SEMI-RECURSIVE ALGORITHM OF PIECEWISE LINEAR APPROXIMATION OF TWO-DIMENSIONAL FUNCTION BY THE METHOD OF WORST SEGMENT DIVIDING

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Consider a two-dimensional continuous function $F(x, y)$ on a rectangle. Fig.1 shows the graph of a piecewise linear approximation of F. This graph consists of triangles, the projections of which on the OXY plane form a triangular mesh.

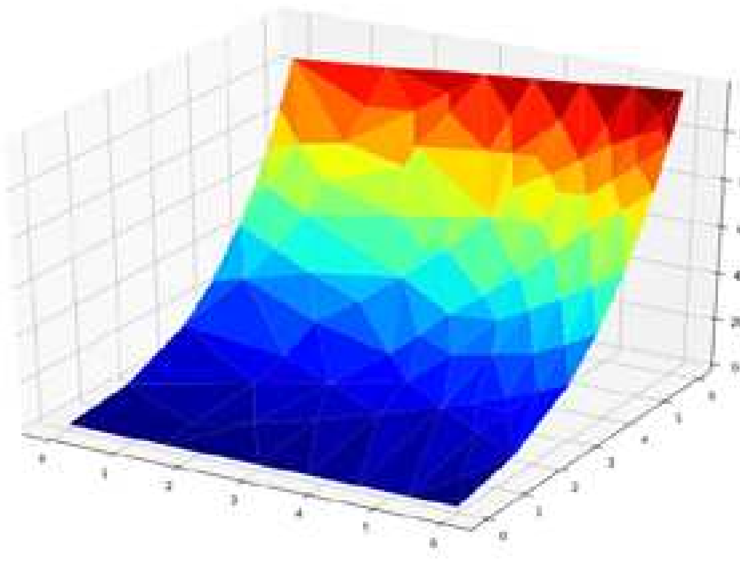

Figure 1

Fig.2 shows the corresponding mesh (or grid, or tessellation).

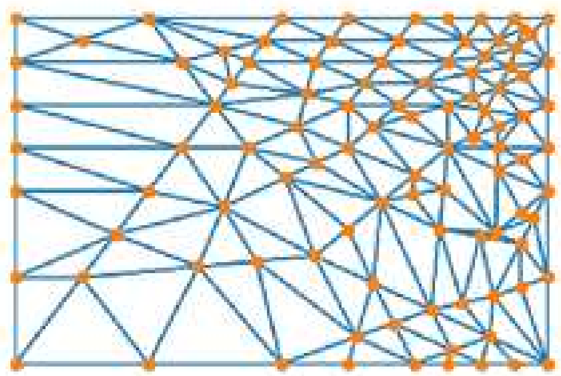

FIGURE 2

Our goal is to minimize the approximation error for a given number of grid knots. Or, minimize the number of grid knots for a given approximation error.

1. The method of dividing of the worst segment

First consider on the rectangle $[a,b] \times [c,d]$ the primary mesh S_4 consisting of 4 vertices: P_1, P_2, P_3, P_4 and two triangles $\Delta P_1P_2P_3$ and $\Delta P_1P_3P_4$. Construct the next point P_5 as follows.

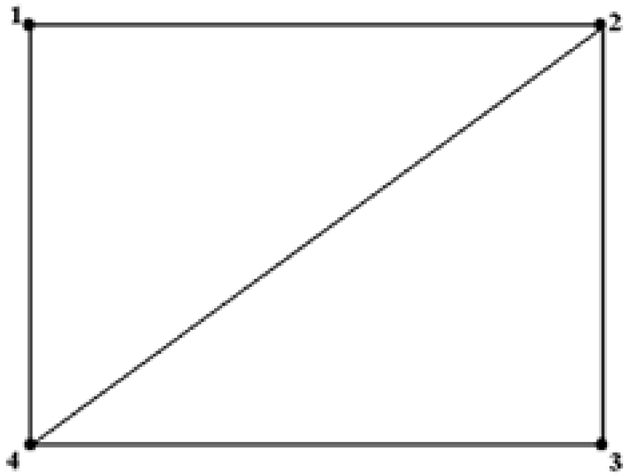

Figure 3

Two knots P_i and P_j are called neighboring and denoted by $P_i \sim P_j$, if they are the endpoints of a side of some triangle of the mesh.

