Entropy production in viscous fluid flows

Armen Shirikyan

Coauthors: V. Jakšić, V. Nersesyan, C.-A. Pillet

Stochastic and Analytic Methods in Mathematical Physics

AUA, Yerevan

September 2 – 7, 2019

Outline

Heuristics

Question: *Can one see the direction of time ?*

Let $\{u_k\}_{k\in\mathbb{Z}}$ be a stationary Markov process in a space X.

Question: *Can one see the difference between the forward and backward evolutions by observing $\{u_k\}_{k\in\mathbb{Z}}$?*

Let us fix an integer $T \geq 1$ and define the path space

$$\boldsymbol{X}_T = X \times X \times \cdots \times X \quad (T+1 \text{ times})$$

and the time-reversal operator

$$\theta_T : \boldsymbol{X}_T \to \boldsymbol{X}_T, \quad [v_0, \dots, v_T] \mapsto [v_T, \dots, v_0].$$

Let μ_T be the law of $[u_0, \dots, u_T]$ regarded as a random variable in $\boldsymbol{X}_T$. Then

$$\mathcal{D}([u_T, \dots, u_0]) = \mu_T \circ \theta_T^{-1}.$$

Heuristics

Question: *Can one see the direction of time ?*

Let $\{u_k\}_{k\in\mathbb{Z}}$ be a stationary Markov process in a space X.

Question: *Can one see the difference between the forward and backward evolutions by observing $\{u_k\}_{k\in\mathbb{Z}}$?*

Let us fix an integer $T \geq 1$ and define the path space

$$\boldsymbol{X}_T = X \times X \times \cdots \times X \quad (T+1 \text{ times})$$

and the time-reversal operator

$$\theta_T : \boldsymbol{X}_T \to \boldsymbol{X}_T, \quad [v_0, \ldots, v_T] \mapsto [v_T, \ldots, v_0].$$

Let μ_T be the law of $[u_0, \ldots, u_T]$ regarded as a random variable in $\boldsymbol{X}_T$. Then

$$\mathcal{D}([u_T, \ldots, u_0]) = \mu_T \circ \theta_T^{-1}.$$

Heuristics

Question: *Can one see the direction of time ?*

Let $\{u_k\}_{k\in\mathbb{Z}}$ be a stationary Markov process in a space X.

Question: *Can one see the difference between the forward and backward evolutions by observing* $\{u_k\}_{k\in\mathbb{Z}}$ *?*

Let us fix an integer $T \geq 1$ and define the path space

$$\boldsymbol{X}_T = X \times X \times \cdots \times X \quad (T+1 \text{ times})$$

and the time-reversal operator

$$\theta_T : \boldsymbol{X}_T \to \boldsymbol{X}_T, \quad [v_0, \ldots, v_T] \mapsto [v_T, \ldots, v_0].$$

Let μ_T be the law of $[u_0, \ldots, u_T]$ regarded as a random variable in $\boldsymbol{X}_T$. Then

$$\mathcal{D}([u_T, \ldots, u_0]) = \mu_T \circ \theta_T^{-1}.$$

Heuristics

Question: *Can one see the direction of time ?*

Let $\{u_k\}_{k\in\mathbb{Z}}$ be a stationary Markov process in a space X.

Question: *Can one see the difference between the forward and backward evolutions by observing $\{u_k\}_{k\in\mathbb{Z}}$?*

Let us fix an integer $T \geq 1$ and define the path space

$$\boldsymbol{X}_T = X \times X \times \cdots \times X \quad (T+1 \text{ times})$$

and the time-reversal operator

$$\theta_T : \boldsymbol{X}_T \to \boldsymbol{X}_T, \quad [v_0, \dots, v_T] \mapsto [v_T, \dots, v_0].$$

Let $\boldsymbol{\mu}_T$ be the law of $[u_0, \dots, u_T]$ regarded as a random variable in $\boldsymbol{X}_T$. Then

$$\mathcal{D}([u_T, \dots, u_0]) = \boldsymbol{\mu}_T \circ \theta_T^{-1}.$$

Hypotheses

We wish to compare the probability measures $\boldsymbol{\mu}_T$ and $\boldsymbol{\mu}_T \circ \theta_T^{-1}$.

Regularity

There is a reference measure ℓ on X such that

$$P(u, \mathrm{d}v) = \rho(u, v)\ell(\mathrm{d}v), \qquad \rho(u, v) > 0 \quad \forall u, v \in X,$$

where $P(u, \mathrm{d}v)$ is the time-1 transition function for $\{u_k\}$.

