Some remark about local large deviations principle

N.D. Vvedenskaya, IITP RAS

Erevan, September, 2019

Thanks to my coauthors

A.V.Logachov, Y.M.Suhov, A.A.Yambartsev

We are interested in probability of rare events.

Examples of such events :

1) Reproduction and reduction of population. Some balance conditions are presumed. What is the probability of sharp growth or decrease of population ? What is the probability of given form of growth or decrease of population ?

2) Supermarket with many self-service devices. What is the probability of many servers being busy at the same time ?

In both examples the intensity of events lineary depends on the state of the system.

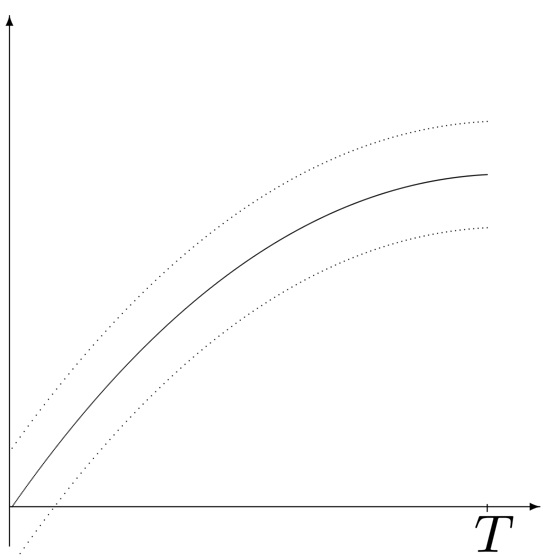
Consider a continuous-time burth and death Markov procces on $\mathbb{Z}^1_+$ with jump lengths $\pm\mathbf{1}$. The jump rates at state $\xi(x) \in \mathbb{Z}_+$ depend on $\xi(\cdot)$ **polynomially**. The aim is to assess the log-asymptotics of probabilities of long excursions of the process confined to a (re-scaled) tube around a given function $f(t)$.

Let $f(s) \in C[0,1]$, $f(0) = 0$, $f(s) > 0$ as $s > 0$.

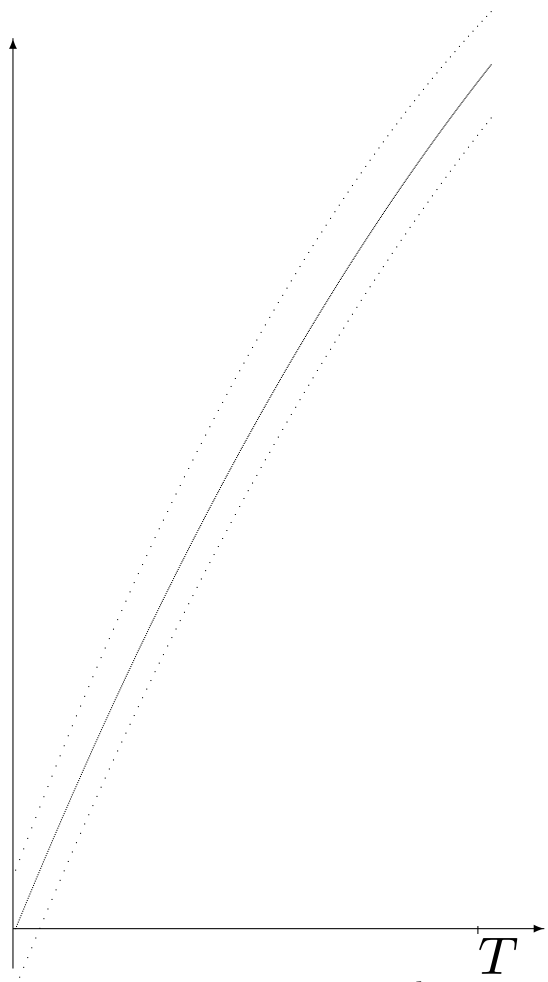
Scaling: $f(t) = T^b f(t/T)$, $0 \leq t \leq T$, b>0.

Parameter $T \to \infty$.

$f(t) = T^b f(t/T)$, $0 \leq t \leq T$



$b = 1, \xi(t) = O(T)$



$b > 1, \xi(t) = O(T^b)$

Let $\xi(0)=0$. As $t>0$

$$\mathbf{P}(\xi(t+\tau_x)=x+1)=\frac{\lambda(x)}{h(x)},\ \ \mathbf{P}(\xi(t+\tau_x)=x-1)=\frac{\mu(x)}{h(x)},$$

exponential distribution of $\xi(\cdot)$-jumps with rate

$h(x):=\lambda(x)+\mu(x)$, max$(l,m)>0$,

$$\lambda(x)=P(x)x^l,\ \mu(x)=Q(m)x^m,$$

$\lim_{x\to\infty}\frac{P(ax)}{P(x)}=\frac{Q(ax)}{Q(x)}=1,\ a>0,\ \lim_{x\to\infty}\frac{P(x)}{Q(x)}\neq 1.$

Existence of such a process as $l\leq 1$ can be established by standard arguments. As $l>1$ the process may "explote"at a random finite time.

But we are interested in events where $\xi(\cdot)$ is inside a bounded domain $U_\varepsilon(f)$ during finite time T. The probability of such events is positive.

