Asymptotic pointwise estimates for heat kernels of convolution type operators

(Large time asymptotics for the distribution of continuous time jump processes in $\mathbb{R}^d$)

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September 2019

A. Grigor'yan , Yu. Kondratiev, A. Piatnitski and E. Zhizhina,
Pointwise estimates for heat kernels of convolution type operators,
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Asymptotics of fundamental solutions for time fractional equations with convolution kernels, Arxiv.org/abs/1907.08677

Outline

- Fundamental solution of non-local operators. Assumptions
- Existing results on heat kernel behaviour.
- Normal and moderate deviation region.
- Large deviation region.
- Extra large deviation region.
- Time fractional equations with convolution kernels.

Convolution type operators

We consider a zero order convolution type non-local operator A in $L^2(\mathbb{R}^d)$, $d \geq 1$. It is defined by

$$Af(x) = \int_{\mathbb{R}^d} a(x-y)\big(f(y) - f(x)\big)dy,$$

where the convolution kernel $a = a(z)$ is a non-negative integrable function, $a : \mathbb{R}^d \mapsto \mathbb{R}^+$. If $\int_{\mathbb{R}^d} a(z)\, dz = 1$ then

$$Af = a * f - f.$$

In this case A is the generator of a continuous time Markov jump process with the jump distribution $a(z)$.

Assumptions

We assume that the convolution kernel $a(\cdot)$ possesses the following properties

- Boundedness

C1

$$a(x) \geq 0; \qquad a(x) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d). \tag{1}$$

- Symmetry

C2

$$a(x) = a(-x) \quad \text{for all } x \in \mathbb{R}^d.$$

- Normalization and second moments

C3

$$\int_{\mathbb{R}^d} a(x)dx = 1, \quad \int_{\mathbb{R}^d} |x|^2 a(x)dx < \infty. \tag{2}$$

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Fast decay of dispersal kernels

We mostly deal with dispersal kernels a that decay at infinity at least exponentially.

(Super)exponential decay. There exists $p \geq 1$ such that

Condition A_p

$$a(x) \leq C_1 e^{-b|x|^p}, \quad b > 0.$$

Fundamental solution

Our goal is to study the large time behaviour of the fundamental solution of the parabolic problem

$$\begin{aligned} \partial_t u(x,t) = Au(x,t) = a * u - u, \quad (x,t) \in \mathbb{R}^d \times (0,+\infty), \\ u(x,0) = \delta(x). \end{aligned} \tag{3}$$

In terms of probability theory:

Let $\xi^0(t)$ be a continuous time jump Markov process with jump intensity equal to 1 and with jump distribution $a(\cdot)$, and assume that $\xi^0(0) = 0$. Then $u(\cdot, t)$ is the law of $\xi^0(t)$.

Fundamental solution

Since A is a bounded operator in $L^2(\mathbb{R}^d)$ we have

$$e^{tA} = e^{-t}\, e^{ta\star} = e^{-t} \sum_{k=0}^{\infty} t^k \frac{a^{\star k}}{k!} = e^{-t}\mathbb{I} + e^{-t} \sum_{k=1}^{\infty} t^k \frac{a^{\star k}}{k!}$$

and

$$u(x,t) = e^{tA}\delta(x) = e^{-t}\delta(x) + e^{-t} \sum_{k=1}^{\infty} \frac{t^k}{k!}\, a^{\star k}(x).$$

Observe that $u(\cdot, t)$ consists of a singular part at zero and a regular part:

$$u(x,t) = e^{-t}\delta(x) + v(x,t), \quad v(\cdot,t) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d).$$

We focus on obtaining point-wise upper and lower bounds for the regular part $v(x,t)$ as $t \to \infty$.

Existing results (some of ....)

- The heat kernel of the classical heat equation in $\mathbb{R}^d$

$$\partial_t u - \Delta u = 0,$$

is given by the Gauss-Weierstrass function

$$p_t(x, y) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

- For a more general parabolic equation

$$\partial_t u - Lu = 0,$$

where L is a uniformly elliptic second-order operator in divergence form, D.G. Aronson ('67) proved the following Gaussian estimates for the heat kernel

$$p_t(x, y) \asymp \frac{C}{t^{d/2}} \exp\left(-\frac{|x-y|^2}{ct}\right).$$

Existing results

- A simplest heat equation with non-local elliptic part is

$$\partial_t u + (-\Delta)^{\alpha/2} u = 0, \quad \text{where } \ 0 < \alpha < 2.$$

The heat kernel satisfies the following estimates A.Bendikov ('94)

$$p_t(x,y) \asymp \frac{C}{t^{d/\alpha}} \left(1 + \frac{|x-y|}{t^{1/\alpha}}\right)^{-(d+\alpha)} \tag{4}$$

Remark

Note that $(-\Delta)^{\alpha/2}$ is an integro-differential operator of the form

$$(-\Delta)^{\alpha/2} f(x) = c_{d,\alpha} \, \mathrm{p.v.} \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x-y|^{d+\alpha}} dy. \tag{5}$$

The heavy tail of the heat kernel in the estimate (4) is a consequence of the heavy integral kernel in (5).