Ergodicity

Both forward and backward processes are ergodic.

In other words, defining $\boldsymbol{X} = X^{\mathbb{Z}}$ and the group of shifts

$$T_n : \boldsymbol{X} \to \boldsymbol{X}, \quad \{v_k\}_{k\in\mathbb{Z}} \mapsto \{v_{k+n}\}_{k\in\mathbb{Z}},$$

we require that the invariant measure $\boldsymbol{\mu} = \mathcal{D}(\{u_k\}_{k\in\mathbb{Z}})$ be ergodic both at $+\infty$ and $-\infty$.

Hypotheses

We wish to compare the probability measures $\boldsymbol{\mu}_T$ and $\boldsymbol{\mu}_T \circ \theta_T^{-1}$.

Regularity

There is a reference measure ℓ on X such that

$$P(u, \mathrm{d}v) = \rho(u, v)\ell(\mathrm{d}v), \qquad \rho(u, v) > 0 \quad \forall u, v \in X,$$

where $P(u, \mathrm{d}v)$ is the time-1 transition function for $\{u_k\}$.

Ergodicity

Both forward and backward processes are ergodic.

In other words, defining $\boldsymbol{X} = X^{\mathbb{Z}}$ and the group of shifts

$$T_n : \boldsymbol{X} \to \boldsymbol{X}, \quad \{v_k\}_{k\in\mathbb{Z}} \mapsto \{v_{k+n}\}_{k\in\mathbb{Z}},$$

we require that the invariant measure $\boldsymbol{\mu} = \mathcal{D}(\{u_k\}_{k\in\mathbb{Z}})$ be ergodic both at $+\infty$ and $-\infty$.

Hypotheses

We wish to compare the probability measures $\boldsymbol{\mu}_T$ and $\boldsymbol{\mu}_T \circ \theta_T^{-1}$.

Regularity

There is a reference measure ℓ on X such that

$$P(u, \mathrm{d}v) = \rho(u, v)\ell(\mathrm{d}v), \qquad \rho(u, v) > 0 \quad \forall u, v \in X,$$

where $P(u, \mathrm{d}v)$ is the time-1 transition function for $\{u_k\}$.

Ergodicity

Both forward and backward processes are ergodic.

In other words, defining $\boldsymbol{X} = X^{\mathbb{Z}}$ and the group of shifts

$$T_n : \boldsymbol{X} \to \boldsymbol{X}, \quad \{v_k\}_{k\in\mathbb{Z}} \mapsto \{v_{k+n}\}_{k\in\mathbb{Z}},$$

we require that the invariant measure $\boldsymbol{\mu} = \mathcal{D}(\{u_k\}_{k\in\mathbb{Z}})$ be ergodic both at $+\infty$ and $-\infty$.

Path measures

The regularity assumption implies that $\mu = \mathcal{D}(u_0)$ satisfies

$$\mu(\mathrm{d}v) = \rho(v)\ell(\mathrm{d}v), \quad \rho(v) = \int_X \rho(z, v)\mu(\mathrm{d}z) > 0.$$

Moreover, the path measure can be written as

$$\boldsymbol{\mu}_T(\mathrm{d}\boldsymbol{v}_T) = \rho(v_0)\rho(v_0, v_1)\dots\rho(v_{T-1}, v_T)\,\ell^{T+1}(\mathrm{d}\boldsymbol{v}_T),$$

where $\boldsymbol{v}_T = [v_0, \dots, v_T]$ and $\ell^{T+1} = \ell \otimes \cdots \otimes \ell$ ($T + 1$ times).

It follows that

$$\log \frac{\mathrm{d}\boldsymbol{\mu}_T}{\mathrm{d}\boldsymbol{\mu}_T \circ \theta_T^{-1}}(\boldsymbol{v}_T) = \sum_{k=1}^{T} \sigma(v_{k-1}, v_k) + \log \rho(v_0) - \log \rho(v_T),$$

where σ is the entropy production observable defined by

$$\boxed{\sigma(u, v) = \log \frac{\rho(u, v)}{\rho(v, u)}}$$

Path measures

The regularity assumption implies that $\mu = \mathcal{D}(u_0)$ satisfies

$$\mu(\mathrm{d}v) = \rho(v)\ell(\mathrm{d}v), \quad \rho(v) = \int_X \rho(z,v)\mu(\mathrm{d}z) > 0.$$

