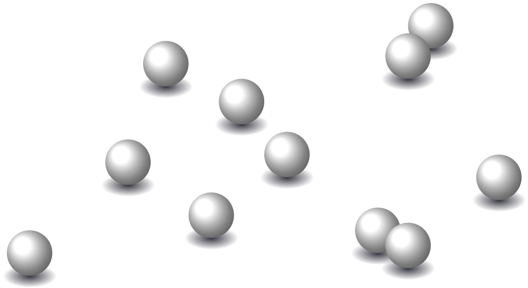
Stochastic and Analytic Methods in Mathematical Physics
Yerevan, September 2–7, 2019

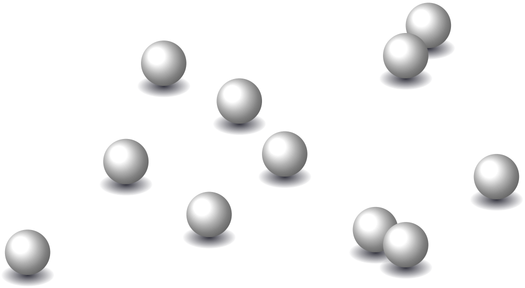
Plan of the talk

1. How to describe point processes
2. Pinning and examples
3. Characterization
4. Key steps
5. Remarks on further examples

How to describe point processes

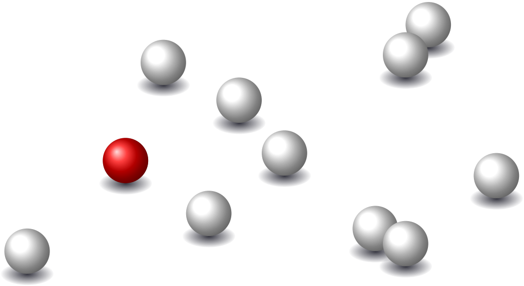


How to describe point processes



static: total number of points and joint law

How to describe point processes



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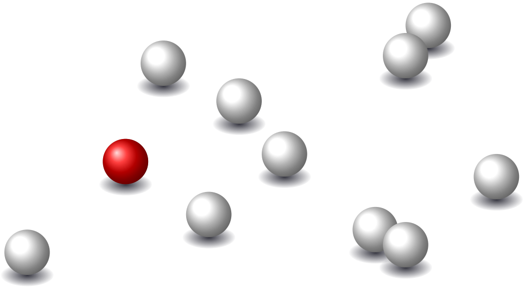
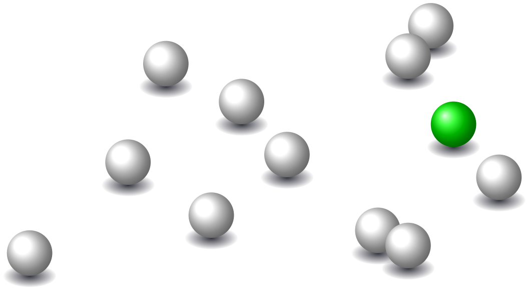
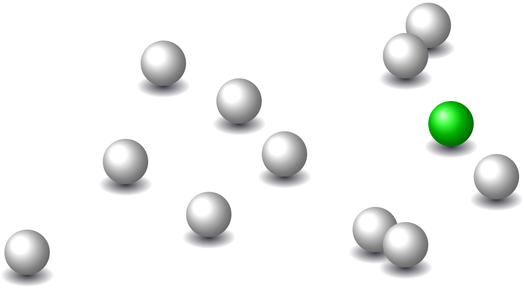


dynamic: how to remove

How to describe point processes



How to describe point processes



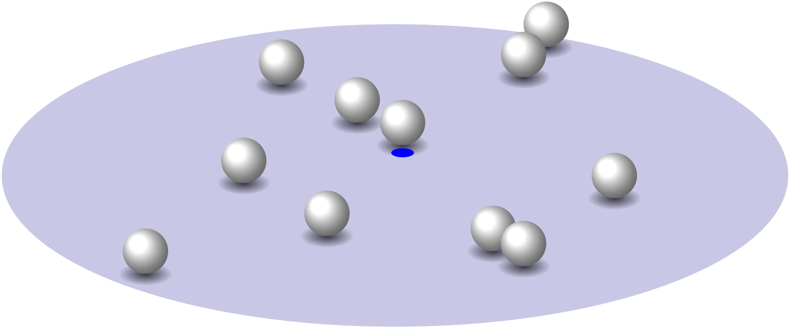
dynamic: how to add

How to describe point processes

- dynamics for Markov process (existence different story)
  - add – birth
  - remove – death
- existence and uniqueness of stationary/reversible distribution

