Asymptotic pointwise estimates for heat kernels of convolution type operators

(Large time asymptotics for the distribution of continuous time jump processes in $\mathbb{R}^d$)

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A. Grigor’yan, Yu. Kondratiev, A. Piatnitski and E. Zhizhina,
*Pointwise estimates for heat kernels of convolution type operators*,
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Yu. Kondratiev, A. Piatnitski and E. Zhizhina,
*Asymptotics of fundamental solutions for time fractional equations with convolution kernels*, Arxiv.org/abs/1907.08677
Outline

• Fundamental solution of non-local operators. Assumptions
• Existing results on heat kernel behaviour.
• Normal and moderate deviation region.
• Large deviation region.
• Extra large deviation region.
• Time fractional equations with convolution kernels.
We consider a zero order convolution type non-local operator $A$ in $L^2(\mathbb{R}^d)$, $d \geq 1$. It is defined by

$$Af(x) = \int_{\mathbb{R}^d} a(x - y)(f(y) - f(x)) \, dy,$$

where the convolution kernel $a = a(z)$ is a non-negative integrable function, $a : \mathbb{R}^d \mapsto \mathbb{R}^+$. If $\int_{\mathbb{R}^d} a(z) \, dz = 1$ then

$$Af = a \ast f - f.$$

In this case $A$ is the generator of a continuous time Markov jump process with the jump distribution $a(z)$. 
**Assumptions**

We assume that the convolution kernel \( a(\cdot) \) possesses the following properties

- **Boundedness**
  
  \[ \text{C1} \]
  
  \[
  a(x) \geq 0; \quad a(x) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d). \tag{1}
  \]

- **Symmetry**
  
  \[ \text{C2} \]
  
  \[
  a(x) = a(-x) \quad \text{for all } x \in \mathbb{R}^d.
  \]

- **Normalization and second moments**
  
  \[ \text{C3} \]
  
  \[
  \int_{\mathbb{R}^d} a(x) \, dx = 1, \quad \int_{\mathbb{R}^d} |x|^2 a(x) \, dx < \infty. \tag{2}
  \]
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  (1)

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\]  

(2)
We mostly deal with dispersal kernels $a$ that decay at infinity at least exponentially.

(Super)exponential decay. There exists $p \geq 1$ such that

**Condition $A_p$**

$$a(x) \leq C_1 e^{-b|x|^p}, \quad b > 0.$$
Our goal is to study the large time behaviour of the fundamental solution of the parabolic problem

$$\partial_t u(x, t) = Au(x, t) = a \ast u - u, \quad (x, t) \in \mathbb{R}^d \times (0, +\infty),$$

$$u(x, 0) = \delta(x).$$

*(3)*

*In terms of probability theory:*

Let $\xi^0(t)$ be a continuous time jump Markov process with jump intensity equal to 1 and with jump distribution $a(\cdot)$, and assume that $\xi^0(0) = 0$. Then $u(\cdot, t)$ is the law of $\xi^0(t)$. 
Fundamental solution

Since $A$ is a bounded operator in $L^2(\mathbb{R}^d)$ we have

$$e^{tA} = e^{-t} e^{ta^*} = e^{-t} \sum_{k=0}^{\infty} t^k \frac{a^* k}{k!} = e^{-t} I + e^{-t} \sum_{k=1}^{\infty} t^k \frac{a^* k}{k!}$$

and

$$u(x, t) = e^{tA} \delta(x) = e^{-t} \delta(x) + e^{-t} \sum_{k=1}^{\infty} \frac{t^k}{k!} a^*(x).$$

Observe that $u(\cdot, t)$ consists of a singular part at zero and a regular part:

$$u(x, t) = e^{-t} \delta(x) + v(x, t), \quad v(\cdot, t) \in L^{\infty}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d).$$

We focus on obtaining point-wise upper and lower bounds for the regular part $v(x, t)$ as $t \to \infty$. 
Existing results (some of ....)

