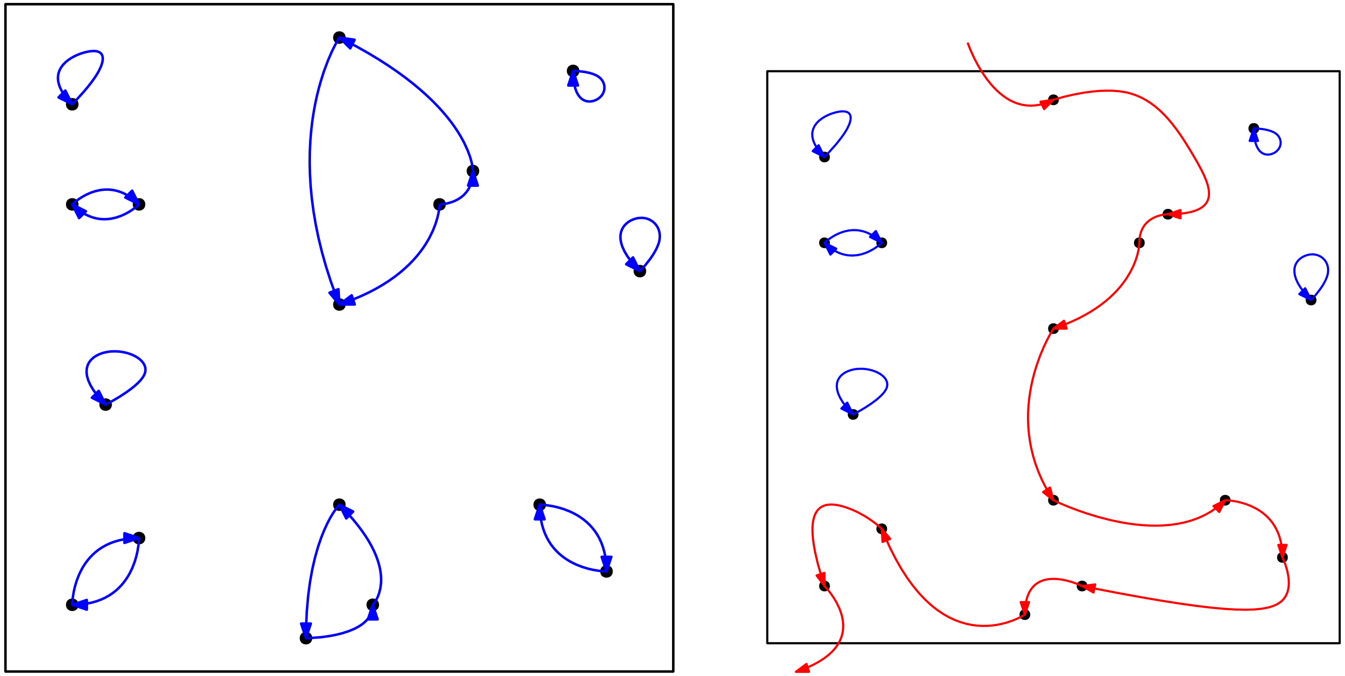
Asymptotics in Loop Percolation

SAMMP in Yerevan

Quirin Vogel, University of Klagenfurt

10th of September 2026



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1+O(1) Asymptotics for Loop Percolation in Five and Higher Dimensions,
V., *Int. Math. Res. Not.*, **9**, 2026.

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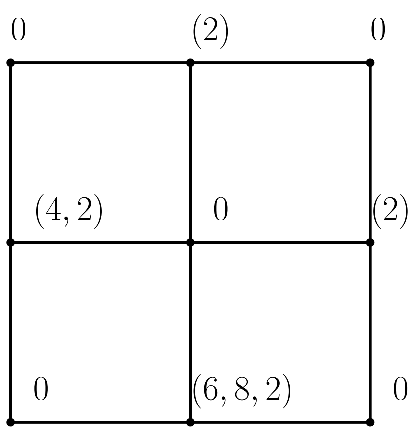

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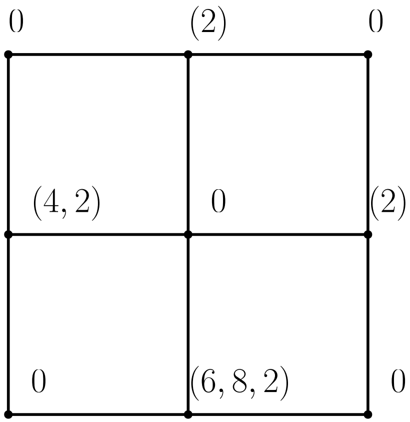