Pinning a point process

Barycentre



Pinning a point process

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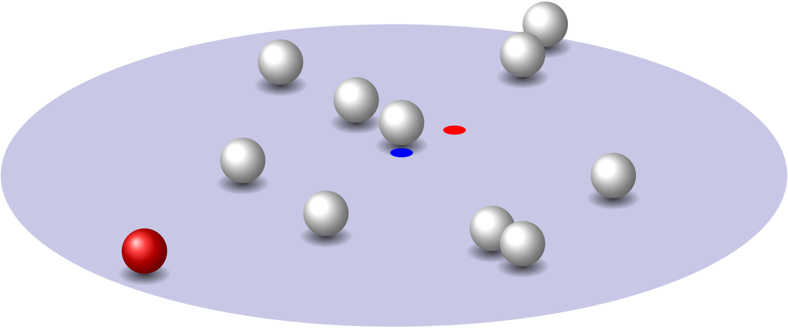

Pinning a point process

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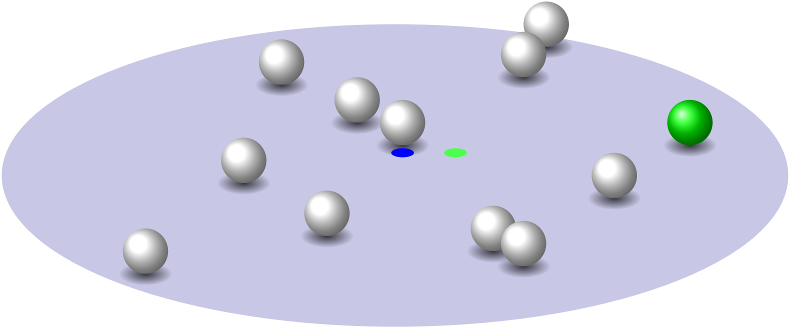
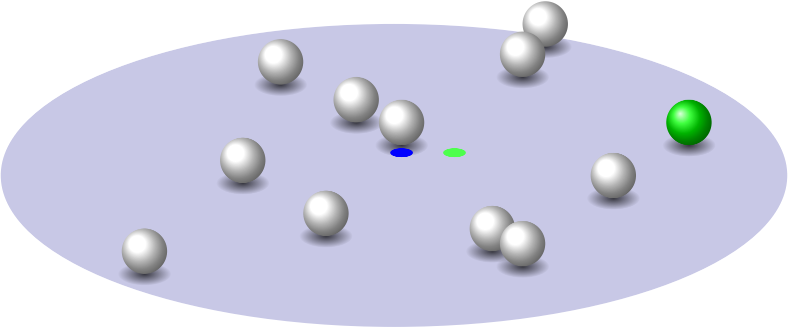


Pinning a point process

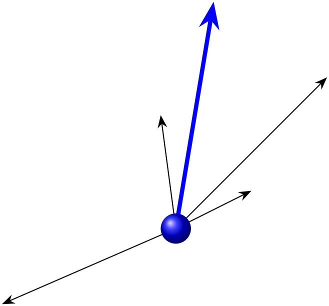
Barycentre



task: modify configuration without disbalancing

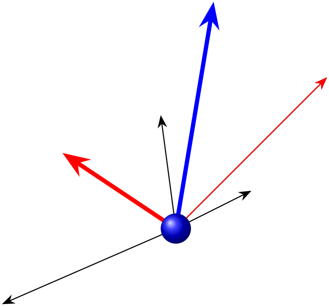
Pinning a point process

First moment



Pinning a point process

First moment



Pinning a point process

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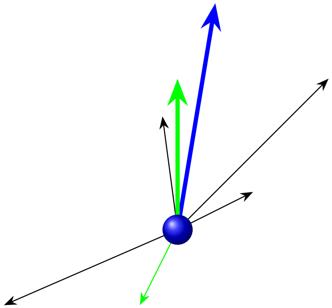

