A simple mathematical model for EPR argument

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Plan

Brief summary of the original Einstein, Podolsky and Rosen argument (1935)
(i.e., in terms of continuous variables)

then

discussion of a simple mathematical model with the (only) purpose of illustrating the argument in a quantitative and rigorous way

Historical context

Formulation of Quantum Mechanics 1925-26

Copenhagen interpretation
Bohr (Solvay and Como 1927), Heisenberg (Chicago lect. 1930)

- ψ gives a complete description

- intrinsic indeterminism

- a quantum system does not have a definite property until we measure that property

- crucial role of the measurement

Intense debate between
Bohr, Heisenberg, Pauli... and Einstein, Schrödinger, De Broglie...

According to Einstein's view: there exists an objective physical reality and our theories try to capture elements of this reality.

In a first period Einstein's criticism was aimed at showing possible violations of the uncertainty principle.

In EPR (1935) the goal changes:
they want to show that the description given by ψ is not complete.

EPR (1935)

They start with a distinction

"*between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates.*"

Objective property of the system vs. physical quantity of the theory

Then they introduce

- Reality Criterion: "*If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of reality corresponding to the physical quantity.*"

(i.e. there exists an objective property of the system, independent of any observation).

Given two separated, non interacting systems, they also assume

- Locality Principle: *Since at the time of measurement the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to the first system.*

(Considered obvious by any physicist of that period).

Then they define

- Completeness: "*every element of the physical reality must have a counterpart in the physical theory.*"

(i.e., QM is complete if ψ describes all elements of physical reality of a given system).

They show that

reality criterion $+$ locality principle $\rightarrow$ QM is not complete

The EPR model

Two particles in $d = 1$ (set $\hbar = 1$, x_0 is a constant)

$$P^{(j)} f(x_j) = \frac{1}{i}\frac{\partial}{\partial x_j} f(x_j)\,, \qquad X^{(j)} f(x_j) = x_j f(x_j)\,, \qquad j = 1,2$$

At least formally

$$\begin{aligned}
u_p(x_1) &:= e^{ipx_1} && \text{then} && P^{(1)} u_p = p\, u_p \\
v_x(x_1) &:= \delta_x(x_1) && \text{then} && X^{(1)} v_x = x\, v_x \\
\psi_{-p}(x_2) &:= e^{-ip(x_2 - x_0)} && \text{then} && P^{(2)} \psi_{-p} = -p\, \psi_{-p} \\
\varphi_{x+x_0}(x_2) &:= \delta_{x+x_0}(x_2) && \text{then} && X^{(2)} \varphi_{x+x_0} = (x + x_0)\, \varphi_{x+x_0}
\end{aligned}$$

$u_p(x_1)\psi_{-p}(x_2)$ is a state where
the first particle has momentum p and the second $-p$

$v_x(x_1)\varphi_{x+x_0}(x_2)$ is a state where
the first particle has position x and the second $x + x_0$.

Then they assume that the system is described by the "entangled" (i.e. non factorized) state

$$\Psi(x_1,x_2)=\int dp\, u_p(x_1)\psi_{-p}(x_2)=\int dx\, v_x(x_1)\varphi_{x+x_0}(x_2)\ \Big(=\delta_{x_0}(x_2-x_1)\Big)$$

$$\Big(\text{and}\quad \hat{\Psi}(p_1,p_2)=e^{ip_1x_0}\delta_0(p_1+p_2)\Big)$$

the particles are at (large) distance x_0, with opposite momentum.

Notice that if QM is complete then there are no elements of reality corresponding to the physical quantities $P^{(j)}$ (or $X^{(j)}$) of each particle.

The proof .

Let us measure $P^{(1)}$ on the first particle and let $\bar{p}$ be the result.

By wave packet collapse

$$\Psi(x_1, x_2) = \int dp\, u_p(x_1)\psi_{-p}(x_2) \to u_{\bar{p}}(x_1)\psi_{-\bar{p}}(x_2)\,,$$

hence we can predict with certainty that the second particle has momentum $-\bar{p}$.