Existing results

• E. Davies ('93) studied the large time behaviour of the heat kernel of the discrete Laplace operator on a countable locally finite connected graph. Sharp upper bound in the case of uniformly bounded degree $\deg(x)$:

$$p_t\left(x,y\right) \leq \exp\left(-ct\Phi\left(\frac{d\left(x,y\right)}{ct}\right)\right), \tag{6}$$

with

$$\Phi(r) = r\ln(r+\sqrt{r^2+1}) - \sqrt{1+r^2}.$$

Since

$$\Phi(r) \sim \frac{r^2}{2} \text{ as } r \to 0 \quad \text{and} \quad \Phi(r) \sim r\ln r \text{ as } r \to \infty,$$

the estimate (6) implies

$$p_t\left(x,y\right) \leq \exp\left(-\frac{d^2\left(x,y\right)}{ct}\right) \quad \text{for small } \frac{d\left(x,y\right)}{t},$$

$$p_t\left(x,y\right) \leq \exp\left(-cd\left(x,y\right)\ln\frac{d\left(x,y\right)}{ct}\right) \quad \text{for large } \frac{d\left(x,y\right)}{t}.$$

Existing results

- C. Brandle, M. Chaves, R. Ferreira and J. Rossi ('11 – '14) considered some special cases of dispersal kernels and obtained (non-sharp) two-sided estimates in the region $|x| \gg t$.

- S. Molchanov and E. Yarovaya ('13) obtained local large deviations upper bounds for a symmetric random walk in $\mathbb{Z}^d$ in the case of super-exponential decay $(p > 1)$ of the transition probabilities $a(z)$.

- A number of sharp large deviation results for jump Markov process have been obtained by A. Mogulskii ('17).

Regions in (x, t) space

The large time behaviour of the studied fundamental solution (heat kernel) depends crucially on the relation between $|x|$ and t. We consider separately four different regions in the (x, t) space. Namely,

I. $|x| \leq rt^{1/2}(1 + o(1))$ (standard deviations region)

II. $|x| = r\, t^{\frac{1+\delta}{2}}(1 + o(1))$, $\delta \in (0, 1)$ (moderate deviations region)

III. $|x| = rt(1 + o(1))$ $(\delta = 1)$ (large deviations region)

IV. $|x| = rt^{\frac{1+\delta}{2}}(1 + o(1))$, $\delta > 1$ ("extra-large" deviations region)

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Normal deviations. The result.

We begin with the case $|x| \sim \sqrt{t}$. For any $r > 0$ in the region $|x| \le r\sqrt{t}$ the local limit theorem holds.

Theorem

Assume that $a(\cdot)$ satisfies conditions C_1–C_3. Then for the function $v(x,t)$ the following asymptotic relation holds as $t \to \infty$:

$$v(x,t) = (2\pi t)^{-\frac{d}{2}} |\det(\sigma)|^{-\frac{1}{2}} \; e^{-\frac{(\sigma^{-1}x,x)}{2t}} \left(1 + o(1)\right),$$

where

$$\sigma^{ij} = \int_{\mathbb{R}^d} x_i x_j \, a(x) \, dx.$$

Moderate deviations. The result.

In the region $\sqrt{t} \ll |x| \ll t$ the following result holds.

Theorem

Assume that conditions C_1–C_3 and A_p, $p \geq 1$, are fulfilled. Then for $x = \mathbf{r} t^{\frac{1+\delta}{2}}(1+o(1))$ with $0 < \delta < 1$ and $\mathbf{r} \in R^d \backslash \{0\}$ the following asymptotic relation holds as $t \to \infty$:

$$v(x,t) = e^{-\frac{(\sigma^{-1}x,x)}{2t}(1+o(1))} = e^{-\frac{1}{2}(\sigma^{-1}\mathbf{r},\mathbf{r})\, t^{\delta}(1+o(1))}.$$

where $o(1)$ tends to zero as $t \to \infty$.