Moreover, the path measure can be written as

$$\boldsymbol{\mu}_T(\mathrm{d}\boldsymbol{v}_T) = \rho(v_0)\rho(v_0,v_1)\dots\rho(v_{T-1},v_T)\,\ell^{T+1}(\mathrm{d}\boldsymbol{v}_T),$$

where $\boldsymbol{v}_T = [v_0,\dots,v_T]$ and $\ell^{T+1} = \ell\otimes\cdots\otimes\ell$ ($T+1$ times). It follows that

$$\log\frac{\mathrm{d}\boldsymbol{\mu}_T}{\mathrm{d}\boldsymbol{\mu}_T\circ\theta_T^{-1}}(\boldsymbol{v}_T) = \sum_{k=1}^{T}\sigma(v_{k-1},v_k) + \log\rho(v_0) - \log\rho(v_T),$$

where σ is the entropy production observable defined by

$$\boxed{\sigma(u,v) = \log\frac{\rho(u,v)}{\rho(v,u)}}$$

Reversible case

Theorem

For any integer $T \geq 1$, the following properties are equivalent.

- *The process $\{u_k\}_{k\in[0,T]}$ is reversible:*

$$\boldsymbol{\mu}_T = \boldsymbol{\mu}_T \circ \theta_T^{-1}$$

- *The transition function satisfies the detailed balance:*

$$\big(Pf, g\big)_\mu = \big(f, Pg\big)_\mu \quad \forall f, g \in L^\infty(X, \mu)$$

- *The mean entropy production vanishes:*

$$\langle\sigma\rangle_{\boldsymbol{\mu}} = \int_{X\times X} \sigma(v_0, v_1)\boldsymbol{\mu}_1(\mathrm{d}v_0, \mathrm{d}v_1) = 0$$

Separation under forward and backward evolutions

We now assume that $\langle\sigma\rangle_{\mu} \neq 0$. Define the time-reversal

$$\theta : \boldsymbol{X} \to \boldsymbol{X}, \quad \{v_k\}_{k\in\mathbb{Z}} \mapsto \{v_{-k}\}_{k\in\mathbb{Z}},$$

and the reversed measure $\tilde{\mu} = \mu \circ \theta^{-1}$. By assumption, μ and $\tilde{\mu}$ are ergodic invariant measures for $\{T_n\}$. Since $\mu \neq \tilde{\mu}$, they are singular. Moreover,

$$\langle\sigma\rangle_{\tilde{\mu}} = -\langle\sigma\rangle_{\mu}.$$

For two equivalent measures λ and ν, define the Stein exponent

$$\mathfrak{s}_\varepsilon(\lambda,\nu) := \inf\{\nu(\Gamma) : \lambda(\Gamma) \geq 1-\varepsilon\}, \quad \varepsilon \in (0,1).$$

Proposition

Under rather general assumptions, for any $\varepsilon \in (0,1)$ we have

$$\lim_{T\to+\infty} \frac{1}{T} \log \mathfrak{s}_\varepsilon(\mu_T, \mu_T \circ \theta_T^{-1}) = -\langle\sigma\rangle_{\mu}.$$

Separation under forward and backward evolutions

We now assume that $\langle\sigma\rangle_{\boldsymbol{\mu}} \neq 0$. Define the time-reversal

$$\theta : \boldsymbol{X} \to \boldsymbol{X}, \quad \{v_k\}_{k\in\mathbb{Z}} \mapsto \{v_{-k}\}_{k\in\mathbb{Z}},$$

and the reversed measure $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu}\circ\theta^{-1}$. By assumption, $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$ are ergodic invariant measures for $\{T_n\}$. Since $\boldsymbol{\mu} \neq \tilde{\boldsymbol{\mu}}$, they are singular. Moreover,

$$\langle\sigma\rangle_{\tilde{\boldsymbol{\mu}}} = -\langle\sigma\rangle_{\boldsymbol{\mu}}.$$

For two equivalent measures λ and ν, define the Stein exponent

$$\mathfrak{s}_\varepsilon(\lambda,\nu) := \inf\big\{\nu(\Gamma) : \lambda(\Gamma) \geq 1-\varepsilon\big\}, \quad \varepsilon\in(0,1).$$

Proposition

Under rather general assumptions, for any $\varepsilon\in(0,1)$ we have

$$\lim_{T\to+\infty}\frac{1}{T}\log\mathfrak{s}_\varepsilon(\boldsymbol{\mu}_T, \boldsymbol{\mu}_T\circ\theta_T^{-1}) = -\langle\sigma\rangle_{\boldsymbol{\mu}}.$$