By reality criterion, there is an element of reality corresponding to the physical quantity $P^{(2)}$ of the second particle.

By locality principle, this element of reality cannot have been created by the measurement on the first particle and therefore it existed from the beginning.

Remark. This would be sufficient to show incompleteness, but they go further.

Starting from the same initial state,
assume to measure $X^{(1)}$ and let $\bar{x}$ be the result.

Then $\Psi(x_1, x_2) = \int dx\, v_x(x_1)\varphi_{x+x_0}(x_2) \rightarrow v_{\bar{x}}(x_1)\varphi_{\bar{x}+x_0}(x_2)$.

As before, there is also an element of reality corresponding to the physical quantity $X^{(2)}$ of the second particle from the beginning.

Conclusion:

from the beginning, there are elements of reality corresponding to both $P^{(2)}$ and $X^{(2)}$ of the second particle, but this fact contradicts the uncertainty principle.

Then $\Psi(x_1, x_2)$ does not provide a complete description of the physical situation, i.e., QM is not complete. $\square$

The story continues...

Bohm (1951) reformulates the argument in terms of spin variables

Bell (1964) finds an inequality which is fulfilled by any local theory and it is violated by QM

Various experiments (Clauser (1972), Aspect (1982), Zeilinger (2018)...) have verified the violation of the inequality

Summarizing

reality criterion, locality principle, completeness
can not hold simultaneously (EPR)
and it is the locality principle that must be abandoned (Bell).

Our model

Two particles in $d = 1$ as in EPR, but we

- define initial state e corresponding time evolution in L^2

- add a simple model of measurement apparatus (spin)

- do not use wave packet reduction

- consider the Schrödinger evolution of the whole system "particle 1 + particle 2 + spin".

- introduce a scaling limit, crucial to highlight the effect

- prove the effect, with an estimate of the error

Hilbert space

$$\mathcal{K} = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes \mathbb{C}^2$$

State

$$\Psi_t(x_1, x_2) \; = \; \begin{pmatrix} \Psi_t^+(x_1, x_2) \\ \Psi_t^-(x_1, x_2) \end{pmatrix}$$

where

$\Psi_t^{\pm} \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$

$x_j, \; j = 1, 2$ denotes the position variable of the $j-$th particle.

$\Psi_t^+(x_1, x_2)$ is associated to the value $\hbar/2$ of the spin,
$\Psi_t^-(x_1, x_2)$ refers to the value $-\hbar/2$.

Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + \gamma V\left(\frac{x_1 - a}{\delta}\right)\sigma_2 = -\frac{\hbar^2}{2m_1}\Delta_1 - \frac{\hbar^2}{2m_2}\Delta_2 + \frac{\hbar\omega}{2}\sigma_3 + \gamma V\left(\frac{x_1 - a}{\delta}\right)\sigma_2$$

particle 1 and the spin interact, particle 2 is free. Moreover

m_1 and m_2 are the masses of the particles

ω is the frequency of the spin

σ_3, σ_2 are the Pauli matrices $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

γ is the coupling constant between particle 1 and the spin

δ is the spatial scale of the range of V

$a > 0$ is the fixed position of the spin.

Initial datum

$$\Psi_0 \;=\; \begin{pmatrix} 0 \\ \Psi_0^- \end{pmatrix}$$

i.e., initially the spin equals $-\hbar/2$, with energy is $-\hbar\omega/2$. Moreover

the two particles are in a maximally entangled state concentrated in position around the origin

$$\Psi_0^-(x_1, x_2) \;=\; N\Big(\phi_P(x_1)\phi_{-P}(x_2) + \phi_{-P}(x_1)\phi_P(x_2)\Big),$$

