Extremal states decomposition in quantum spin systems

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Stochastic and Analytic Methods in Mathematical Physics
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♢ Quantum spins, decomposition in extremal states

♢ Results for the complete graph, for the Heisenberg / XY / quantum interchange models

♢ The random interchange model and the joint distribution of cycle lengths

Setting

Spin parameter $S \in \frac{1}{2}\mathbb{N}$, domain $\Lambda \Subset \mathbb{Z}^d$

Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda}$; $\dim \mathcal{H} = (2S+1)^{|\Lambda|}$

Hamiltonian $H_\Lambda = -\sum_{\substack{x,y \in \Lambda \\ \|x-y\|=1}} h_{xy}$, where

$$h_{xy} = \begin{cases} S_x^1 S_y^1 + S_x^2 S_y^2 + \Delta S_x^3 S_y^3 & \text{for XY, Heisenberg} \\ T_{xy} & \text{for quantum interchange} \end{cases}$$

$T_{xy}|\varphi_x\rangle \otimes |\varphi_y\rangle = |\varphi_y\rangle \otimes |\varphi_x\rangle$ is the transposition operator (For $S = \frac{1}{2}$, we have $T_{xy} = 2\vec{S}_x \cdot \vec{S}_y + \frac{1}{2}$, **[Tóth '93]**)

Finite-volutme Gibbs state: $\langle \cdot \rangle_{\beta,\Lambda} = \frac{1}{Z(\beta,\Lambda)} \mathrm{Tr} \cdot \mathrm{e}^{-\beta H_\Lambda}$

Infinite-volume Gibbs state: $\langle \cdot \rangle_\beta \lim_{\Lambda \Uparrow \mathbb{Z}^d} \langle \cdot \rangle_{\beta,\Lambda}$

Extremal state decomposition

Let $\mathcal{G}_\beta$: set of infinite-volume states (KMS condition) at inverse temperature β

The goal is to characterise $\mathcal{G}_\beta$

General results: $\mathcal{G}_\beta$ is a Choquet simplex, i.e. a compact convex subset of a normed space where every point is given by a convex combination of extremal points. That is, there exists an index set ψ_β and a probability measure μ_β on ψ_β s.t.

$$\langle \cdot \rangle_\beta = \int_{\psi_\beta} \langle \cdot \rangle_\psi \mathrm{d}\mu(\psi)$$

Infinite-volume states are *clustering*:

$$\lim_{\|x\| \to \infty} \big[\langle A \tau_x B \rangle_\psi - \langle A \rangle_\psi \langle \tau_x B \rangle_\psi \big] = 0$$

(Think of a ferromagnetic state)

"Spin-density" Laplace transform of μ

Tom Spencer's suggestion: Look at the "spin-density" Laplace transform of μ

Let A an operator on $\mathbb{C}^{2S+1}$ and $A_x = A \otimes \mathbb{1}_{\Lambda \setminus \{x\}}$

Define $\Phi_\beta(h) = \lim_{\Lambda \Uparrow \mathbb{Z}^d} \big\langle \, \mathrm{e}^{\frac{h}{|\Lambda|} \sum_x A_x} \, \big\rangle_{\beta,\Lambda}$

We expect that

$$
\begin{aligned}
\lim_{\Lambda \Uparrow \mathbb{Z}^d} \Big\langle \, \mathrm{e}^{\frac{h}{|\Lambda|} \sum_{x \in \Lambda} A_x} \, \Big\rangle_{\beta,\Lambda} &\overset{(?)}{=} \lim_{\boldsymbol{\Lambda} \Uparrow \mathbb{Z}^d} \lim_{\boldsymbol{\Lambda'} \Uparrow \mathbb{Z}^d} \Big\langle \, \mathrm{e}^{\frac{h}{|\boldsymbol{\Lambda}|} \sum_{x \in \boldsymbol{\Lambda}} A_x} \, \Big\rangle_{\beta,\boldsymbol{\Lambda'}} \\
&= \lim_{\Lambda \Uparrow \mathbb{Z}^d} \int_{\psi_\beta} \Big\langle \, \mathrm{e}^{\frac{h}{|\Lambda|} \sum_x A_x} \, \Big\rangle_\psi \mathrm{d}\mu(\psi) \\
&= \int_{\psi_\beta} \mathrm{e}^{h \langle A_0 \rangle_\psi} \, \mathrm{d}\mu(\psi)
\end{aligned}
$$