Pinning a point process

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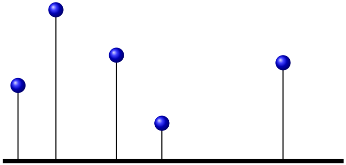

Pinning a point process

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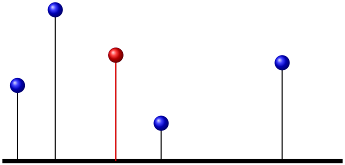

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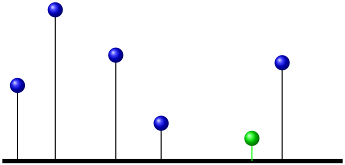

Pinning a point process

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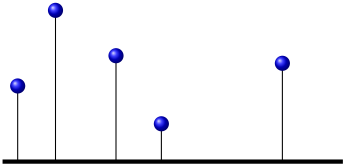

task: modify measure without changing total mass

Application to simulation

Monte-Carlo simulation

via birth-and-death processes

- simple
- functionals $\mathfrak{b}$ and $\mathfrak{m}$ not invariant
- $\{\mathfrak{b} = a\}$ unlikeli

Simulationof pinned point configuration

Rejection method more or less unfeasible

Application to simulation

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Simulationof pinned point configuration

Rejection method more or less unfeasible

Previous Work

G. Conforti, T. Kosenkova, S. Rœlly:
Conditioned point processes with application to Lévy bridges (2018)

- large class of Poisson processes and pinning of first moment
- application to Lévy bridges
- idea:
  - remove two distinct points and add a suitably chosen one
  - remove one point and add one point together with a suitably chosen one

number of points increases/decreases by one

Transformations and Invariance

Tranformations of point configurations

for point configuration μ and points $x, y \in \mu$,

$$\mu \mapsto \mu - \delta_x - \delta_y + \delta_z$$

Invariance

functionals $\mathfrak{b}$ and $\mathfrak{m}$ shall remain invariant

$$\mathfrak{b}(\mu) = \mathfrak{b}(\mu - \delta_x - \delta_y + \delta_z) \quad \Leftrightarrow \quad z = x + y - \mathfrak{b}\mu;$$
$$\mathfrak{m}(\mu) = \mathfrak{m}(\mu - \delta_x - \delta_y + \delta_z) \quad \Leftrightarrow \quad z = x + y.$$

- for any two points, the one is determined
- further functionals?

State space and conditions

Choose ...

- state space
  - $\mathbb{R}^d$
  - $\mathbb{Z}^d$ or any grid
- pinning of
  - barycentre $\mathfrak{b}$
  - first moment $\mathfrak{m}$
- focus on $\mathbb{R}^d$ and $\mathfrak{b}$

IBPF(π)

A point process N is a Papangelou process if

$$\mathbf{E}\left[\int h(x,N)N(\mathrm{d}x)\right] = \mathbf{E}\left[\int h(x,N+\delta_x)\pi(N,\mathrm{d}x)\right]$$

rate of removal $N(\mathrm{d}x)$, rate of addition $\pi(N,\mathrm{d}x)$

Examples

1. Poisson process $\pi(\mu,\mathrm{d}x) = \rho(\mathrm{d}x)$
2. Gibbs process $\pi(\mu,\mathrm{d}x) = \exp\big(-U(x\,|\,\mu)\big)\mathrm{d}x$

1. Absolute continuity of π
   - density denoted by π
   - rules out some examples with e.g. reinforcement
2. Compatibility with transformation
   - $\pi^{(2)}(\mu - \delta_z; z - y - \mathfrak{b}\mu, y) > 0$ implies $\pi(\mu - \delta_z, z) > 0$