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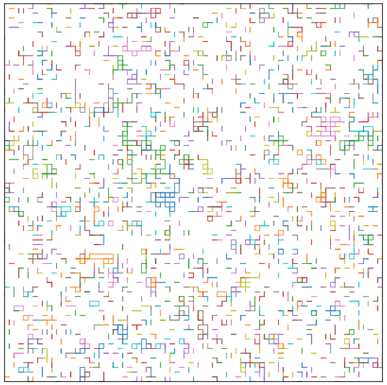

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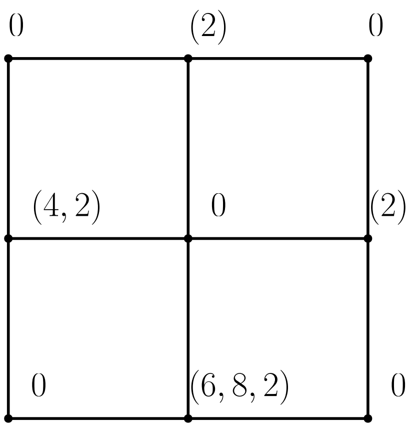

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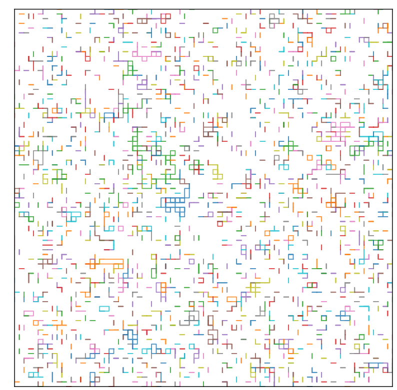

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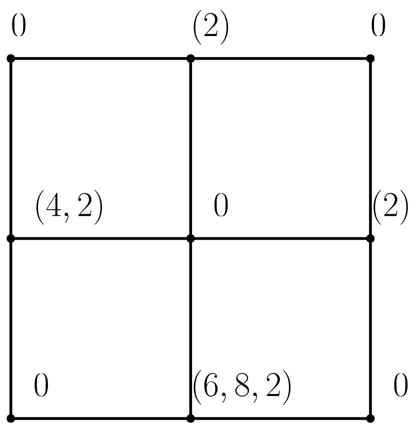

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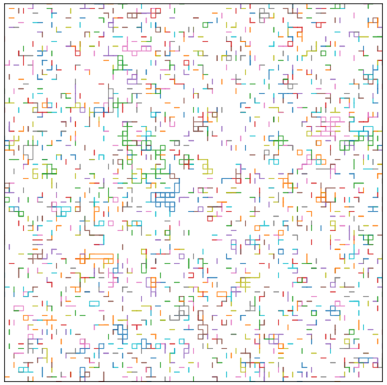

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Going back to **[Symanzik 69], [Lawler & Werner 04], [LeJan 08]**. Recently **[Lupu, Sapozhnikov, Chang,...]**.

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Chang & Sapozhnikov, Probab. Theory Relat. Fields (2016), Conjecture
For $\alpha < \alpha_c$ and $d \geq 5$

1. there exists $C_d > 0$ with
$$\mathbb{P}_\alpha(0 \leftrightarrow \partial B_n) = C_d n^{2-d}(1 + o(1))$$
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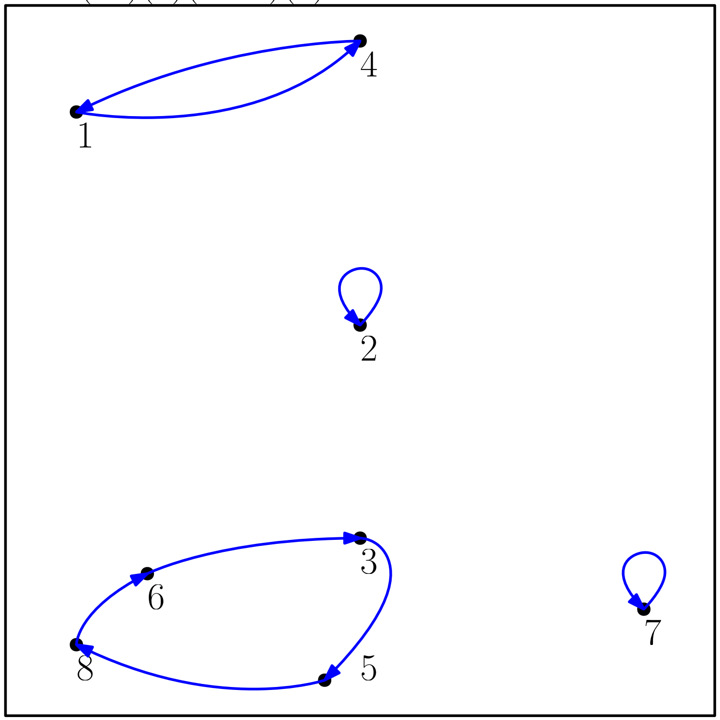
For $d = 4$, the *log-log* conjecture