Our models have continuous symmetry (XY: U(1); Heisenberg: SU(2); interchange: $\mathrm{SU}(2S+1)$), given by the group G

We expect that the set of extremal states is given by a *single* state $\langle\cdot\rangle_0$ and its orbit by G

Then $\psi_\beta \equiv G$ and $\langle A\rangle_\psi = \langle U_\psi^{-1} A U_\psi\rangle_0$. Then

$$\Phi_\beta(h) = \int_{\psi_\beta} \mathrm{e}^{h\langle U_\psi^{-1} A U_\psi\rangle_0}\, \mathrm{d}\mu(\psi)$$

where μ is the Haar measure on G

Given an explicit ψ_β, this is usually easy to calculate

Notice that $\Phi_\beta(h)$ does not depend much on the dimension d of $\mathbb{Z}^d$

β small: If $\int\langle A\rangle_\psi \mathrm{d}\mu(\psi) = 0$, then $\Phi_\beta(h) = 1$ for all h, as can be proved by cluster expansion, see e.g. **[Poghosyan et.al.]**

Model 1 — Heisenberg ferromagnet

On the complete graph, $\Lambda = \{1, \dots, n\}$: $H_\Lambda = -\frac{1}{n} \sum_{x,y=1}^n \vec{S}_x \cdot \vec{S}_y$

Theorem [Björnberg, Fröhlich, U '19]
There exists $m^*(\beta) \geqslant 0$ such that

$$\lim_{n\to\infty} \Big\langle \exp \frac{h}{n} \sum_{x=1}^n S_x^{(3)} \Big\rangle_{\beta,n} = \frac{\sinh(hm^*(\beta))}{hm^*(\beta)}$$

Further, $m^*(\beta) > 0$ iff $\beta > \beta_{\rm c} = \frac{3/2}{S^2+S}$

Remark: For $S = \frac{1}{2}$, this goes back to **[Tóth '91; Penrose '92]**

The proof: Since $H_\Lambda \simeq \frac{1}{n} (\sum_x \vec{S}_x)^2$, we can use the theory of additions of spins and asymptotic analysis

Symmetry breaking

Symmetry breaking: $G = \psi_\beta = \mathbb{S}^2$. For $\vec{a} \in \mathbb{S}^2$:

$$\langle \cdot \rangle_{\vec{a}} = \lim_{\eta \to 0+} \lim_{n \to \infty} \langle \cdot \rangle_{H_\Lambda - \eta \sum_x \vec{a} \cdot \vec{S}_x}$$

Let $m^*(\beta) = \langle S_1^{(3)} \rangle_{\vec{e}_3}$. Then

$$\begin{aligned} \Phi_\beta(h) &= \int_{\mathbb{S}^2} \mathrm{e}^{h \langle \vec{a} \cdot \vec{S}_1 \rangle_{\vec{e}_3}} \, \mathrm{d}\vec{a} \\ &= \int_{\mathbb{S}^2} \mathrm{e}^{h a_3 m^*(\beta)} \, \mathrm{d}\vec{a} \\ &= \frac{\sinh(h m^*(\beta))}{h m^*(\beta)} \end{aligned}$$

Model 2 — Quantum XY

Theorem **[Björnberg, Fröhlich, U '19]**
For the same $m^*(\beta) \geqslant 0$ as before, we have

$$\lim_{n\to\infty} \Big\langle \exp \frac{h}{n} \sum_{x=1}^{n} S_x^{(1)} \Big\rangle_{\beta,n} = \sum_{k\,\geqslant\,0} \frac{1}{(k!)^2} \big(\tfrac12 m^*(\beta)\big)^{2k}$$

As before, $m^*(\beta) > 0$ iff $\beta > \beta_c = \frac{3/2}{S^2+S}$

We can also check that

$$\Phi_\beta(h) = \int_{\mathbb{S}^1} e^{h\langle \vec{a}\cdot\vec{S}_1\rangle_{\vec{e}_1}} \, d\vec{a} = \sum_{k\,\geqslant\,0} \frac{1}{(k!)^2} \big(\tfrac12 m^*(\beta)\big)^{2k}$$