Large deviations region.

In order to formulate the result in the region $|x| \sim t$ we should first introduce a number of auxiliary quantities.

Let X be a random vector in $\mathbb{R}^d$ with distribution $a(\cdot)$. If condition A_p is fulfilled with some $p \geq 1$ then X has finite exponential moment

$$\Lambda(\gamma) = \mathbb{E}e^{\gamma \cdot X}$$

at least for small enough $\gamma \in \mathbb{R}^d$.

We define the cumulant generating function $L(\gamma) = \ln \Lambda(\gamma)$, and introduce $I(r)$, $r \in \mathbb{R}^d$, as its Legendre transform:

$$I(r) = \sup_{\gamma} (\gamma \cdot r - L(\gamma)), \quad r, \gamma \in R^d.$$

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Large deviations region.

Denote by ξ_r a positive solution of the equation

$$\ln \xi = I(\xi r) - \xi r \cdot \nabla I(\xi r), \quad \xi \in R,$$

Lemma

Let $a(x)$ satisfy conditions C_1–C_3 and A_p with some $p \geq 1$. Then for any $r \in R^d \backslash \{0\}$ the above equation has a unique solution ξ_r, and $0 < \xi_r < 1$.

We introduce the *rate* function

$$\Phi(r) = 1 - \frac{1}{\xi_r}\big(1 + \ln \xi_r - I(\xi_r r)\big).$$

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Large deviations region: logarithmic asymptotics.

The case $p = 1$: Condition L_1

$$\mathbb{E}|X|e^{bX\cdot\theta} = \infty \quad \text{for any} \quad \theta \in S^{d-1}.$$

Theorem (Large time asymptotics)

Let conditions C_1–C_3 and A_p, with some $p \geq 1$ be fulfilled , and assume additionally that in the case $p = 1$ condition L_1 holds. Then for $x = rt(1 + o(1))$ with $r \in R^d \setminus \{0\}$ we have

$$v(x,t) = e^{-\Phi(r)t(1+o(1))} \quad \text{as} \quad t \to \infty.$$

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Properties of $\Phi(r)$.

$$\Phi(0) = 0, \qquad \Phi(r) > 0, \quad \text{if } \; r \neq 0,$$

Φ is a convex function, and the following limit relations hold:

$$\Phi(r) = \tfrac{1}{2}\,\sigma^{-1} r \cdot r\,(1 + o(1)), \qquad \text{as } \; r \to 0;$$

$$\Phi(r) \to \infty, \qquad \text{as } \; r \to \infty.$$

If $a(x)$ has a finite support, then

$$\Phi(r) \geq c|r|\ln|r|, \qquad \text{as } \; |r| \to \infty.$$

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Behaviour of $\Phi(r)$ at infinity.

If condition

Condition L_p

$$C_2 e^{-b|x|^p} \le a(x) \le C_1 e^{-b|x|^p}, \quad p \ge 1.$$

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1) If $p = 1$, then

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Extra-large deviations region.

In the region $|x| \gg t$ we obtain only a point-wise upper bound for the heat kernel.

Theorem

Assume that $a(x)$ satisfies conditions C_1–C_3 and A_p with some $p \geq 1$. Then for $|x| = rt^{\frac{1+\delta}{2}}(1+o(1))$ with $\delta > 1$ and $r > 0$ the following asymptotic upper bound holds:

$$v(x,t) \leq e^{-c_p t^{\frac{\delta+1}{2}} (\ln t)^{\frac{p-1}{p}} (1+o(1))}, \quad \text{as } t \to \infty,$$

where the constant $c_p = c_p(b,r)$ depends on b, r and p.

If $a(\cdot)$ has a finite support, then for $|x| = rt^{\frac{\delta+1}{2}}(1+o(1))$ with $\delta > 1$ we have

$$v(x,t) \leq e^{-\tilde{c}\, t^{\frac{\delta+1}{2}} \ln t (1+o(1))}, \quad \text{as } t \to \infty,$$

where $\tilde{c}$ depends on r, δ and the support of $a(\cdot)$.

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Conclusions.

In the case when $a(x)$ has a finite support the estimate on $v(x,t)$ in the region of of extra-large deviations $|x| = rt^{\varkappa}(1+o(1))$ with $\varkappa > 1$ can be written as

$$v(x,t) \leq e^{-\tilde{k}\,|x|\,\ln\frac{|x|}{t}}.$$

Let us compare with the estimate by E. Davies ('93)

$$p_t\,(x,y) \leq \exp\left(-cd\,(x,y)\ln\frac{d\,(x,y)}{ct}\right) \quad \text{for large } \frac{d\,(x,y)}{t}.$$

Conclusions.