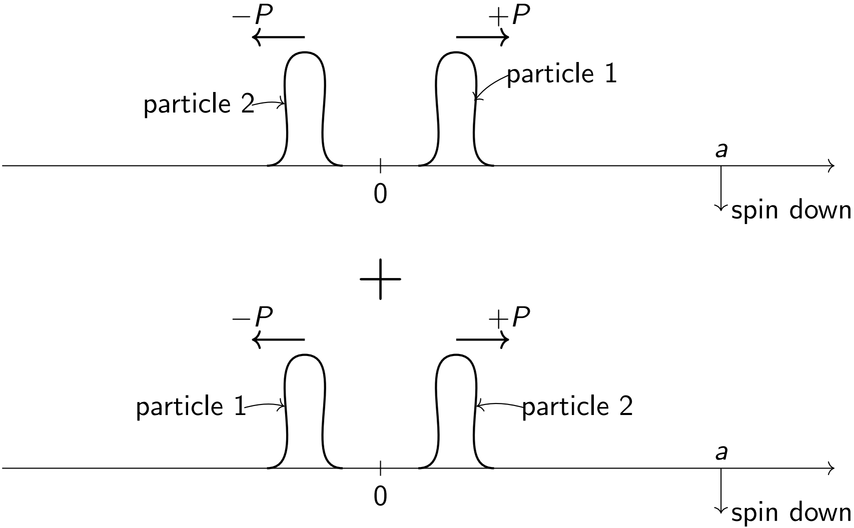
where

$$\phi_P(x) \;=\; \frac{e^{-\frac{x^2}{2\sigma^2} + i\frac{P}{\hbar}x}}{\pi^{1/4}\sqrt{\sigma}} \;=\; e^{i\frac{P}{\hbar}x}\frac{f(x/\sigma)}{\sqrt{\sigma}}$$

and

$$N := (2 + 2e^{-\frac{2P^2\sigma^2}{\hbar^2}})^{-\frac{1}{2}}$$

For t small, the state is the superposition:



The scaling

$$\hbar = \varepsilon^2 \quad \omega = \varepsilon^{-1} \quad \gamma = \varepsilon^2 \quad \delta = \varepsilon \quad \sigma = \varepsilon \quad P = a = O(1)$$

where ε is a small positive parameter. We also set $m_1 = m_2 = 1$. Therefore

Semiclassical regime: $\hbar \to 0$ and short-range interaction: $\delta \to 0$, hence

the single-particle states in the initial datum are semi-classical and

$$t = t_c := a/P$$

is the (classical) collision time of particle 1 with the spin placed in $x = a$.

The coupling is weak: $\gamma \to 0$.

Quasi-elastic regime for particle 1: $P = O(1)$, $\hbar\omega = \varepsilon$

Rescaled Hamiltonian

$$\mathcal{H}^{\varepsilon} = \mathcal{H}_0^{\varepsilon} + \varepsilon^2 \, V^{\varepsilon} \, \sigma_2 = -\frac{\varepsilon^4}{2}\Delta_1 - \frac{\varepsilon^4}{2}\Delta_2 + \frac{\varepsilon}{2}\sigma_3 + \varepsilon^2 \, V^{\varepsilon} \, \sigma_2$$

where V^{ε} is the multiplication operator by $V\left(\frac{x_1 - a}{\varepsilon}\right)$, $\; V \in \mathcal{S}(\mathbb{R})$.

Rescaled initial state

$$\Psi_0^{-\varepsilon}(x_1, x_2) \;=\; N^{\varepsilon}\left(\phi_P^{\varepsilon}(x_1)\phi_{-P}^{\varepsilon}(x_2) + \phi_{-P}^{\varepsilon}(x_1)\phi_P^{\varepsilon}(x_2)\right),$$

where

$$\phi_P^{\varepsilon}(x) \;=\; \frac{e^{-\frac{x^2}{2\varepsilon^2} + i\frac{P}{\varepsilon^2}x}}{\pi^{1/4}\sqrt{\varepsilon}} \;=\; \frac{1}{\sqrt{\varepsilon}} f\left(\frac{x}{\varepsilon}\right) e^{i\frac{P}{\varepsilon^2}x}$$

and $N^{\varepsilon} = (2 + 2e^{-\frac{2P^2}{\varepsilon^2}})^{-\frac{1}{2}}$

The result

Theorem. Let $\mathcal{U}(t) = e^{-i\frac{t}{\varepsilon^2}\mathcal{H}}$, $\mathcal{U}_0(t) = e^{-i\frac{t}{\varepsilon^2}\mathcal{H}_0}$, $t > t_c = a/P$.