Model 3 — Quantum interchange

On the complete graph $\Lambda = \{1, \dots, n\}$: $H_\Lambda = -\frac{1}{n} \sum_{x,y=1}^{n} T_{xy}$ with T_{xy} the transposition operator

Theorem **[Björnberg, Fröhlich, U '19]**
There exists $m^*(\beta) \geqslant 0$ such that

$$\lim_{n\to\infty} \Big\langle \exp \frac{h}{n} \sum_{x=1}^{n} S_x^{(3)} \Big\rangle_{\beta,n} = \Big(\frac{\sinh(hm^*(\beta))}{hm^*(\beta)} \Big)^{2S}$$

Further, $m^*(\beta) > 0$ iff $\beta > \beta_{\rm c} = \begin{cases} 2 & \text{if } S = \frac{1}{2} \\ \frac{4S}{2S-1} \log(2S) & \text{if } S \geqslant 1 \end{cases}$

The proof is easy for $S = \frac{1}{2}$ (Heisenberg model) but not for $S \geqslant 1$. It uses representation theory for the symmetric group, results from **[Alon, Kozma '13]** and **[Berestycki, Kozma '15]**, and also symmetric polynomials

Symmetry breaking of quantum interchange model

We expect that extremal states are given by

$$\langle\cdot\rangle_Q = \lim_{\eta\to0+}\lim_{n\to\infty}\langle\cdot\rangle_{H_\Lambda-\eta\sum_x Q_x},$$

where Q is a rank-1 projector in $\mathbb{S}^{2S+1}$

The group is then $\mathbb{CP}^{2S}$ (i.e. complex vectors of $2S+1$ components, of norm 1, where $\vec{a}\sim\vec{b}$ if $\vec{a}=\mathrm{e}^{\mathrm{i}\alpha}\,\vec{b}$ for some α)

We have $\int_{\mathbb{CP}^{2S}}\mathrm{e}^{\langle A_1\rangle_{\vec{a}}}\,\mathrm{d}\vec{a}=\int_{\mathcal{U}(2S+1)}\mathrm{e}^{\langle U^{-1}A_1U\rangle_0}\,\mathrm{d}U$

Using the **Harish-Chandra-Itzykson-Zuber formula**, and further calculations, we get that $\Phi_\beta(h)$ is indeed given by the formula of the theorem

Tóth's representation of spin $\frac{1}{2}$ Heisenberg ferromagnet

Arbitrary finite graph $(\Lambda, \mathcal{E})$. On each edge $e \in \mathcal{E}$ is associated the interval $[0, \beta]$

Independent Poisson point processes on each interval. Notation: $\rho(\mathrm{d}\omega)$

$\mathcal{L}(\omega)$: set of loops

Partition function:
$Z = \int 2^{|\mathcal{L}(\omega)|} \rho(\mathrm{d}\omega)$

Probability measure:
$\frac{1}{Z} 2^{|\mathcal{L}(\omega)|} \rho(\mathrm{d}\omega)$

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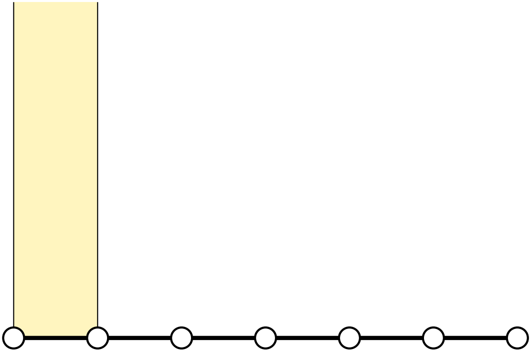

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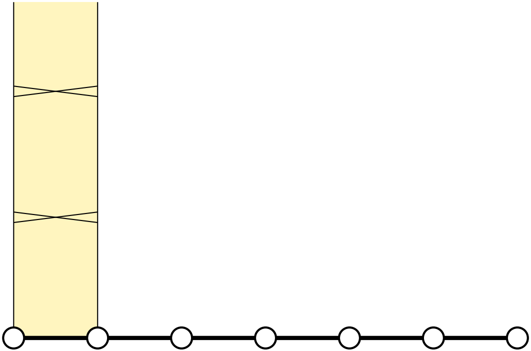