The condition for $\xi(\cdot) \in U_\varepsilon(f)$ are formulated in terms of re-scaled process

$$\xi_{1,b}(t) = \frac{\xi(tT)}{T^b}, \ (\xi_{1,1}(t) = \frac{\xi(tT)}{T} \text{ as } b = 1),$$

$$\xi_{1,b} \in U_\varepsilon(f) \ \longleftrightarrow \ sup_{t\in[0,1]}|f(t) - \xi_{1,b}(t)| \le \varepsilon.$$

LLDP :

Let $\psi(t)$ be a function: $\lim_{t\to\infty} \psi(t) = \infty$. Let $I = I(f)$ be LD rate functional.

The processes $\xi_{1,b}$ satisfy the LLDP in space $(\mathbb{D}[0,1], \rho)$ with the rate functional I and the normalizing function $\psi(T)$ if $\forall \ f \in \mathbb{D}[0,1]$

$$\lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}(\xi_{1,b} \in U_\varepsilon(f))$$

$$= \lim_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}(\xi_{1,b} \in U_\varepsilon(f)) = -I(f).$$

We used the method of [1],[2] and got that in case $b < b(l+m)/2+1$ ($T^b \ll T^{b(l+m)/2+1}$)

$$\psi = \phi(P,Q)T^{\max(l,m)+1},\ I = -\int_0^1 f^{\max(l,m)}(t)dt.$$

in case $b > b(l+m)/2+1$ ($T^b \gg T^{b(l+m)/2+1}$)

$$\psi = \phi_1(P,Q)R(T)T^b,\ I = -\max_{0\le t\le 1} f'(t),$$

$R(T) = b \ln T$.

What brings such difference ?

Too "high"trajectory, too many steps....

Consider the expression for

$$\mathbf{P}(\xi(\cdot) \in \mathbf{U}_\varepsilon(\mathbf{f}))$$

$$= \mathbb{E}\prod_{n,i} e^{-(PT^{bl}f^l(t_i)+QT^{bm}f^m(t_i))(t_i-t_{i-1})} \times \nu(t_i),$$

$$n \geq n^+ = \max_{0<t<1} |f'(t)|bT^b \ln T, \quad 0 < i \leq n,$$

$$\nu(t_i) = \begin{cases} \lambda(\xi(t_{i-1})), & \text{if } \xi(t_i) > \xi(t_{i-1}); \\ \mu(\xi(t_{i-1})), & \text{if } \xi(t_i) < \xi(t_{i-1}). \end{cases}$$

$\sum_i (PT^{bl}f^l(t_i)+QT^{bm}f^m(t_i))(t_i-t_{i-1}) \to \phi(P,Q)T^{b\max(l,m)+1}I(f)$, $I(f) = \int_0^1 f^{\max(l,m)}dt$.

$$\lim_{T\to\infty} \mathbb{E}\prod_{n,i} \nu(t_i) = \sum_{n\geq n^+} \frac{[\phi_0(P,Q)T^{b(l+m)/2+1}I_1(f)]^n}{n!}$$

The sum $\sum_{n\geq n^+} \frac{[\phi_0(P,Q)T^{b(l+m)/2+1}I_1(f)]^n}{n!} \sim e^{\phi_0(P,Q)T^{b(l+m)/2+1}I_1(f)}$ as $n^+ < O(T^{b(l+m)/2+1})$.

and $\sum_{n\geq n^+} \frac{[\phi_1(P,Q)T^{b(l+m)/2+1}I_1(f)]^n}{n!} \sim \left(\frac{e\phi_0(P,Q)T^{b(l+m)/2+1}I_1(f)}{n^+}\right)^{n^+} \frac{1}{\sqrt{2\pi n^+}}$ as $n^+ > O(T^{b(l+m)/2+1})$

(We do not present here the case $T^b \sim T^{b(l+m)/2+1}$ where happeneds a "jump").

Comare $T^{b\,\max(l,m)}, T^{b(l+m)/2}$ and $T^{n^+} \sim T^b \ln T$.

The expressions for $\mathbf{P}(\xi_T \in U_\varepsilon(f))$ demonstrate probability of "ordinary"(upper line) and "super"(lower line) LLDP :

$$\lim_{T\to\infty} \ln[\mathbf{P}(\xi(\cdot) \in \mathbf{U}_\varepsilon(\mathbf{f}))] =$$

$$= \begin{cases} -\psi(T)T^{\max(l,m)+1}\int_0^1 f^{\max(l,m)}dt, & as\ b(l+m)/2+1 > b; \\ -\psi_1 T^b \ln T \max_{0<t<1}|f'(t)|, & as\ b(l+m)/2+1 < b, \end{cases}$$

$(\psi, \psi_1$ depend on $P(T^b), Q(T^b))$.

References

1. A. Mogulsky, E. Pechersky, A. Yambartsev. *Large deviations for excursions of non-homogeneous Markov processes, Electronic Commun. Probab., 2014. V. 19, PP 1-8.*

2. N.D.Vvedenskaya, A.V.Logachev, Y.M.Suhov, A.A.Yambartsev. *Local large deviations for inhomogeneous birth-and-death processes, Probl. Inf. Tr., 2018, V. 54, № 3, Pp 73-91*

THANK YOU