Then

$$\Psi_t := \mathcal{U}(t)\Psi_0 = \mathcal{U}_0(t)\Psi_0 + \varepsilon\,\mathcal{U}_0(t)\Psi_1 + \mathcal{R}(t)$$

where

$$\Psi_1(x_1, x_2) = \begin{pmatrix} A(x_1)\,\phi_{-P}(x_2) \\ 0 \end{pmatrix}$$

$$A(x) = \frac{\sqrt{2\pi}}{P} e^{i\frac{a}{2P^3}} e^{i\frac{x}{\varepsilon^2}(P-\frac{\varepsilon}{P})} \widehat{V}\left(-\frac{1}{P}\right) \frac{f(-\frac{a}{P^2} + \frac{x}{\varepsilon})}{\sqrt{\varepsilon}}$$

$$\|\mathcal{R}(t)\|_{\mathcal{K}} < C\,\varepsilon^2 .$$

Notice: the spin flips only if particle 1 has positive momentum and particle 2 has negative momentum.

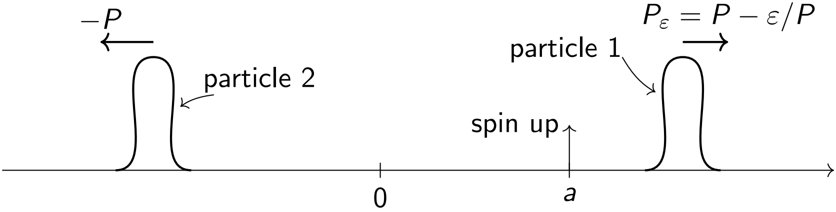
For $t > a/P$ (and up to $O(\varepsilon^2)$) $\mathcal{U}(t)\Psi_0$ is the sum of

the unperturbed evolution $\mathcal{U}_0(t)\Psi_0(x)$ (with spin down)

$$+$$

the first order correction :

$$\varepsilon\,\mathcal{U}_0(t)\Psi_1 = \varepsilon\,\mathcal{U}_0(t)\begin{pmatrix} A_{P_\varepsilon}(x_1)\cdot\Phi_{-P}(x_2) \\ 0 \end{pmatrix} =$$



By Born's rule, we can compute

- probability to find spin up: $\mathcal{P}_u = \alpha\,\varepsilon^2 + O(\varepsilon^3)$

  where $\alpha = \pi P^{-2}\,|\hat{V}(P^{-1})|^2$

- probability to find spin down: $\mathcal{P}_d = 1 + O(\varepsilon)$

- probability to find spin up and particle 2 with negative momentum: $\mathcal{P}_{-p,u} = \alpha\,\varepsilon^2 + O(\varepsilon^3)$

- probability to find spin down and particle 2 with negative momentum: $\mathcal{P}_{-p,d} = \frac{1}{2} + O(\varepsilon)$

Corollary Let $t > t_c$. Then

$$\frac{\mathcal{P}_{-p,u}}{\mathcal{P}_u} = 1 + O(\varepsilon)\,, \qquad \frac{\mathcal{P}_{-p,d}}{\mathcal{P}_d} = \frac{1}{2} + O(\varepsilon)$$

Conclusion

• If we measure the spin for $t > t_c$ and we find up then we can predict with probability one that particle 2 has momentum $-p$

hence, particle 2 has a definite momentum $-P$.

Recall: at $t = 0$ particle 2 does not have a definite momentum.

• If we measure the spin for $t > t_c$ and we find down then the situation for particle 2 remains unchanged.

• We use Schrödinger eq., a scaling limit (crucial) and Born's rule, the argument does not require the wave packet collapse.

A word on the proof

Perturbative expansion by iteration of Duhamel formula

Analysis of the terms of the expansion via stationary and non stationary phase arguments

Estimates in L^2