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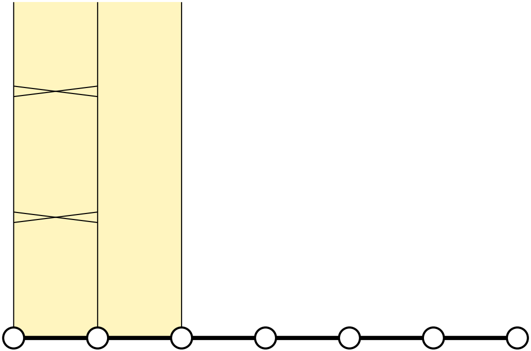

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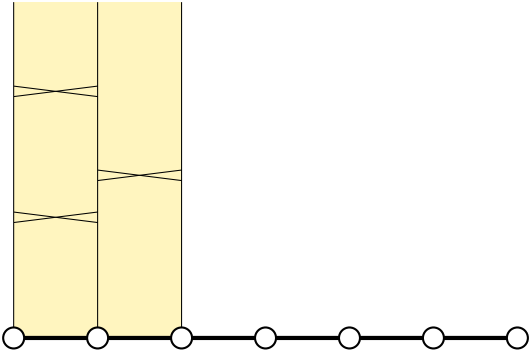

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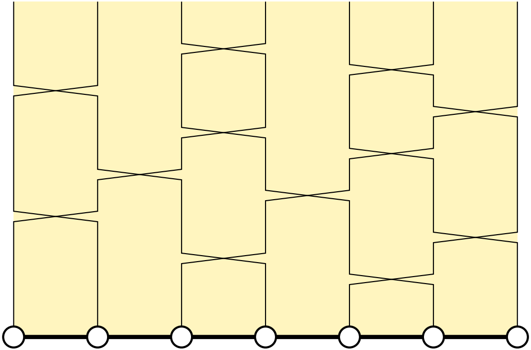

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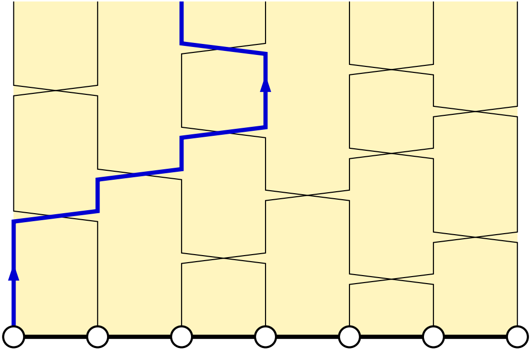

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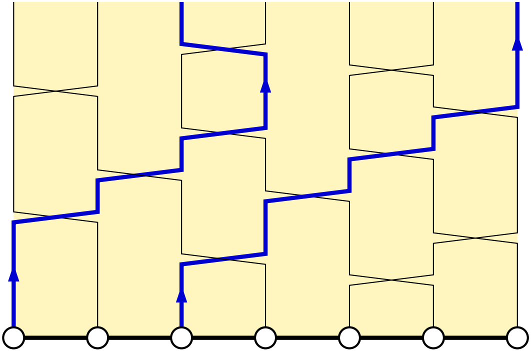

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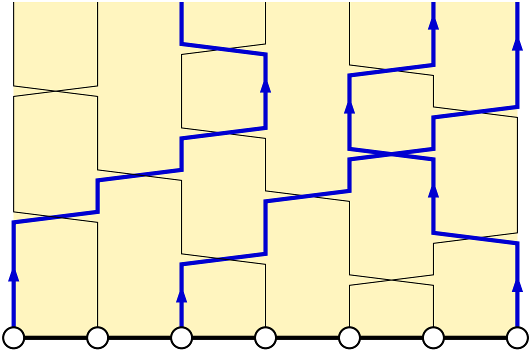

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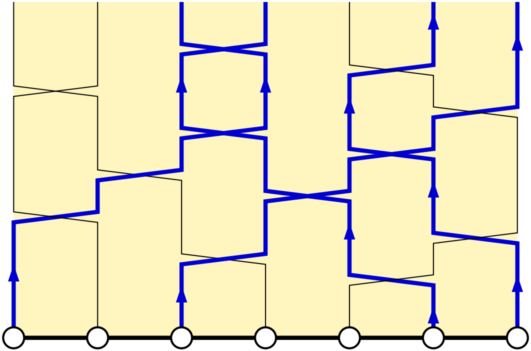