$$\mathbb{P}_\alpha(0 \leftrightarrow \partial B_n) = n^{-2+o(1)} \,.$$

Connection to the Bose gas

$\sigma = (14)(2)(3586)(7)$



1. Fix a domain $\Lambda \subset \mathbb{Z}^d$
2. Throw $N \propto \alpha |\Lambda| \beta^{-d/2}$ points into the box
3. Sample a permutation of N objects with weight

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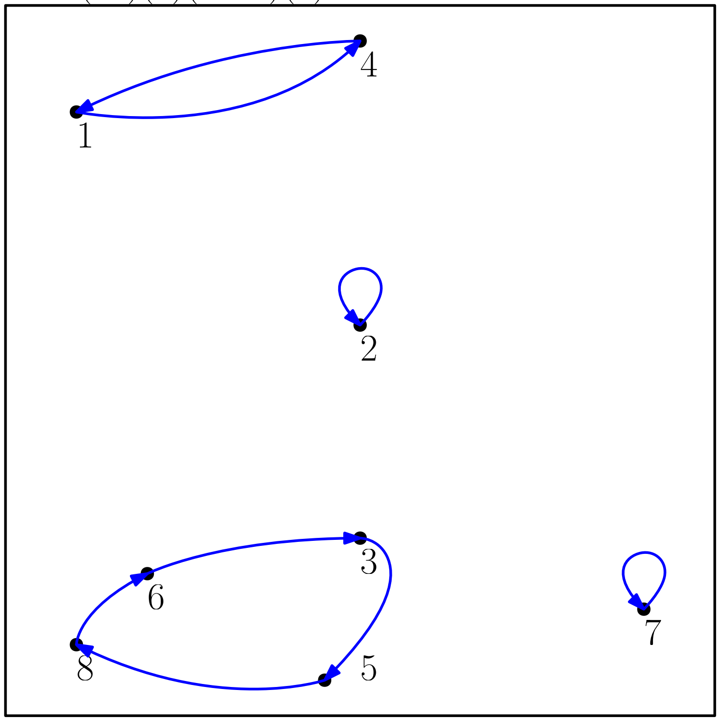

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5. Send $\beta \to 0$

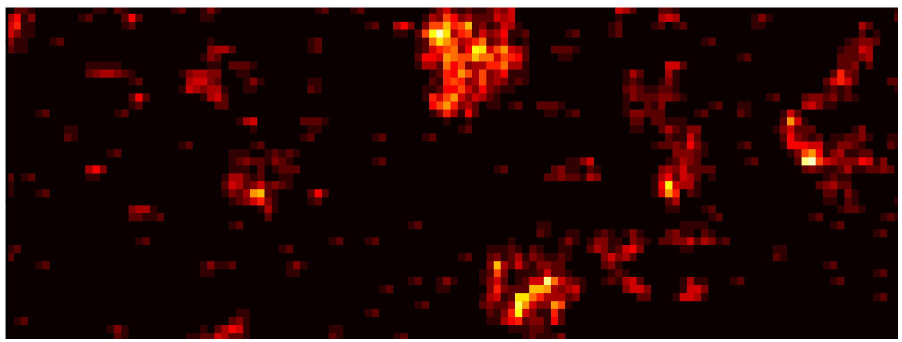
Obtain $\mathbb{P}_\alpha$!

Connection to the Gaussian Free Field

LeJan's Isomorphism

For $\alpha = \alpha_{\mathrm{G}}$, the *total occupation time* has the same distribution as a square of a Gaussian free field

$$\{\mathcal{L}_x\}_{x \in \Lambda} \stackrel{d}{=} \{\phi_x^2\}_{x \in \Lambda}$$