- The heat kernel of the classical heat equation in $\mathbb{R}^d$

$$
\partial_t u - \Delta u = 0,
$$

is given by the Gauss-Weierstrass function

$$
p_t(x, y) = \frac{1}{(4\pi t)^{d/2}} \exp \left( -\frac{|x - y|^2}{4t} \right).
$$

- For a more general parabolic equation

$$
\partial_t u - Lu = 0,
$$

where $L$ is a uniformly elliptic second-order operator in divergence form, D.G. Aronson ('67) proved the following Gaussian estimates for the heat kernel

$$
p_t(x, y) \asymp \frac{C}{t^{d/2}} \exp \left( -\frac{|x - y|^2}{ct} \right).
$$
Existing results

• A simplest heat equation with non-local elliptic part is

$$\partial_t u + (-\Delta)^{\alpha/2} u = 0, \quad \text{where } 0 < \alpha < 2.$$ 

The heat kernel satisfies the following estimates A.Bendikov ('94)

$$p_t(x, y) \asymp \frac{C}{t^{d/\alpha}} \left(1 + \frac{|x - y|}{t^{1/\alpha}}\right)^{-(d+\alpha)} \quad (4)$$

Remark

Note that $$(-\Delta)^{\alpha/2}$$ is an integro-differential operator of the form

$$(-\Delta)^{\alpha/2} f(x) = c_{d,\alpha} \text{ p.v.} \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+\alpha}} dy. \quad (5)$$

The heavy tail of the heat kernel in the estimate (4) is a consequence of the heavy integral kernel in (5).
Existing results

- E. Davies (’93) studied the large time behaviour of the heat kernel of the discrete Laplace operator on a countable locally finite connected graph. Sharp upper bound in the case of uniformly bounded degree \( \text{deg}(x) \):

\[
    p_t (x, y) \leq \exp \left( - t \Phi \frac{d(x, y)}{ct} \right),
\]

with

\[
    \Phi(r) = r \ln(r + \sqrt{r^2 + 1}) - \sqrt{1 + r^2}.
\]

Since

\[
    \Phi(r) \sim \frac{r^2}{2} \quad \text{as } r \to 0 \quad \text{and} \quad \Phi(r) \sim r \ln r \quad \text{as } r \to \infty,
\]

the estimate (6) implies

\[
    p_t (x, y) \leq \exp \left( - \frac{d^2(x, y)}{ct} \right) \quad \text{for small } \frac{d(x, y)}{t},
\]

\[
    p_t (x, y) \leq \exp \left( -cd(x, y) \ln \frac{d(x, y)}{ct} \right) \quad \text{for large } \frac{d(x, y)}{t}.
\]
Existing results

- **C. Brandle, M. Chaves, R. Ferreira and J. Rossi ('11 – '14)** considered some special cases of dispersal kernels and obtained (non-sharp) two-sided estimates in the region $|x| \gg t$.

- **S. Molchanov and E. Yarovaya ('13)** obtained local large deviations upper bounds for a symmetric random walk in $\mathbb{Z}^d$ in the case of super-exponential decay ($p > 1$) of the transition probabilities $a(z)$.

- A number of sharp large deviation results for jump Markov process have been obtained by **A. Mogulskii ('17)**.
Regions in \((x, t)\) space

The large time behaviour of the studied fundamental solution (heat kernel) depends crucially on the relation between \(|x|\) and \(t\). We consider separately \textbf{four different regions in the \((x, t)\) space}. Namely,

I. \(|x| \leq rt^{1/2}(1 + o(1))\) (standard deviations region)

II. \(|x| = rt^{1+\delta/2} (1 + o(1)), \delta \in (0, 1)\) (moderate deviations region)

III. \(|x| = rt(1 + o(1)) (\delta = 1)\) (large deviations region)

IV. \(|x| = rt^{1+\delta/2} (1 + o(1)), \delta > 1\) ("extra-large" deviations region)
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IV. \(|x| = rt^{1+\delta}(1 + o(1)), \delta > 1\) (”extra-large” deviations region)
Normal deviations. The result.

We begin with the case $|x| \sim \sqrt{t}$. For any $r > 0$ in the region $|x| \leq r\sqrt{t}$ the local limit theorem holds.