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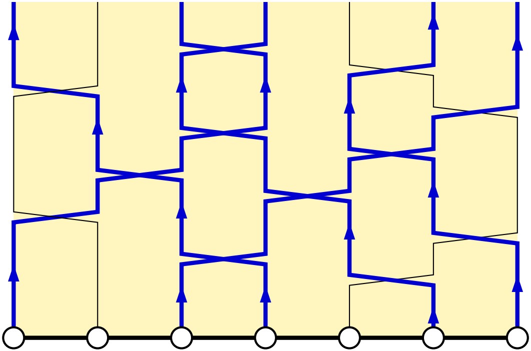

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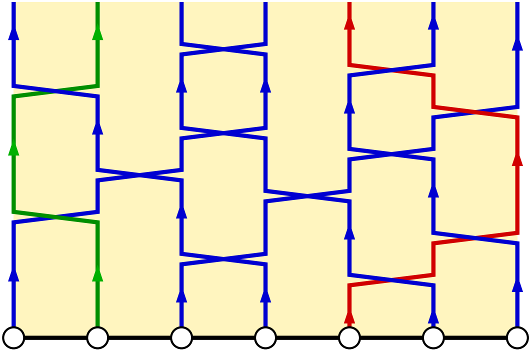

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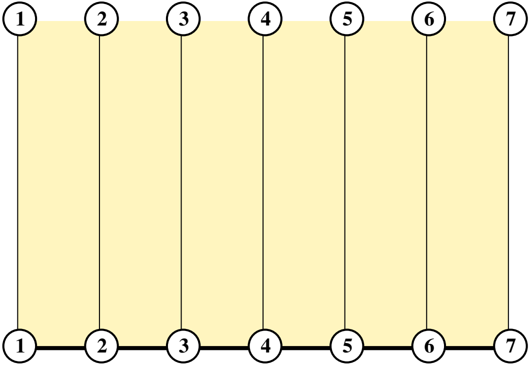
Random interchange model

Stochastic process that gives lattice permutations



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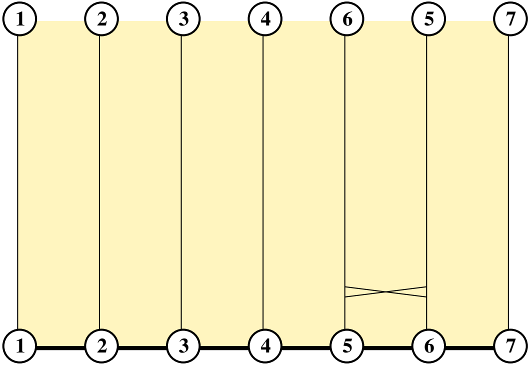

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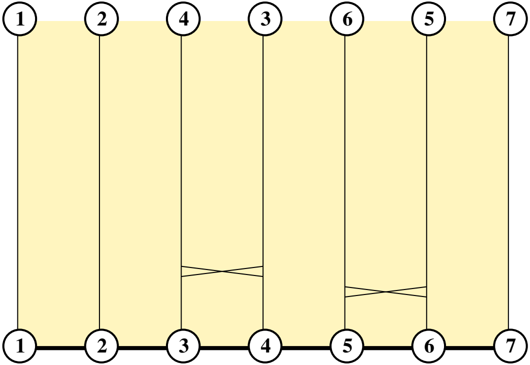

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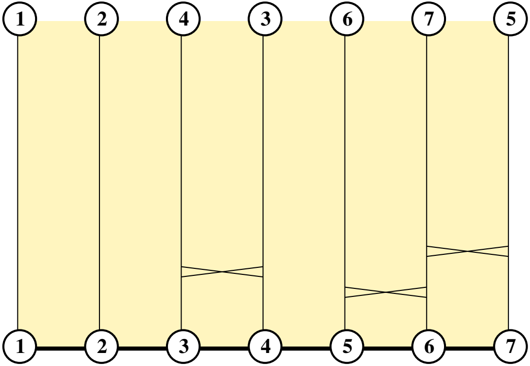

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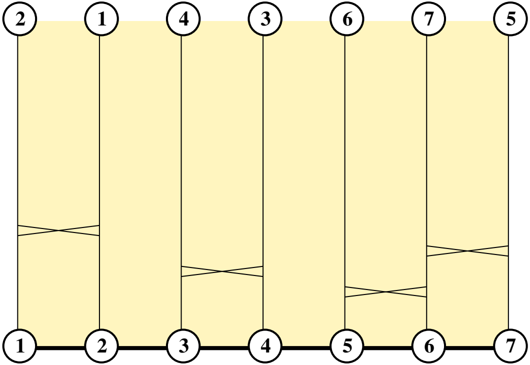