IBPF-2(σ)

Let N be a solution of IBPF(π), then

$$\mathbf{E}\left[\int h(x,y,N)N^{(2)}(\mathrm{d}x,\mathrm{d}y)\right]$$
$$= \mathbf{E}\left[\int h(x,y,N+\delta_{z-y-\mathfrak{b}}+\delta_y-\delta_z)\sigma(N,y,z)\mathrm{d}yN(\mathrm{d}z)\right]$$

where

$$\sigma(N,y,z) = \frac{\pi^{(2)}(N-\delta_z, z-y-\mathfrak{b}, y)}{\pi(N-\delta_z, z)}, \qquad z \in N.$$

- σ measures preference
- IBPF-2 is no characterization

Pinning and properties

Let N be a finite point process, $a \in \mathbb{R}^d$

- $\mathcal{L}(\mathfrak{b}) =: \tau$, distribution on $\mathbb{R}^d \cup \{o\}$ and absolutely continuous on $\mathbb{R}^d$
- N^a is N conditioned on $\{\mathfrak{b} = a\}$
- $N^a \neq 0$ a.s. for τ-a.e. $a \in \mathbb{R}^d$

IBPF-2(σ) for pinned Papangelou processes

If N satisfies IBPF-2(σ), then N^a satisfies IBPF-2(σ) for τ-a.e. $a \in \mathbb{R}^d$.

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IBPF-2(σ) for pinned Papangelou processes

If N satisfies IBPF-2(σ), then N^a satisfies IBPF-2(σ) for τ-a.e. $a \in \mathbb{R}^d$.

Characterization

Let Q be the distribution of some point process N such that

1. Q solves IBPF-2,
2. $Q(\mathfrak{b} \neq a) = 0$ for some $a \in \mathbb{R}$,
3. kernel π is positive and compatible,

then Q is the distribution of a pinned Papangelou process.

- simple dynamic leaving $\mathfrak{b}$ invariant

Diminished point process

For point process N, let N^- be N with a uniformly chosen point removed.

- diminution destroys pinning
- $\mathcal{L}\big((N^a)^-\big) \ll \mathcal{L}(N)$ with density
$$\frac{\pi(\mu, (\mu(X)+1)a - \mu(X)\cdot\mathfrak{b}(\mu))}{\tau(a)}$$
- $(\mu(X)+1)a - \mu(X)\cdot\mathfrak{b}(\mu))$ is location of removed point

Recover diminished law

If Q solves IBPF-2(σ), Q is pinned to $a \in \mathbb{R}^d$ and π is positive. Then $Q^- \ll \mathcal{L}(N)$ with density

$$\frac{\pi(\mu, (\mu(X)+1)a - \mu(X) \cdot \mathfrak{b}(\mu))}{\tau(a)}.$$

Recover law (CKR)

If Q is pinned to some $a \in \mathbb{R}$ and $Q^- = \mathcal{L}\big((N^a)^-\big)$, then $Q = \mathcal{L}(N^a)$.

Recover diminished law

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Recover law (CKR)

If Q is pinned to some $a \in \mathbb{R}$ and $Q^- = \mathcal{L}\big((N^a)^-\big)$, then $Q = \mathcal{L}(N^a)$.

First moment $\mathfrak{m}$

- state space $\mathbb{R}^d$ and $\mathfrak{m}$
  - pinning on $\mathfrak{m} = a \neq 0$ rules out empty configurations
- invariance in case of $z = x + y$
- assumptions
  - absolute continuity condition
  - compatibility condition
- adjustments in IBPF-2(σ)
  - $z - y + \mathfrak{b}$ replaced by $z - y$
  - (also in σ)

First moment $\mathfrak{m}$

- state space $\mathbb{Z}^d$ and $\mathfrak{m}$
- not allowed in CKR
- absolute continuity with respect to counting measure
- adjustment in proof of IBPF-2(σ)

Finally

- derived IBPF-2 for pinned Papangelou process
- IBPF-2+pinning characterizes law
- description of all laws of IBPF-2?