**Theorem**

Assume that $a(\cdot)$ satisfies conditions $C_1$–$C_3$. Then for the function $v(x, t)$ the following asymptotic relation holds as $t \to \infty$:

$$v(x, t) = (2\pi t)^{-\frac{d}{2}} |\det(\sigma)|^{-\frac{1}{2}} e^{-\frac{(\sigma^{-1}x, x)}{2t}} (1 + o(1)),$$

where

$$\sigma^{ij} = \int_{\mathbb{R}^d} x_i x_j a(x) \, dx.$$
In the region $\sqrt{t} \ll |x| \ll t$ the following result holds.

**Theorem**

Assume that conditions $C_1 \rightarrow C_3$ and $A_p$, $p \geq 1$, are fulfilled. Then for $x = rt^{1+\delta/2} (1 + o(1))$ with $0 < \delta < 1$ and $r \in \mathbb{R}^d \setminus \{0\}$ the following asymptotic relation holds as $t \rightarrow \infty$:

$$v(x, t) = e^{-\frac{(\sigma^{-1}x, x)}{2t}(1+o(1))} = e^{-\frac{1}{2} (\sigma^{-1}r, r) t^\delta (1+o(1))}.$$  

where $o(1)$ tends to zero as $t \rightarrow \infty$. 

In order to formulate the result in the region $|x| \sim t$ we should first introduce a number of auxiliary quantities.

Let $X$ be a random vector in $\mathbb{R}^d$ with distribution $a(\cdot)$. If condition $A_p$ is fulfilled with some $p \geq 1$ then $X$ has finite exponential moment

$$\Lambda(\gamma) = \mathbb{E} e^{\gamma \cdot X}$$

at least for small enough $\gamma \in \mathbb{R}^d$.

We define the cumulant generating function $L(\gamma) = \ln \Lambda(\gamma)$, and introduce $I(r)$, $r \in \mathbb{R}^d$, as its Legendre transform:

$$I(r) = \sup_{\gamma} (\gamma \cdot r - L(\gamma)), \quad r, \gamma \in \mathbb{R}^d.$$
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$$I(r) = \sup_{\gamma} (\gamma \cdot r - L(\gamma)), \quad r, \gamma \in \mathbb{R}^d.$$
Large deviations region.

Denote by $\xi_r$ a positive solution of the equation

$$\ln \xi = I(\xi_r) - \xi_r \cdot \nabla I(\xi_r), \quad \xi \in R,$$

Lemma

Let $a(x)$ satisfy conditions $C_1$–$C_3$ and $A_p$ with some $p \geq 1$. Then for any $r \in R^d \setminus \{0\}$ the above equation has a unique solution $\xi_r$, and $0 < \xi_r < 1$.

We introduce the rate function

$$\Phi(r) = 1 - \frac{1}{\xi_r} (1 + \ln \xi_r - I(\xi_r)).$$
Large deviations region.

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**Let** $a(x)$ **satisfy conditions** $C_1$–$C_3$ **and** $A_p$ **with some** $p \geq 1$. **Then for any** $r \in R^d \setminus \{0\}$ **the above equation has a unique solution** $\xi_r$, **and** $0 < \xi_r < 1$.

We introduce the *rate* function

$$\Phi(r) = 1 - \frac{1}{\xi_r} \left(1 + \ln \xi_r - I(\xi_r r)\right).$$
The case \( p = 1 \): Condition \( L_1 \)

\[
\mathbb{E}|X|e^{bX \cdot \theta} = \infty \quad \text{for any } \theta \in S^{d-1}.
\]

Theorem (Large time asymptotics)

Let conditions \( C_1 - C_3 \) and \( A_p \), with some \( p \geq 1 \) be fulfilled, and assume additionally that in the case \( p = 1 \) condition \( L_1 \) holds. Then for \( x = rt(1 + o(1)) \) with \( r \in \mathbb{R}^d \setminus \{0\} \) we have

\[
v(x, t) = e^{-\Phi(r)t(1+o(1))} \quad \text{as } t \to \infty.
\]

where

\[
\Phi(r) = 1 - \frac{1}{\xi_r} \left(1 + \ln \xi_r - I(\xi_r r)\right).
\]
Large deviations region: logarithmic asymptotics.