The pairs of neighboring knots we call *the edges* of the mesh. For any edge with vertices P_i and P_j we calculate the difference $|z_i - z_j|$, where $z_i = F(x_i, y_i)$ is the value of our function F at the point P_i, and (x_i, y_i) are Cartesian coordinates of vertex P_i. The edge (P_i, P_j), for which the difference $|z_i - z_j|$ is maximal, we call *the worst segment* of mesh.

We add the new vertex in the middle of the worst segment. If the worst segment is an inner edge (diagonal), then we add two new edges (see Fig. 4).

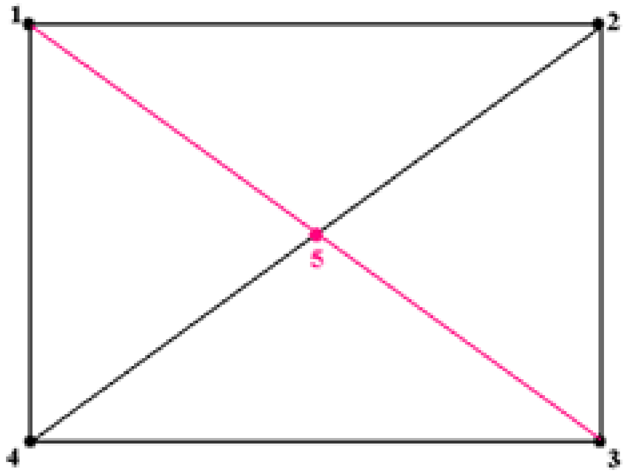

Figure 4

If the worst segment is a boundary edge, then we add one new edge (see Fig. 5).

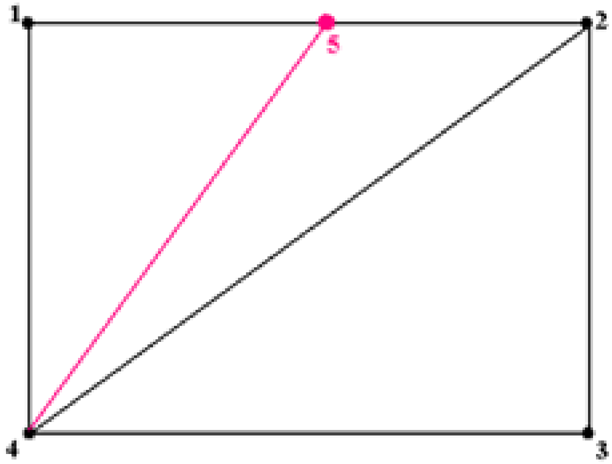

FIGURE 5

An algorithm is called *recursive*, if each subsequent step does not lead to changes in the parameters obtained in the past. In the case of the construction of meshes with variable number of knots, the algorithm will be recursive if the addition of each new knot leaves the old knots and old edges in place.

Algorithm of dividing of the worst segment is recursive.

By $F_n(x, y)$ we denote the piecewise linear approximation of function $F(x, y)$ using triangular mesh with n vertices. The approximation error

$$E_n = \max_{x,y} |F(x, y) - F_n(x, y)|$$

approximately equals

$$E_n \approx \varepsilon_n = \max_{P_i \sim P_j} |F(x_i, y_j) - F(x_i, y_j)|.$$

For any continuous function $F(x, y)$ we have

$$\varepsilon_n \downarrow 0, \quad \text{as } n \to \infty.$$

For any number of vertices n we have (highly likely)

$$\varepsilon_n \text{ for recursive algorithm } \leq \varepsilon_n \text{ for regular lattice.}$$

2. Delaunay triangulation

A triangle mesh $\{D_j\}_{j=1}^m$ is called *Delaunay triangulation* with knots $M_n = \{P_i\}_{i=1}^n$, if for any triangle D_j

$$int\ K(D_j) \cap \{P_i\}_{i=1}^n = \emptyset, \quad j = 1, ..., m,$$

where $K(D)$ is the circumscribing circle of triangle D.