Separation under forward and backward evolutions

We now assume that $\langle\sigma\rangle_{\boldsymbol{\mu}} \neq 0$. Define the time-reversal

$$\theta : \boldsymbol{X} \to \boldsymbol{X}, \quad \{v_k\}_{k\in\mathbb{Z}} \mapsto \{v_{-k}\}_{k\in\mathbb{Z}},$$

and the reversed measure $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} \circ \theta^{-1}$. By assumption, $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$ are ergodic invariant measures for $\{T_n\}$. Since $\boldsymbol{\mu} \neq \tilde{\boldsymbol{\mu}}$, they are singular. Moreover,

$$\langle\sigma\rangle_{\tilde{\boldsymbol{\mu}}} = -\langle\sigma\rangle_{\boldsymbol{\mu}}.$$

For two equivalent measures λ and ν, define the Stein exponent

$$\mathfrak{s}_\varepsilon(\lambda,\nu) := \inf\bigl\{\nu(\Gamma) : \lambda(\Gamma) \geq 1-\varepsilon\bigr\}, \quad \varepsilon \in (0,1).$$

Proposition

Under rather general assumptions, for any $\varepsilon \in (0,1)$ we have

$$\lim_{T\to+\infty} \frac{1}{T} \log \mathfrak{s}_\varepsilon(\boldsymbol{\mu}_T, \boldsymbol{\mu}_T \circ \theta_T^{-1}) = -\langle\sigma\rangle_{\boldsymbol{\mu}}.$$

Large deviations and separation of measures

Consider the mean entropy production over a time interval $[0, T]$:

$$\langle\sigma\rangle_T = T^{-1}\sum_{k=1}^{T}\sigma(u_{k-1}, u_k).$$

Given $\theta \in \mathbb{R}$, introduce the Hoeffding exponent

$$\mathfrak{h}_\theta = \inf\left\{\lim_{T\to\infty} T^{-1}\log \boldsymbol{\mu}_T\circ\theta_T^{-1}(\Gamma_T) : \limsup_{T\to\infty} T^{-1}\log\boldsymbol{\mu}_T(\Gamma_T^c) < -\theta\right\},$$

where the infimum is taken over $\Gamma_T \in \boldsymbol{X}_T$ for which $\exists$ limit.

Proposition

Suppose that $\langle\sigma\rangle_T$ satisfies the LDP with some good rate function $I : \mathbb{R} \to [0, +\infty]$. Then

$$\mathfrak{h}_\theta = \inf_{\alpha\in(0,1]}\frac{(1-\alpha)\theta + \mathrm{e}(\alpha)}{\alpha},$$

where e is the Legendre transform of I.

Large deviations and separation of measures

Consider the mean entropy production over a time interval $[0, T]$:

$$\langle\sigma\rangle_T = T^{-1}\sum_{k=1}^{T}\sigma(u_{k-1}, u_k).$$

Given $\theta \in \mathbb{R}$, introduce the Hoeffding exponent

$$\mathfrak{h}_\theta = \inf\Big\{\lim_{T\to\infty} T^{-1}\log \boldsymbol{\mu}_T\circ\theta_T^{-1}(\Gamma_T) : \limsup_{T\to\infty} T^{-1}\log \boldsymbol{\mu}_T(\Gamma_T^c) < -\theta\Big\},$$

where the infimum is taken over $\Gamma_T \in \boldsymbol{X}_T$ for which $\exists$ limit.

Proposition

Suppose that $\langle\sigma\rangle_T$ satisfies the LDP with some good rate function $I : \mathbb{R} \to [0, +\infty]$. Then

$$\mathfrak{h}_\theta = \inf_{\alpha\in(0,1]} \frac{(1-\alpha)\theta + \mathrm{e}(\alpha)}{\alpha},$$

where e is the Legendre transform of I.

Summary of conclusions

- The entropy production observable σ is well defined under the regularity hypothesis for the transition function.
- One cannot distinguish between the forward and backward evolutions iff the mean entropy production $\langle\sigma\rangle_\mu$ vanishes.
- If the forward and backward processes are ergodic, then the mean entropy production determines the exponential rate of separation of forward and backward path measures.
- In concrete physical models, the quantity

$$T^{-1}\log\frac{\mathrm{d}\mu_T}{\mathrm{d}\mu_T\circ\theta_T^{-1}}(\boldsymbol{v}_T)\sim\frac{1}{T}\sum_{k=1}^{T}\sigma(v_{k-1},v_k)$$

coincide, up to (non-negligible!!!) boundary terms, with the time average of various relevant observable.