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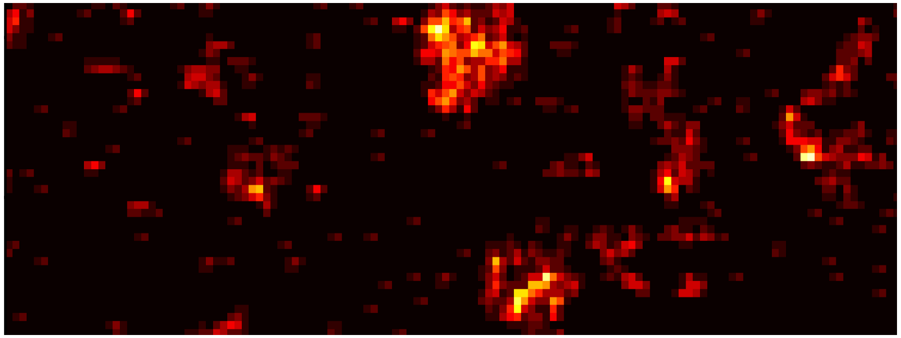

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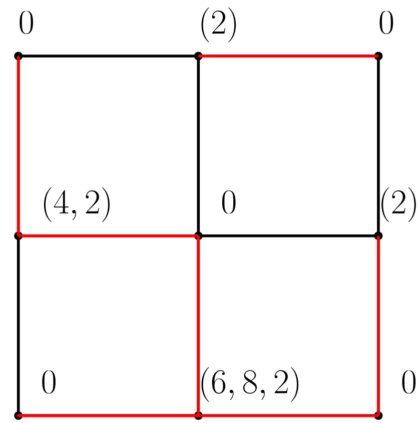
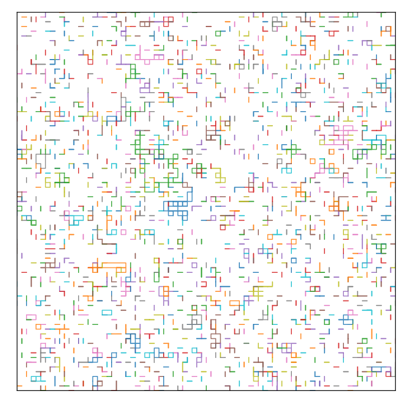
(Edge-law isomorphic to random current model with Gaussian weights and generalization, **[Werner 16], [Lupu 23]**)

Summary

- At each site N_x particles, $\mathrm{Poi}(\alpha)$
- Each particle i has length ℓ_i
- Simple random walk of length ℓ_i
- Bose-gas on $\mathbb{Z}^d$ with $\beta \to 0$
- Gaussian-Free field connection via local-times

Goal: Understand one-point and two-point function!





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[V. 26] For any $\alpha \in (0, \alpha_{\#})$, as $n \to \infty$

$$\mathbb{P}_\alpha(0 \leftrightarrow \partial B_n) = \alpha \frac{d\Gamma(d/2)}{(d-2)\pi^{d/2}} \mathbb{E}_\alpha[\mathrm{Cap}(\mathcal{C}_0)] n^{2-d}(1 + o(1))\,.$$

Furthermore, $\mathbb{P}_\alpha(A \mid 0 \leftrightarrow \partial B_n)$ is asymptotically equivalent to adding on large loop to the subcritical loop soup *independently*:

Sample the subcritical loop-soup. Then, sample a loop that connects the $O(1)$-cluster to the outside.

- $\alpha =$intensity
- $\frac{d\Gamma(d/2)}{(d-2)\pi^{d/2}} =$Green's constant
- $\mathrm{Cap}(A) = \sum_{x\in A} \mathbb{P}_x(H_A = \infty)$

Compare with: H. Duminil-Copin, A. Raoufi and V. Tassion: Calculation for the Poisson–Boolean model, one-loop domination, *Annales Henri Lebesgue*, 3, (2020): 677-700

Proof: Loop estimates

Connectivity Lemma If $\mathrm{diam}(K) = o(n)$, then as $n \to \infty$

$$\mu(\exists \omega\colon K \overset{\omega}{\leftrightarrow} \partial B_n) = \frac{d\Gamma(d/2)}{(d-2)\pi^{d/2}} \mathrm{Cap}(K) n^{2-d}(1 + o(1))\,,$$

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Long loop concentration For $\alpha < \alpha_\#$

$$\lim_{n\to\infty} \mathbb{P}_\alpha \left(\exists ! \omega_{\mathrm{long}} \in \mathcal{C}_0\colon \mathrm{diam}(\omega_{\mathrm{long}}) \gtrsim n \mid 0 \leftrightarrow \partial B_n\right) = 1\,.$$

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Proof idea: use tree domination to disprove the case of no long loops or at least two:

Connecting with short loops: $\mathbb{P}_\alpha \left(\forall \omega \in \mathcal{C}_0 \colon \mathrm{diam}(\omega_{\mathrm{long}}) \leq m \text{ and } 0 \leftrightarrow \partial B_n\right) = \mathcal{O}\left(\mathrm{e}^{-c\frac{n}{m}}\right)$

Long loops: $\mathbb{P}_\alpha \left(\exists \omega_1 \neq \omega_2 \colon \omega_i \in \mathcal{C}_0 \text{ and } \mathrm{diam}(\omega_i) \geq m\right) = \mathcal{O}(m^{6-2d})$.