The case $p = 1$: Condition $L_1$

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$$\Phi(r) = 1 - \frac{1}{\xi_r} \left(1 + \ln \xi_r - I(\xi_r r)\right).$$
Properties of $\Phi(r)$.

$\Phi(0) = 0$,  \hspace{1cm} $\Phi(r) > 0$,  \hspace{1cm} if  $r \neq 0$,

$\Phi$ is a convex function, and the following limit relations hold:

$$\Phi(r) = \frac{1}{2} \sigma^{-1} r \cdot r \ (1 + o(1)), \hspace{1cm} \text{as} \hspace{0.5cm} r \to 0;$$

$$\Phi(r) \to \infty, \hspace{1cm} \text{as} \hspace{0.5cm} r \to \infty.$$ 

If $a(x)$ has a finite support, then

$$\Phi(r) \geq c|r| \ln |r|, \hspace{1cm} \text{as} \hspace{0.5cm} |r| \to \infty.$$
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Behaviour of $\Phi(r)$ at infinity.

If condition

**Condition $L_p$**

\[ C_2 e^{-b|x|^p} \leq a(x) \leq C_1 e^{-b|x|^p}, \quad p \geq 1. \]

holds, then

**Lemma**

1) If $p = 1$, then

\[ \Phi(r) = b|r| (1 + o(1)), \quad as \quad |r| \to \infty. \]

2) If $p > 1$, then

\[ \Phi(r) = \frac{p}{p-1} \left( b(p-1) \right)^{1/p} |r| (\ln |r|)^{\frac{p-1}{p}} (1 + o(1)), \quad as \quad |r| \to \infty. \]
Behaviour of $\Phi(r)$ at infinity.

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Extra-large deviations region.

In the region $|x| \gg t$ we obtain only a point-wise upper bound for the heat kernel.

**Theorem**

Assume that $a(x)$ satisfies conditions $C_1 - C_3$ and $A_p$ with some $p \geq 1$. Then for $|x| = rt^{1+\delta} (1 + o(1))$ with $\delta > 1$ and $r > 0$ the following asymptotic upper bound holds:

$$v(x,t) \leq e^{-c_p t^{\frac{\delta+1}{2}} (\ln t)^{\frac{p-1}{p}}} (1+o(1)),$$

as $t \to \infty$,

where the constant $c_p = c_p(b, r)$ depends on $b, r$ and $p$.

If $a(\cdot)$ has a finite support, then for $|x| = rt^{\frac{\delta+1}{2}} (1 + o(1))$ with $\delta > 1$ we have

$$v(x,t) \leq e^{-\tilde{c} t^{\frac{\delta+1}{2}}} \ln t(1+o(1)),$$

as $t \to \infty$,

where $\tilde{c}$ depends on $r, \delta$ and the support of $a(\cdot)$. 
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Assume that $a(x)$ satisfies conditions $C_1$–$C_3$ and $A_p$ with some $p \geq 1$. Then for $|x| = rt^{1+\delta} \left(1 + o(1)\right)$ with $\delta > 1$ and $r > 0$ the following asymptotic upper bound holds:

$$v(x,t) \leq e^{-c_p t^{\frac{\delta+1}{2}} \left(\ln t\right)^{\frac{p-1}{p}} \left(1+o(1)\right)}, \quad \text{as } t \to \infty,$$

where the constant $c_p = c_p(b,r)$ depends on $b$, $r$ and $p$.

If $a(\cdot)$ has a finite support, then for $|x| = rt^{\frac{\delta+1}{2}} \left(1 + o(1)\right)$ with $\delta > 1$ we have

$$v(x,t) \leq e^{-\tilde{c} t^{\frac{\delta+1}{2}} \ln t(1+o(1))}, \quad \text{as } t \to \infty,$$

where $\tilde{c}$ depends on $r$, $\delta$ and the support of $a(\cdot)$. 
Conclusions.