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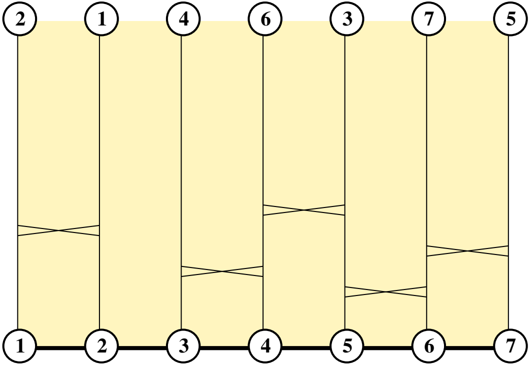

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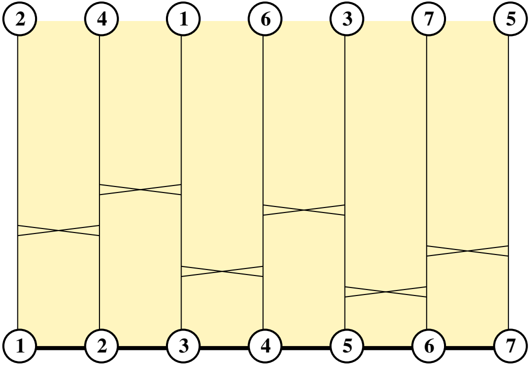

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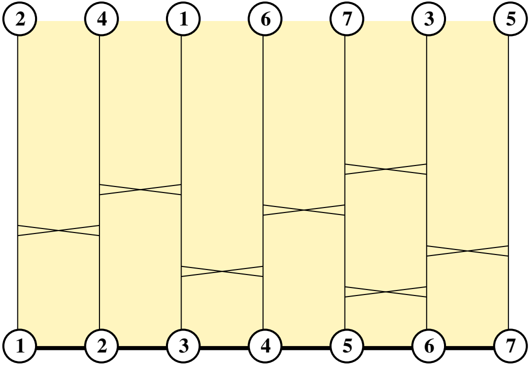

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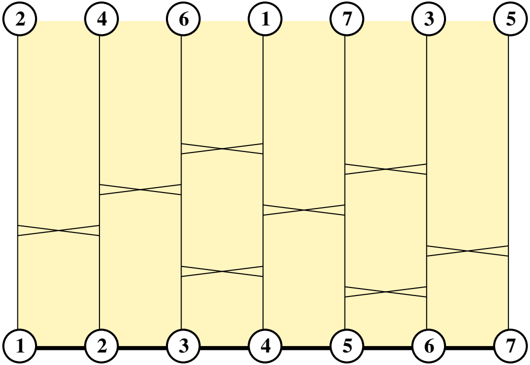

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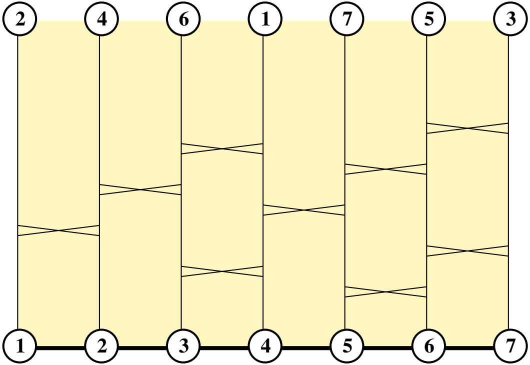

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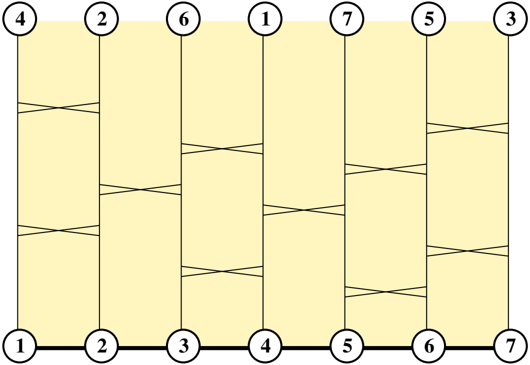