The Delaunay triangulation with the system of knots M_n we denote by $D(M_n)$. In 1934 B. N. Delaunay was proved that for any finite set of points M_n there exists a Delaunay triangulation $D(M_n)$ (not necessarily unique).

The mesh resulting from the operation of the recursive algorithm of the worst segment dividing is not necessarily a Delaunay triangulation. This depends on the function $F(x, y)$.

The mesh in Fig. 4 is Delaunay triangulation, however the mesh in Fig. 5 is not Delaunay triangulation.

3. Semi-recursive algorithm

The operation of replacing a longer diagonal with a shorter one in a tetragon is called *the "flip" operation.*

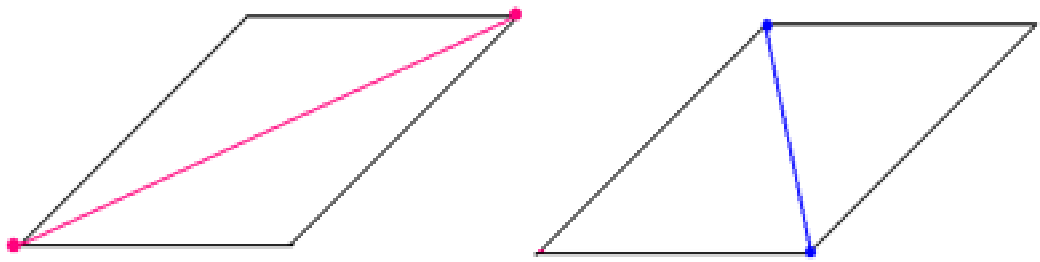

Figure 6

We have constructed a semi-recursive (or highly likely recursive) algorithm for constructing a piecewise linear approximation of a two-dimensional function by dividing the worst segment. When adding a new vertex, all previous vertices and almost all edges remain in their places. One edge may change if the "flip" operation is applicable.

Theorem 1. *For any approximated function F and any number of vertices, the mesh resulting from the operation of the semi-recursive algorithm is a Delaunay triangulation.*

For any number of vertices n and any continuous function F we have also

$$\varepsilon_n \text{ for semi-recursive algorithm } \leq \varepsilon_n \text{ for recursive algorithm.}$$

4. Applications to mathematical physics

Let $F(x, y)$ be an unknown solution of a boundary value problem

$$\begin{cases} L(F(x,y)) = 0, & (x,y) \in D, \\ F(x,y) = f, & (x,y) \in \partial D, \end{cases}$$

where L is a differential operator acting on rectangle D with boundary ∂D.

Assume that we can construct the piecewise linear approximation of F for given mesh S_n, and we can determine the worst segment of the mesh S_n. Then using the semi-recursive algorithm we obtain new method of solution of boundary value problem using the meshes with variable number of vertices.

Theorem 2. (H.Sukiasyan, 2008) *To numerical solution of the Maxwell equation for magnetic field by the finite elements method the best mesh is Delaunay triangulation.*

By Theorems 1 and 2 we obtain

Theorem 3. *To construction of the mesh for numerical solution of the Maxwell equation for magnetic field by the finite elements method the best algorithm is the semi-recursive algorithm.*

Example. We solve the Maxwell equation for magnetic field using the semi-recursive algorithm. The Figure 7 shows the solution,