In the case when \( a(x) \) has a finite support the estimate on \( v(x, t) \) in the region of extra-large deviations \( |x| = r t^{\varkappa} (1 + o(1)) \) with \( \varkappa > 1 \) can be written as

\[
v(x, t) \leq e^{-\tilde{k} |x| \ln \frac{|x|}{t}}.
\]

Let us compare with the estimate by E. Davies ('93)

\[
 p_t (x, y) \leq \exp \left( -c d (x, y) \ln \frac{d (x, y)}{ct} \right) \quad \text{for large } \frac{d (x, y)}{t}.
\]
Conclusions.

Let us compare the large time behaviour of the classical heat kernel and the heat kernel for a jump Markov process in different regions. The classical heat kernel

$$p_t(x, y) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x - y|^2}{4t}\right).$$

In the region $|x| \ll t$ (normal and moderate deviations regions) the logarithmic large time asymptotics of $v(x, t)$ and $p_t(x)$ coincide.

If $x = rt$, $r \in \mathbb{R}^d$, (large deviations area) then

$$\log p_t(x) = -\frac{r^2}{4} t(1 + o(1)), \quad \log v(x, t) = -\Phi(r)t(1 + o(1)).$$

Since for large $|r|

$$\Phi(r) \sim |r|(\ln |r|)^{\frac{p-1}{p}} \ll |r|^2, \quad p \geq 1,$$

we conclude that in the large and extra-large deviations regions the heat kernel of the jump Markov process has much heavier tails than that of the diffusion.
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Let us compare the large time behaviour of the classical heat kernel and the heat kernel for a jump Markov process in different regions. The classical heat kernel

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If \(x = rt, r \in \mathbb{R}^d\), (large deviations area) then

\[ \log p_t(x) = -\frac{r^2}{4} t (1 + o(1)), \quad \log v(x, t) = -\Phi(r) t (1 + o(1)). \]

Since for large \(|r|\)

\[ \Phi(r) \sim |r| (\ln |r|)^{\frac{p-1}{p}} \ll |r|^2, \quad p \geq 1, \]

we conclude that in the large and extra-large deviations regions the heat kernel of the jump Markov process has **much heavier tails** than that of the diffusion.
Fundamental solutions of time fractional parabolic type equations with convolution kernels

We study a solution $w_\alpha(x, t)$ of the following fractional time parabolic problem:

**Time fractional non-local problem, $0 < \alpha < 1$**

\[
\partial_t^\alpha w_\alpha = a \ast w_\alpha - w_\alpha \\
w_\alpha |_{t=0} = \delta_0
\]

Here $\partial_t^\alpha$ is the fractional derivative (the Caputo derivative) of the order $\alpha$, $0 < \alpha < 1$, and $a(x)$ is a convolution kernel.
We assume that

- **F1**
  \[a(x) \geq 0; \quad a(x) = a(-x);\]

- **F2**
  \[a(x) \in C_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d); \quad \int_{\mathbb{R}^d} a(x) dx = 1;\]

- **F3**
  \[a(x) \leq C_p e^{-b|x|^p}, \quad p > 1.\]
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We assume that

- **F1**

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- **F2**

  \[ a(x) \in C_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d); \quad \int_{\mathbb{R}^d} a(x)\,dx = 1; \]

- **F3**

  \[ a(x) \leq C_p e^{-b|x|^p}, \quad p > 1. \]
The solution $w_\alpha(x, t)$ admits the following representation in terms of the fundamental solution $u(x, t)$ of a nonlocal heat equation:

$$w_\alpha(x, t) = \int_0^\infty u(x, r)d_r \mathbb{P}(S_r \geq t) = \int_0^\infty u(x, r)G_t^\alpha(r)dr,$$

Here $S = \{S_r, r \geq 0\}$ is the $\alpha$-stable subordinator with the Laplace transform

$$\mathbb{E} e^{-\lambda S_r} = e^{-r\lambda^\alpha}$$

and

$$G_t^\alpha(r) = d_r \Pr\{V_t^\alpha \leq r\}$$

is the density of the inverse $\alpha$-stable subordinator $V_t^\alpha$. 