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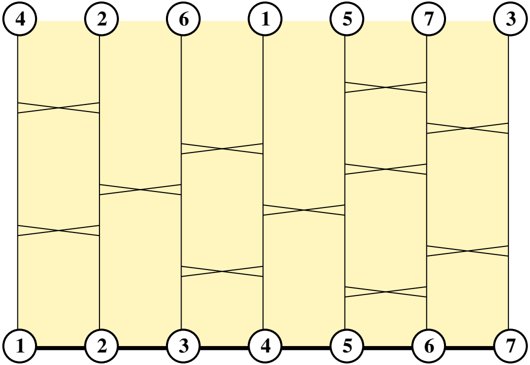

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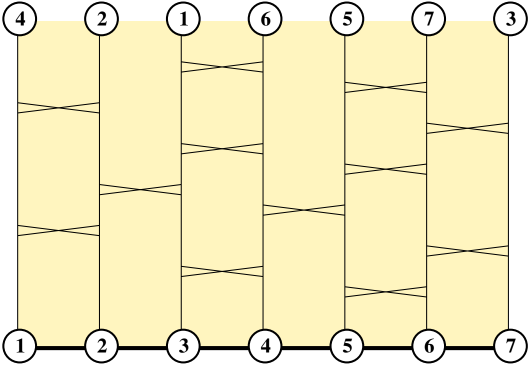

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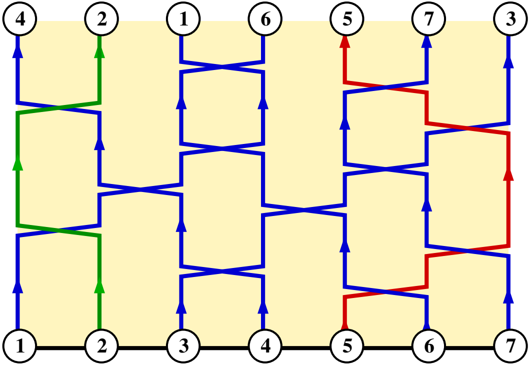


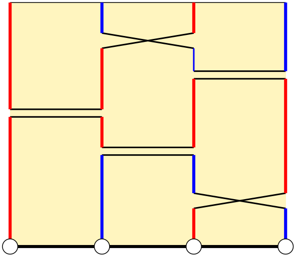
Extension of the representation

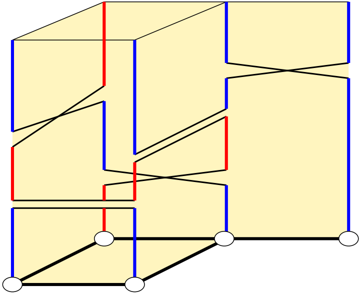
Let ρ be independent Poisson point processes on $\bigtimes_{e \in \mathcal{E}} [0, \beta]$, where:

- **crosses** appear with intensity u
- **double bars** appear with intensity $1 - u$

$\mathcal{L}(\omega)$: set of loops of the realisation ω

Relevant probability measure: $\boxed{\frac{1}{Z} \theta^{|\mathcal{L}(\omega)|} \rho(\mathrm{d}\omega)}$ with $\theta > 0$





Relations between quantum spins and random loops

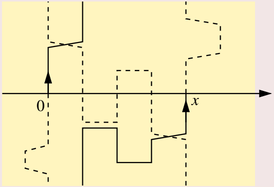
Hamiltonian $H^{(u)} = - \sum_{\|x-y\|=1} \Big(S_x^1 S_y^1 + (2u-1) S_x^2 S_y^2 + S_x^3 S_y^3 \Big)$

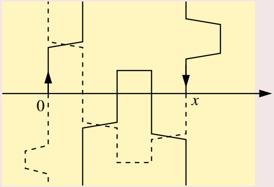
Theorem **[Tóth '93; Aizenman, Nachtergaele '94; U '13]**

$$\mathrm{Tr}\, \mathrm{e}^{-\beta H^{(u)}} = \int 2^{|\mathcal{L}(\omega)|} \rho(\mathrm{d}\omega)$$

$$\langle S_0^1 S_x^1 \rangle_\beta = \langle S_0^3 S_x^3 \rangle_\beta = \tfrac14 \mathbb{P}_{L,\beta,u,2}\big(0 \leftrightarrow x\big)$$

$$\langle S_0^2 S_x^2 \rangle_\beta = \tfrac14 \big[\mathbb{P}\big(0 \leftrightarrow x, \text{same direction}\big) - \mathbb{P}\big(0 \leftrightarrow x, \text{opposite dir.}\big) \big]$$