The entropy production observable is an important object

Summary of conclusions

- The entropy production observable σ is well defined under the regularity hypothesis for the transition function.
- One cannot distinguish between the forward and backward evolutions iff the mean entropy production $\langle\sigma\rangle_{\mu}$ vanishes.
- If the forward and backward processes are ergodic, then the mean entropy production determines the exponential rate of separation of forward and backward path measures.
- In concrete physical models, the quantity

$$T^{-1}\log\frac{\mathrm{d}\mu_T}{\mathrm{d}\mu_T\circ\theta_T^{-1}}(\boldsymbol{v}_T)\sim\frac{1}{T}\sum_{k=1}^{T}\sigma(v_{k-1},v_k)$$

coincide, up to (non-negligible!!!) boundary terms, with the time average of various relevant observable.

The entropy production observable is an important object

Summary of conclusions

- The entropy production observable σ is well defined under the regularity hypothesis for the transition function.
- One cannot distinguish between the forward and backward evolutions iff the mean entropy production $\langle\sigma\rangle_{\mu}$ vanishes.
- If the forward and backward processes are ergodic, then the mean entropy production determines the exponential rate of separation of forward and backward path measures.
- In concrete physical models, the quantity

$$T^{-1} \log \frac{d\mu_T}{d\mu_T \circ \theta_T^{-1}}(\mathbf{v}_T) \sim \frac{1}{T}\sum_{k=1}^{T} \sigma(v_{k-1}, v_k)$$

coincide, up to (non-negligible!!!) boundary terms, with the time average of various relevant observable.

The entropy production observable is an important object

Summary of conclusions

- The entropy production observable σ is well defined under the regularity hypothesis for the transition function.
- One cannot distinguish between the forward and backward evolutions iff the mean entropy production $\langle\sigma\rangle_{\mu}$ vanishes.
- If the forward and backward processes are ergodic, then the mean entropy production determines the exponential rate of separation of forward and backward path measures.
- In concrete physical models, the quantity

$$T^{-1}\log\frac{d\mu_T}{d\mu_T\circ\theta_T^{-1}}(\boldsymbol{v}_T)\sim\frac{1}{T}\sum_{k=1}^{T}\sigma(v_{k-1},v_k)$$

coincide, up to (non-negligible!!!) boundary terms, with the time average of various relevant observable.

The entropy production observable is an important object

Summary of conclusions

- The entropy production observable σ is well defined under the regularity hypothesis for the transition function.
- One cannot distinguish between the forward and backward evolutions iff the mean entropy production $\langle\sigma\rangle_{\mu}$ vanishes.
- If the forward and backward processes are ergodic, then the mean entropy production determines the exponential rate of separation of forward and backward path measures.
- In concrete physical models, the quantity

$$T^{-1}\log\frac{d\boldsymbol{\mu}_T}{d\boldsymbol{\mu}_T\circ\theta_T^{-1}}(\boldsymbol{v}_T)\sim\frac{1}{T}\sum_{k=1}^{T}\sigma(v_{k-1},v_k)$$

coincide, up to (non-negligible!!!) boundary terms, with the time average of various relevant observable.

The entropy production observable is an important object

Navier–Stokes equations with smooth forcing

Let us consider the Navier–Stokes system on $\mathbb{T}^2$:

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0. \tag{1}$$

The noise is assumed to be smooth in x, while its dependence on time should be such that the family of solutions of (1) form a Markov process. Under some non-degeneracy hypotheses, the latter has a unique stationary measure, and our goal is to study entropic fluctuations for the corresponding stationary trajectory.

The validity of the level-3 LDP was proved for various type of random perturbations, and in the discrete-time setting the rate function is given by the Donsker–Varadhan entropy formula:

$$I(\lambda) = \int_{\boldsymbol{H}_-} \operatorname{Ent}\big(\lambda(\boldsymbol{u}, \cdot) \,|\, P(u_0, \cdot)\big) \lambda_-(\mathrm{d}\boldsymbol{u}), \tag{2}$$

where $\boldsymbol{H}_- = H^{\mathbb{Z}_-}$, and $P(u, \mathrm{d}v)$ is the time-1 transition function.

Navier–Stokes equations with smooth forcing

Let us consider the Navier–Stokes system on $\mathbb{T}^2$:

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0. \tag{1}$$

The noise is assumed to be smooth in x, while its dependence on time should be such that the family of solutions of (1) form a Markov process. Under some non-degeneracy hypotheses, the latter has a unique stationary measure, and our goal is to study entropic fluctuations for the corresponding stationary trajectory.