Using the representation for the Laplace transform of $V_t^\alpha$:

$$\mathbb{E}e^{-\lambda V_t^\alpha} = E_\alpha(-\lambda t^\alpha), \quad E_\alpha \text{ is the Mittag-Leffler function}$$

and

$$u(x, t) = e^{-t} \delta_0(x) + v(x, t) \quad \text{with} \quad v(x, t) = \sum_{k=1}^{\infty} \frac{a^k(x)}{k!} t^k e^{-t}$$

we obtain

$$w_\alpha(x, t) = E_\alpha(-t^\alpha) \delta_0(x) + p_\alpha(x, t),$$

where the regular part of $w_\alpha(x, t)$ equals

$$p_\alpha(x, t) = \sum_{k=1}^{\infty} \frac{a^k(x)}{k!} t^{\alpha k} E_\alpha^{(k)}(-t^\alpha)$$
Using the representation for the Laplace transform of $V_t^\alpha$:

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$$p_\alpha(x, t) = \sum_{k=1}^{\infty} \frac{a^*_k(x)}{k!} t^{\alpha k} E_\alpha^{(k)}(-t^\alpha)$$
We consider separately the following regions:

- **|x| is bounded**

- **(Subnormal deviations)**

  \[ 1 \ll |x| \ll t^{\alpha/2} \]

  or equivalently, \( |x(t)| \to \infty \) such that there exists an increasing function \( r(t) \), \( r(0) = 0 \), \( \lim_{t \to \infty} r(t) = +\infty \) that

  \[ r(t) \leq |x| \leq (r(t) + 1)^{-1}t^{\alpha/2} \]

  for all sufficiently large \( t \).

- **(Normal deviations)** \( x = vt^{\alpha/2}(1 + o(1)) \), where \( v \) is an arbitrary vector in \( \mathbb{R}^d \setminus \{0\} \).

- **(Moderate deviations)** \( x = vt^\beta(1 + o(1)) \) with \( \frac{\alpha}{2} < \beta < 1 \) and \( v \in \mathbb{R}^d \setminus \{0\} \).

- **(Large deviations)** \( x = vt(1 + o(1)) \) with \( v \in \mathbb{R}^d \setminus \{0\} \).

- **(Extra large deviations)** \( |x| \gg t \), i.e. \( \lim_{t \to \infty} \frac{|x(t)|}{t} = \infty \).
Asymptotics for \( p_\alpha(x, t) \) as \( t \to \infty \)

- If \( |x| \) is bounded, then

\[
\begin{align*}
c_- t^{\frac{-\alpha}{2}} & \leq p(x, t) \leq c_+ t^{\frac{-\alpha}{2}} & \text{if} \ d = 1, \\
c_- t^{-\alpha} \log t & \leq p(x, t) \leq c_+ t^{-\alpha} \log t & \text{if} \ d = 2, \\
c_- t^{-\alpha} & \leq p(x, t) \leq c_+ t^{-\alpha} & \text{if} \ d \geq 3.
\end{align*}
\]

- If \( 1 \ll |x| \ll t^{\frac{\alpha}{2}} \), then

**Subnormal deviations**

\[
\begin{align*}
c_- t^{\frac{-\alpha}{2}} & \leq p(x, t) \leq c_+ t^{\frac{-\alpha}{2}} & \text{if} \ d = 1, \\
c_- t^{-\alpha} \log \left( \frac{t^\alpha}{|x|^2} \right) & \leq p(x, t) \leq c_+ t^{-\alpha} \log \left( \frac{t^\alpha}{|x|^2} \right) & \text{if} \ d = 2, \\
c_- t^{-\alpha} |x|^{2-d} & \leq p(x, t) \leq c_+ t^{-\alpha} |x|^{2-d} & \text{if} \ d \geq 3.
\end{align*}
\]
Asymptotics for \( p_\alpha(x, t) \) as \( t \to \infty \)