Joint distribution of the lengths of the loops

Given realisation ω, let $\ell_1(\omega), \ell_2(\omega), \dots, \ell_{K(\omega)}(\omega)$ be the lengths of the loops in decreasing order. This is a **random partition** of the interval $[0,1]$:

$$\Big(\frac{\ell_1(\omega)}{|\Lambda|}, \frac{\ell_2(\omega)}{|\Lambda|}, \dots, \frac{\ell_{K(\omega)}(\omega)}{|\Lambda|} \Big)$$

Small loops have lengths of order 1, so the random partition looks like:

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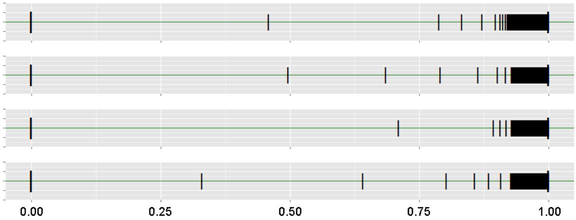
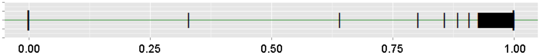


Universal behaviour for $d \geqslant 3$

Here: random interchange model ($u = 1$, $\theta = 1$)



Conjecture: There exists $m = m(\beta) \in [0, 1)$ such that

- $\displaystyle\sum_{j=1}^{k} \frac{\ell_j(\omega)}{L^d} \underset{k\to\infty}{\overset{L\to\infty}{\longrightarrow}} m$ a.s.
- $\displaystyle\Big(\frac{\ell_1(\omega)}{L^d}, \dots, \frac{\ell_k(\omega)}{L^d}\Big) \underset{k\to\infty}{\overset{L\to\infty}{\longrightarrow}} \mathrm{PD}_1[0, m]$ in distribution

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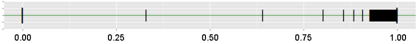

More general model, $u \in [0,1]$, $\theta > 0$

Conjecture: There exists $m = m(\beta, u) \in [0,1)$ such that

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  where $\boxed{\vartheta = \theta}$ for $u = 0, 1$ and $\boxed{\vartheta = \theta/2}$ for $u \in (0,1)$

Proof for annealed spatial permutations **[Betz, U '11]**

Numerical results :

- **[Grosskinsky, Lovisolo, U '12]** for lattice permutations
- **[Nahum, Chalker et.al. '13]** for O(N) loop models (they also calculate moments of joint distribution using "supersymmetry")
- **[Barp^2, Briol, U '15]** for random loops

Rigorous results for the random interchange model on the complete graph

Let $q(t) = \frac{1}{\theta}\big(e^{-St} + e^{-(S-1)t} + \dots + e^{St}\big) = \dfrac{\sinh(\frac{\theta}{2}t)}{\theta \sinh(\frac{1}{2}t)}$ where $2S+1=\theta$

Theorem [Björnberg, Fröhlich, U '19]
Let $\theta = 2, 3, 4, \dots$. For all $h \in \mathbb{C}$, we have

$$\lim_{n\to\infty} \mathbb{E}_{n,\beta,\theta}\Big[\prod_{i \geqslant 1} q(\tfrac{h}{n}\ell_i)\Big] = \Big[\frac{\sinh(\frac{1}{2}hm^*(\beta))}{\frac{1}{2}hm^*(\beta)}\Big]^{2S}$$

$$= \mathbb{E}_{\mathrm{PD}(\theta)}\Big[\prod_{i \geqslant 1} q(hm^*(\beta)X_i)\Big]$$

In the case $\theta = 1$, stronger results have been proved **[Schramm '05]**

Interesting question: How good is this characterisation of the random partition? (Partial results in **[Björnberg, Mailler, Mörters, U '19]**)

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- Universal behaviour of loop soups in $d \geqslant 3$: **Poisson-Dirichlet**
- The spin-density Laplace transform of quantum spins is equal to the expectation of certain functions in the random partitions formed by loop lengths

THANK YOU!