The validity of the level-3 LDP was proved for various type of random perturbations, and in the discrete-time setting the rate function is given by the Donsker–Varadhan entropy formula:

$$\boldsymbol{I}(\boldsymbol{\lambda}) = \int_{\boldsymbol{H}_-} \operatorname{Ent}\big(\lambda(\boldsymbol{u}, \cdot) \,|\, P(u_0, \cdot)\big) \boldsymbol{\lambda}_-(\mathrm{d}\boldsymbol{u}), \tag{2}$$

where $\boldsymbol{H}_- = H^{\mathbb{Z}_-}$, and $P(u, \mathrm{d}v)$ is the time-1 transition function.

Difficulties: linear case

Even in the linear case, one cannot justify the above calculation. For instance, for the discrete-time system

$$u_k = S(u_{k-1}) + \eta_k, \quad k \geq 1,$$

the transition function has the form

$$P(u_0, \cdot) = \ell\big(\cdot - S(u_0)\big), \quad \ell \text{ is the law of } \eta_k.$$

Thus, the measures $P(u_0, \cdot)$ and $P(u_1, \cdot)$ are equivalent if and only if the difference $S(u_0) - S(u_1)$ is an admissible shift for ℓ. In the case of Gaussian measures, this is equivalent to a lower bound on the covariance operator. A similar claim is true when comparing the forward and backward stationary laws.

Various approches to entropic fluctuations

- Finite-dimensional approximations

Projecting the Navier–Stokes to the subspace spanned by the first N eigenvectors of the Stokes operator, we obtain an ODE with a quadratic nonlinearity. For the resulting system, one can apply well-known methods to treat the questions mentioned in the Introduction; see **Baiesi–Maes (2005)**.

However, passing to the infinite-dimensional limit is not easy. For instance, one can prove that both truncated and full Navier–Stokes systems satisfy the level-3 LDP with good rate functions given by the Legendre transform of the pressures. On the other hand, it is not known if the rate functions for the truncated problems converge to that for the Navier–Stokes system.

Various approches to entropic fluctuations

- Rough noise

If the noise is irregular in the space variables, and the equation possesses some strong dissipativity properties, one can prove that the laws for the forward and backward stationary processes are equivalent, so that the logarithmic density is well defined. One can then carry out the whole programme mentioned above.

- Motion of a Lagrangian particle

Let us consider the ODE

$$\dot{y} = u(t, y), \quad y(t) \in \mathbb{T}^2, \tag{3}$$

where $u(t, x)$ is a regular solution of the stochastic NS system. One can establish a number of results on the long-time behaviour of trajectories and define the concept of entropy production as the logarithmic density of forward/backward stationary trajectory.

Various approches to entropic fluctuations

- Rough noise

If the noise is irregular in the space variables, and the equation possesses some strong dissipativity properties, one can prove that the laws for the forward and backward stationary processes are equivalent, so that the logarithmic density is well defined. One can then carry out the whole programme mentioned above.

- Motion of a Lagrangian particle

Let us consider the ODE

$$\dot{y} = u(t, y), \quad y(t) \in \mathbb{T}^2, \tag{3}$$

where $u(t, x)$ is a regular solution of the stochastic NS system. One can establish a number of results on the long-time behaviour of trajectories and define the concept of entropy production as the logarithmic density of forward/backward stationary trajectory.

Lagrangian formulation

Let us consider the ODE

$$\dot{y} = u(t, y), \quad y(t) \in \mathbb{T}^2, \tag{4}$$

where $u(t, x)$ is a stationary (in the probabilistic sense) solution of the Navier–Stokes system with random forcing:

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0. \tag{5}$$

We assume that η is a bounded random process, smooth in both variables and piecewise independent:

$$\eta(t, x) = \sum_{k=1}^{\infty} \mathbb{I}_{[k-1,k)}(t) \eta_k(t - k + 1, x), \tag{6}$$

where $\{\eta_k\}$ are i.i.d. random variables in $L^2([0, 1] \times \mathbb{T}^2)$. We assume in addition that the mean values in x is zero.

Lagrangian formulation

Let us consider the ODE

$$\dot{y} = u(t, y), \quad y(t) \in \mathbb{T}^2, \tag{4}$$

where $u(t, x)$ is a stationary (in the probabilistic sense) solution of the Navier–Stokes system with random forcing:

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = \eta(t, x), \quad \mathrm{div}\, u = 0. \tag{5}$$

We assume that η is a bounded random process, smooth in both variables and piecewise independent:

$$\eta(t, x) = \sum_{k=1}^{\infty} \mathbb{I}_{[k-1,k)}(t) \eta_k(t - k + 1, x), \tag{6}$$

where $\{\eta_k\}$ are i.i.d. random variables in $L^2([0,1] \times \mathbb{T}^2)$. We assume in addition that the mean values in x is zero.