- If \( |x| \) is bounded, then

\[
\begin{align*}
&c_- t^{-\alpha \frac{x}{2}} \leq p(x, t) \leq c_+ t^{-\alpha \frac{x}{2}} \quad \text{if } d = 1, \\
&c_- t^{-\alpha \log t} \leq p(x, t) \leq c_+ t^{-\alpha \log t} \quad \text{if } d = 2, \\
&c_- t^{-\alpha} \leq p(x, t) \leq c_+ t^{-\alpha} \quad \text{if } d \geq 3.
\end{align*}
\]

- If \( 1 \ll |x| \ll t^{\alpha \frac{x}{2}} \), then

Subnormal deviations

\[
\begin{align*}
&c_- t^{-\alpha \frac{x}{2}} \leq p(x, t) \leq c_+ t^{-\alpha \frac{x}{2}} \quad \text{if } d = 1, \\
&c_- t^{-\alpha \log \left( \frac{t_\alpha}{|x|^2} \right)} \leq p(x, t) \leq c_+ t^{-\alpha \log \left( \frac{t_\alpha}{|x|^2} \right)} \quad \text{if } d = 2, \\
&c_- t^{-\alpha |x|^{2-d}} \leq p(x, t) \leq c_+ t^{-\alpha |x|^{2-d}} \quad \text{if } d \geq 3.
\end{align*}
\]
If \( x = vt^{\alpha/2}(1 + o(1)) \) with \( v \in \mathbb{R}^d \setminus \{0\} \), then

**Normal deviations**

\[
p(x, t) = t^{-\frac{d\alpha}{2}} \int_{0}^{\infty} W_{\alpha}(s) \Psi(v, s) \, ds \left(1 + o(1)\right).
\]

where

\[
\Psi(v, s) = \frac{1}{|\text{det}\sigma|^{1/2}(2\pi s)^{d/2}} \exp\left(-\frac{(\sigma^{-1}v, v)}{s}\right)
\]

and \( W_{\alpha}(s) \) is the Wright function that is expressed via the density \( G_t^\alpha(r) \) of the inverse subordinator.
If \( x = vt^{\alpha/2}(1 + o(1)) \) with \( v \in \mathbb{R}^d \setminus \{0\} \), then

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p(x, t) = t^{-\frac{d\alpha}{2}} \int_0^\infty W_\alpha(s) \Psi(v, s) \, ds \, (1 + o(1)).
\]

where

\[
\Psi(v, s) = \frac{1}{|\det\sigma|^{1/2}(2\pi s)^{d/2}} \exp \left( - \frac{(\sigma^{-1}v, v)}{s} \right)
\]

and \( W_\alpha(s) \) is the Wright function that is expressed via the density \( G_\alpha^\alpha(r) \) of the inverse subordinator.
- If $x = vt^\beta (1 + o(1))$ with $\frac{\alpha}{2} < \beta < 1$ and $v \in \mathbb{R}^d \setminus \{0\}$, then

**Moderate deviations**

$$p(x, t) = \exp \{ - K_v t^{\frac{2\beta - \alpha}{2 - \alpha}} (1 + o(1)) \}$$

with a constant $K_v$ depending on $\alpha$ and $v$.

- If $x = vt(1 + o(1))$ with $v \in \mathbb{R}^d \setminus \{0\}$, then

**Large deviations**

$$p(x, t) = \exp \{ - F(v)t(1 + o(1)) \}.$$

- If $|x| \gg t$, then

**Extra-large deviations**

$$p(x, t) \leq \exp \{ - c_+ |x| \left( \log \left| \frac{x}{t} \right| \right)^{\frac{p-1}{p}} \}.$$
• If $x = vt^\beta(1 + o(1))$ with $\frac{\alpha}{2} < \beta < 1$ and $v \in \mathbb{R}^d \setminus \{0\}$, then

**Moderate deviations**

$$p(x, t) = \exp \left\{ - K_v t^{\frac{2\beta - \alpha}{2 - \alpha}} (1 + o(1)) \right\}$$

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Conclusions

1. In the region of large deviations $|x| \sim t$ the form of the asymptotics of $p(x, t)$ is the same as for $v(x, t)$, but the rate functions are different.

2. In the region of extra large deviations the asymptotic upper bounds for $p(x, t)$ and for $v(x, t)$ are the same.