Let us compare the large time behaviour of the classical heat kernel and the heat kernel for a jump Markov process in different regions. The classical heat kernel

$$p_t(x,y) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

In the region $|x| \ll t$ (normal and moderate deviations regions) the logarithmic large time asymptotics of $v(x,t)$ and $p_t(x)$ coincide.

If $x = rt$, $r \in \mathbb{R}^d$, (large deviations area) then

$$\log p_t(x) = -\frac{r^2}{4} t(1+o(1)), \qquad \log v(x,t) = -\Phi(r) t(1+o(1)).$$

Since for large $|r|$

$$\Phi(r) \sim |r| (\ln|r|)^{\frac{p-1}{p}} \ll |r|^2, \quad p \geq 1,$$

we conclude that in the large and extra-large deviations regions the heat kernel of the jump Markov process has **much heavier tails** than that of the diffusion.

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Fundamental solutions of time fractional parabolic type equations with convolution kernels

We study a solution $w_\alpha(x,t)$ of the following fractional time parabolic problem:

Time fractional non-local problem, $0 < \alpha < 1$

$$\begin{aligned} \partial_t^\alpha w_\alpha &= a * w_\alpha - w_\alpha \\ w_\alpha\big|_{t=0} &= \delta_0 \end{aligned}$$

Here ∂_t^α is the fractional derivative (the Caputo derivative) of the order $\alpha, 0 < \alpha < 1$, and $a(x)$ is a convolution kernel.

We assume that

- **F1**

$$a(x) \geq 0; \qquad a(x) = a(-x);$$

- **F2**

$$a(x) \in C_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d); \qquad \int_{\mathbb{R}^d} a(x)dx = 1;$$

- **F3**

$$a(x) \leq C_p e^{-b|x|^p}, \quad \mathbf{p > 1}.$$

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Representation for w_α

The solution $w_\alpha(x,t)$ admits the following representation in terms of the fundamental solution $u(x,t)$ of a nonlocal heat equation:

Z.-Q.Chen '17, Z.-Q.Chen, P.Kim, T.Kumagai, J. Wang '18

$$w_\alpha(x,t) = \int_0^\infty u(x,r) d_r \mathbb{P}(S_r \geq t) = \int\limits_0^\infty u(x,r) G_t^\alpha(r) dr,$$

Here $S = \{S_r,\ r \geq 0\}$ is the α-stable subordinator with the Laplace transform

$$\mathbb{E} e^{-\lambda S_r} = e^{-r\lambda^\alpha}$$

and

$$G_t^\alpha(r) = d_r \,\mathrm{Pr}\{V_t^\alpha \leq r\}$$

is the density of the inverse α-stable subordinator V_t^α.

Using the representation for the Laplace transform of V_t^α:

$$\mathbb{E}e^{-\lambda V_t^\alpha} = E_\alpha(-\lambda t^\alpha), \quad E_\alpha \text{ is the Mittag-Leffler function}$$

and

$$u(x,t) = e^{-t}\delta_0(x) + v(x,t) \quad \text{with} \quad v(x,t) = \sum_{k=1}^{\infty} \frac{a^{*k}(x)}{k!}\, t^k\, e^{-t}$$

we obtain

$$w_\alpha(x,t) \;=\; E_\alpha(-t^\alpha)\delta_0(x) \;+\; p_\alpha(x,t),$$

where the regular part of $w_\alpha(x,t)$ equals

$$p_\alpha(x,t) \;=\; \sum_{k=1}^{\infty} \frac{a^{*k}(x)}{k!}\, t^{\alpha k}\, E_\alpha^{(k)}(-t^\alpha)$$

Using the representation for the Laplace transform of V_t^α:

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$$u(x,t) = e^{-t}\delta_0(x) + v(x,t) \quad \text{with} \quad v(x,t) = \sum_{k=1}^{\infty} \frac{a^{*k}(x)}{k!}\, t^k\, e^{-t}$$

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We consider separately the following regions:

- $|x|$ is bounded
- (Subnormal deviations)

$$1 \ll |x| \ll t^{\frac{\alpha}{2}}$$

  or equivalently, $|x(t)| \to \infty$ such that there exists an increasing function $r(t)$, $r(0) = 0$, $\lim_{t\to\infty} r(t) = +\infty$ that $r(t) \le |x| \le (r(t)+1)^{-1} t^{\alpha/2}$ for all sufficiently large t.
- (Normal deviations) $x = vt^{\alpha/2}(1 + o(1))$, where v is an arbitrary vector in $\mathbb{R}^d \setminus \{0\}$.
- (Moderate deviations) $x = vt^{\beta}(1 + o(1))$ with $\frac{\alpha}{2} < \beta < 1$ and $v \in \mathbb{R}^d \setminus \{0\}$.
- (Large deviations) $x = vt(1 + o(1))$ with $v \in \mathbb{R}^d \setminus \{0\}$.
- (Extra large deviations) $|x| \gg t$, i.e. $\lim_{t\to\infty} \frac{|x(t)|}{t} = \infty$.

Asymptotics for $p_\alpha(x,t)$ as $t \to \infty$

- If $|x|$ is bounded, then

$$\begin{aligned} \mathsf{c}_- t^{-\frac{\alpha}{2}} \le p(x,t) \le \mathsf{c}_+ t^{-\frac{\alpha}{2}} & \quad \text{if } d = 1, \\ \mathsf{c}_- t^{-\alpha} \log t \le p(x,t) \le \mathsf{c}_+ t^{-\alpha} \log t & \quad \text{if } d = 2, \\ \mathsf{c}_- t^{-\alpha} \le p(x,t) \le \mathsf{c}_+ t^{-\alpha} & \quad \text{if } d \ge 3. \end{aligned}$$

- If $1 \ll |x| \ll t^{\frac{\alpha}{2}}$, then

Subnormal deviations

$$\begin{aligned} \mathsf{c}_- t^{-\frac{\alpha}{2}} \le p(x,t) \le \mathsf{c}_+ t^{-\frac{\alpha}{2}} & \quad \text{if } d = 1, \\ \mathsf{c}_- t^{-\alpha} \log\left(\frac{t^\alpha}{|x|^2}\right) \le p(x,t) \le \mathsf{c}_+ t^{-\alpha} \log\left(\frac{t^\alpha}{|x|^2}\right) & \quad \text{if } d = 2, \\ \mathsf{c}_- t^{-\alpha}|x|^{2-d} \le p(x,t) \le \mathsf{c}_+ t^{-\alpha}|x|^{2-d} & \quad \text{if } d \ge 3. \end{aligned}$$

Asymptotics for $p_\alpha(x,t)$ as $t \to \infty$

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&\mathsf{c}_- t^{-\frac{\alpha}{2}} \le p(x,t) \le \mathsf{c}_+ t^{-\frac{\alpha}{2}} && \text{if } d = 1,\\
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- If $x = vt^{\alpha/2}(1 + o(1))$ with $v \in \mathbb{R}^d \setminus \{0\}$, then

Normal deviations

$$p(x,t) = t^{-\frac{d\alpha}{2}} \int_0^\infty W_\alpha(s)\Psi(v,s)\,ds\,\big(1 + o(1)\big).$$

where

$$\Psi(v,s) = \frac{1}{|\mathrm{det}\sigma|^{1/2}(2\pi s)^{d/2}} \exp\Big(-\frac{(\sigma^{-1}v, v)}{s}\Big)$$

and $W_\alpha(s)$ is the Wright function that is expressed via the density $G_t^\alpha(r)$ of the inverse subordinator.

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- If $x = vt^{\beta}(1+o(1))$ with $\frac{\alpha}{2} < \beta < 1$ and $v \in \mathbb{R}^d \setminus \{0\}$, then

Moderate deviations

$$p(x,t) = \exp\Big\{ -K_v\, t^{\frac{2\beta-\alpha}{2-\alpha}}(1+o(1))\Big\}$$

with a constant K_v depending on α and v.

- If $x = vt(1+o(1))$ with $v \in \mathbb{R}^d \setminus \{0\}$, then

Large deviations

$$p(x,t) = \exp\Big\{ -\mathtt{F}(v)t(1+o(1))\Big\}.$$

- If $|x| \gg t$, then

Extra-large deviations

$$p(x,t) \le \exp\Big\{ -\mathrm{c}_+|x|\Big(\log\Big|\frac{x}{t}\Big|\Big)^{\frac{p-1}{p}}\Big\}$$

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Conclusions

1. In the region of large deviations $|x| \sim t$ the form of the asymptotics of $p(x,t)$ is the same as for $v(x,t)$, but the rate functions are different.

2. In the region of extra large deviations the asymptotic upper bounds for $p(x,t)$ and for $v(x,t)$ are the same.