Decomposability hypothesis

We assume that the laws ℓ of η_k is decomposable:

(D) *There is an orthonormal basis $\{e_j(t,x)\}$ in $L^2([0,1]\times\mathbb{T}^2)$ that consists of smooth functions such that*

$$\eta_k(t,x)=\sum_{j=1}^{\infty} b_j\xi_{jk}e_j(t,x) \tag{7}$$

where $\{b_j\}$ is sequence of non-zero numbers going to zero sufficiently fast, ξ_{jk} are independent random variables whose laws are smooth, supported by $[-1,1]$, and positive on some interval $[-\delta,\delta]$.

This hypotheses ensures that the Navier–Stokes system (5) has a unique stationary distribution μ. In the following theorem, we fix an arbitrary stationary solution $u(t,x)$ for (5).

Large deviation principle

Theorem

There is a $\mathbb{T}^2$-valued random process $\{z_t, t \geq 0\}$ whose almost every trajectory satisfies (4)*, and the following assertions hold.*

Stationarity. *The process $\{z_t\}$ is stationary.*

Convergence. *Let $p \in \mathbb{T}^2$ be an arbitrary initial point and let $y_t(p)$ be the corresponding trajectory of* (4)*. Then, for any $s \geq 1$, the law of $(y_t(p), \dots, y_{t+s}(p))$ converges exponentially fast in the total variation norm, as $t \to \infty$, to that of $(z_0, \dots, z_s)$.*

Large deviations. *For any $p \in \mathbb{T}^2$, the empirical measures*

$$\nu_t^p = \frac{1}{t}\sum_{k=0}^{t-1} \delta_{\boldsymbol{y}_k(p)}, \quad \boldsymbol{y}_k(p) = (y_l(p), l \geq p),$$

satisfy LDP with some good rate function $\boldsymbol{I} : \mathcal{P}(\mathbb{T}^{\mathbb{Z}_+}) \to [0, +\infty]$.

Large deviation principle

Theorem

There is a $\mathbb{T}^2$-valued random process $\{z_t, t \geq 0\}$ whose almost every trajectory satisfies (4)*, and the following assertions hold.*

Stationarity. *The process $\{z_t\}$ is stationary.*

Convergence. *Let $p \in \mathbb{T}^2$ be an arbitrary initial point and let $y_t(p)$ be the corresponding trajectory of* (4)*. Then, for any $s \geq 1$, the law of $(y_t(p), \dots, y_{t+s}(p))$ converges exponentially fast in the total variation norm, as $t \to \infty$, to that of $(z_0, \dots, z_s)$.*

Large deviations. *For any $p \in \mathbb{T}^2$, the empirical measures*

$$\nu_t^p = \frac{1}{t} \sum_{k=0}^{t-1} \delta_{\boldsymbol{y}_k(p)}, \quad \boldsymbol{y}_k(p) = (y_l(p), l \geq p),$$

satisfy LDP with some good rate function $\boldsymbol{I} : \mathcal{P}(\mathbb{T}^{\mathbb{Z}_+}) \to [0, +\infty]$.

Large deviation principle

Theorem

There is a $\mathbb{T}^2$-valued random process $\{z_t, t \geq 0\}$ whose almost every trajectory satisfies (4)*, and the following assertions hold.*

Stationarity. *The process $\{z_t\}$ is stationary.*

Convergence. *Let $p \in \mathbb{T}^2$ be an arbitrary initial point and let $y_t(p)$ be the corresponding trajectory of* (4)*. Then, for any $s \geq 1$, the law of $(y_t(p), \dots, y_{t+s}(p))$ converges exponentially fast in the total variation norm, as $t \to \infty$, to that of $(z_0, \dots, z_s)$.*

Large deviations. *For any $p \in \mathbb{T}^2$, the empirical measures*

$$\nu_t^p = \frac{1}{t} \sum_{k=0}^{t-1} \delta_{\boldsymbol{y}_k(p)}, \quad \boldsymbol{y}_k(p) = (y_l(p), l \geq p),$$

satisfy LDP with some good rate function $\boldsymbol{I} : \mathcal{P}(\mathbb{T}^{\mathbb{Z}_+}) \to [0, +\infty]$.