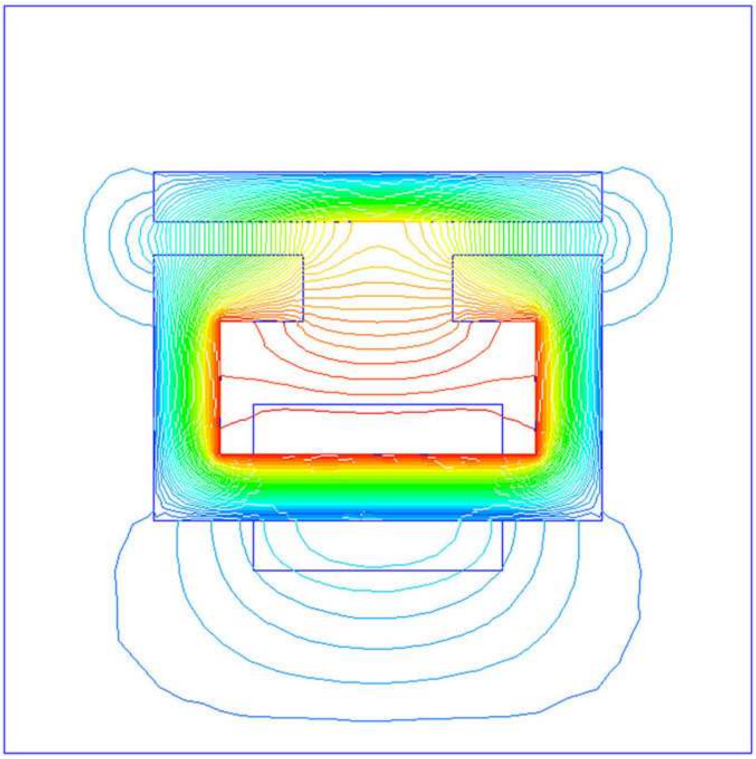

FIGURE 7

While the Figure 8 shows the corresponding mesh.

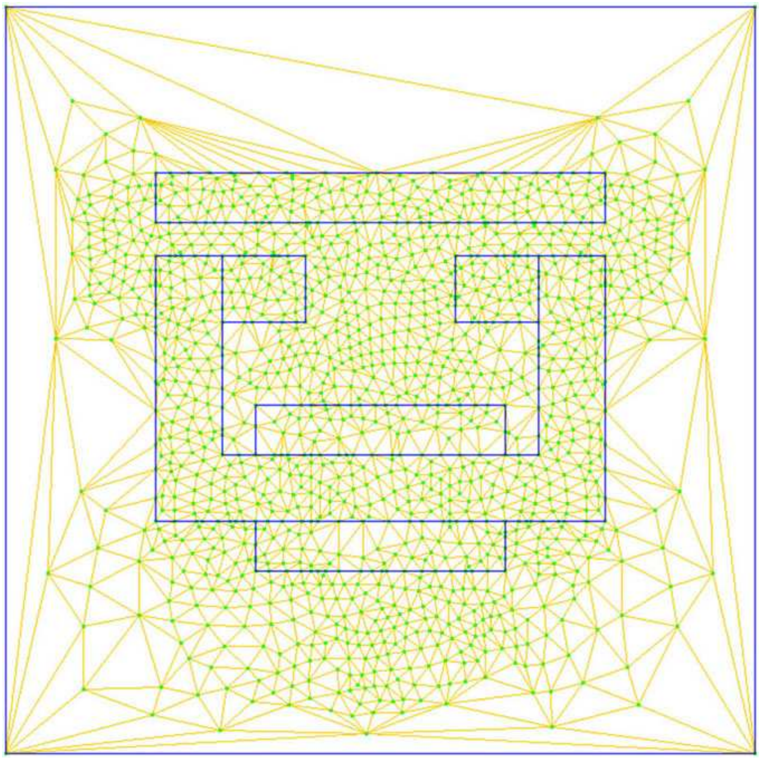

FIGURE 8