Mecke identity and proof

Mecke identity For a Poisson point process $\eta = \sum_\omega \delta_\omega$ with intensity measure μ

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\mathbb{P}_\alpha(0 \leftrightarrow \partial B_n) &\sim \mathbb{P}_\alpha(0 \leftrightarrow \partial B_n \text{ and } \exists! \omega_{\mathrm{long}}) \\
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Important connection to *random walk range capacity*:

$$\mathrm{Cap}(\mathcal{R}_n) \sim \begin{cases} \gamma_d n & \text{if } d \geq 5\,, \\ \gamma_4 \frac{n}{\log(n)} & \text{if } d = 4\,, \\ \chi n^{1/2} & \text{if } d = 3\,, \end{cases}$$

see e.g. Asselah, Schapira, Sousi. *Ann. Probab.* 47(3): 1447-1497.

The two-point function

[V. 26] For any $\alpha \in (0, \alpha_{\#})$, as $|x| \to \infty$

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Technical Lemma

$$\lim_{|x| \to \infty} \mathbb{P}_\alpha(\mathrm{path}(0 \to x) \subset B_{|x|^{1+\varepsilon}} \mid 0 \leftrightarrow x) = 1\,,$$

for any $\varepsilon > 0$.

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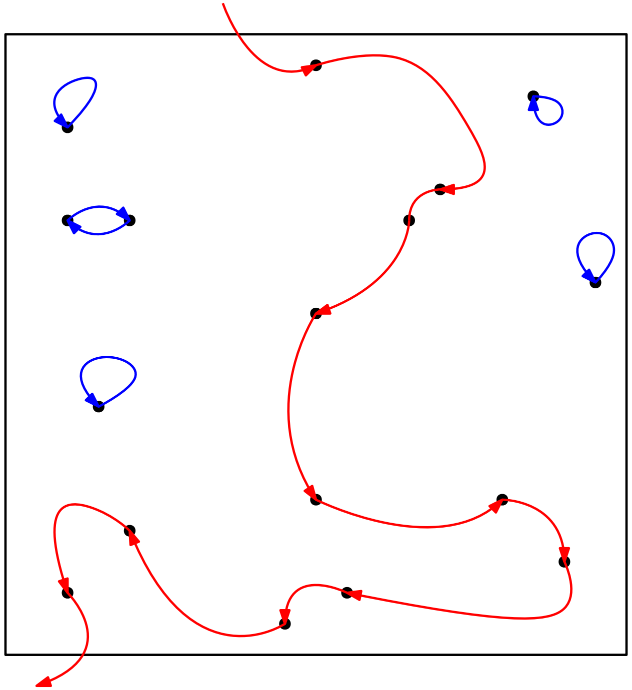
6. Can this help us with the Bose gas?

Thank you for your attention



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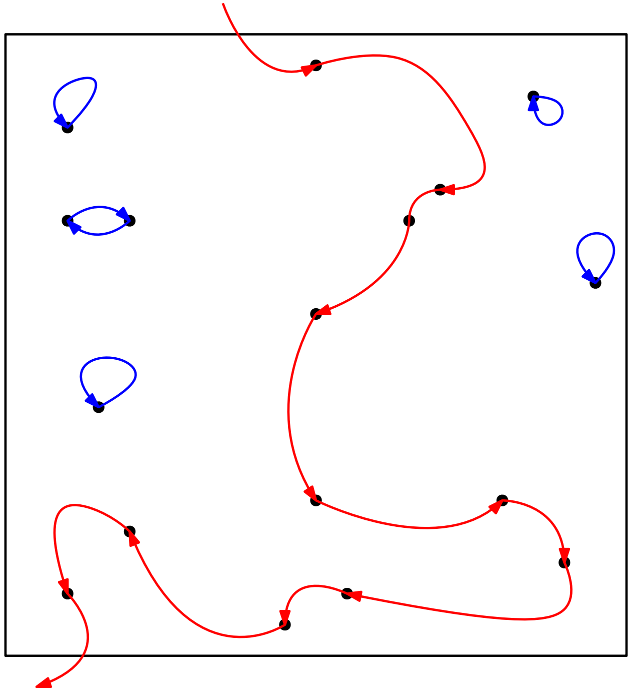

Any questions?