Large deviation principle

Theorem

There is a $\mathbb{T}^2$-valued random process $\{z_t, t \geq 0\}$ whose almost every trajectory satisfies (4)*, and the following assertions hold.*

Stationarity. *The process $\{z_t\}$ is stationary.*

Convergence. *Let $p \in \mathbb{T}^2$ be an arbitrary initial point and let $y_t(p)$ be the corresponding trajectory of* (4). *Then, for any $s \geq 1$, the law of $(y_t(p), \dots, y_{t+s}(p))$ converges exponentially fast in the total variation norm, as $t \to \infty$, to that of $(z_0, \dots, z_s)$.*

Large deviations. *For any $p \in \mathbb{T}^2$, the empirical measures*

$$\boldsymbol{\nu}_t^p = \frac{1}{t}\sum_{k=0}^{t-1} \delta_{\boldsymbol{y}_k(p)}, \quad \boldsymbol{y}_k(p) = (y_l(p), l \geq p),$$

satisfy LDP with some good rate function $\boldsymbol{I} : \mathcal{P}(\mathbb{T}^{\mathbb{Z}_+}) \to [0, +\infty]$.

Entropy production

We now assume that

$$\eta(t,x) = a\sum_{j=1}^{\infty} b_j \xi_{jk} e_j(t,x), \quad |a| \geq 1.$$

Theorem

For any $t \geq 1$, the law of $(z_1, \ldots, z_t)$ has a smooth density $\rho_t(x_1, \ldots, x_t)$ with respect to the Lebesgue measure on $\mathbb{T}^{2t}$. Moreover, there is $a_0 > 0$ such that, for $|a| \geq a_0$, the density ρ_t is strictly positive, and the entropy production

$$\sigma_t(\boldsymbol{x}^t) = \log \frac{\rho_t(x_1, \ldots, x_t)}{\rho_t(x_t, \ldots, x_1)}, \quad \boldsymbol{x}^t := (x_1, \ldots, x_t), \tag{8}$$

satisfies the following inequality for any $t \geq 1$:

$$-C \leq t^{-1}\sigma(\boldsymbol{x}^t) \leq C \quad \text{for all } \boldsymbol{x}^t \in \mathbb{T}^{2t}$$

Entropy production

We now assume that

$$\eta(t,x) = a\sum_{j=1}^{\infty} b_j \xi_{jk} e_j(t,x), \quad |a| \geq 1.$$

Theorem

For any $t \geq 1$, the law of $(z_1,\dots,z_t)$ has a smooth density $\rho_t(x_1,\dots,x_t)$ with respect to the Lebesgue measure on $\mathbb{T}^{2t}$. Moreover, there is $a_0 > 0$ such that, for $|a| \geq a_0$, the density ρ_t is strictly positive, and the entropy production

$$\sigma_t(\boldsymbol{x}^t) = \log \frac{\rho_t(x_1,\dots,x_t)}{\rho_t(x_t,\dots,x_1)}, \quad \boldsymbol{x}^t := (x_1,\dots,x_t), \tag{8}$$

satisfies the following inequality for any $t \geq 1$:

$$-C \leq t^{-1}\sigma(\boldsymbol{x}^t) \leq C \quad \text{for all } \boldsymbol{x}^t \in \mathbb{T}^{2t}.$$

Programme

One can single out the following questions arising in the study of entropic fluctuations:

(a) LDP for the occupation measures of trajectories.

(b) Level-3 fluctuation relation for the rate function.

(c) Well-posedness of the entropy production and its relation with the physical notion of transport.

(d) Law of large numbers for the time average of the entropy production.

(e) Strict positivity of the mean entropy production.

(f) Local and global LDP for the time average of the entropy production.

In the case of a Lagrangian particle, we have studied (a) and (c). The remaining points are open and seem to be difficult problems.

Some references

Evans–Searles (1994), Gallavotti–Cohen (1995): Fluctuation relation in numerical experiments and deterministic systems.

Kurchan (1998), Lebowitz–Spohn (1999): GC relation for some classes of stochastic dynamics.

Maes (1999): Interpretation of GC as a Gibbs property.

Eckmann–Pillet–Rey-Bellet (1999), Rey-Bellet–Thomas (2002): GC in chains of anharmonic oscillators.

Ruelle (1999), Gaspard (2005), Rondoni–Mejia Monasterio (2007), Chetrite–Gawędzki (2008), Jakšić–Pillet–Rey-Bellet (2011): Review papers on various aspects of FR

Baiesi–Maes (2005): Interpretation of the Navier–Stokes equations as a heat conduction network

Bodineau–Lefevere (2008), Barato–Chetrite (2015), Cuneo–Jakšić–Pillet–AS (2017): Level-3 LDP and FR

Jakšić–Nersesyan–Pillet–AS (2015, 2019): Randomly forced PDEs