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About Nonlinear Affine Semigroups in Hilbert Space

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Résumé

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1. Nonlinear semigroups [Kömura (1967), Kato (1967)]

• Similarly to *linear* strongly continuous (C_0)-semigroups $\{T(t)\}_{t\geq 0}$ on a vector space X the *nonlinear* C_0-semigroups: $\{S(t)x\}_{t\geq 0,\, x\in\mathfrak{D}}$, have their origine in differentiable solutions $\{x(t,x_0)\}_{t\geq 0,\, x_0\in\mathfrak{D}} \subset X$ of *nonlinear* abstract Cauchy problem (ACP):

$$\begin{cases} \partial_t\, x(t) + F(x(t)) = 0 \quad \text{for} \quad t \in \mathbb{R}_0^+ := \{t \in \mathbb{R} : t \geq 0\}\,, \\ \{x(t,x_0)\}_{t\geq 0,\, x\in\mathfrak{D}} := \{x(t) : x(t)|_{t=0} = x_0 \in \mathfrak{D} \subseteq X\}\,, \end{cases}$$

• Any *unique* solution (*orbit*) of ACP has for $t, \tau \geq 0$ a *semigroup property*. Consecutive evolutions during $t \geq 0$, then $\tau \geq 0$ yield a *composition*: $x(\tau, x(t,x)) = x(\tau + t, x)$, into *one-step* evolution.

• If we consider solutions of ACP as generated for $t \geq 0$ by *initial conditions* $x \in \mathfrak{D}$, then the mapping: $x \mapsto S(t)x = x(t,x)$, and $\{S(t)\}_{t\geq 0}$ is a *linear*, or *nonlinear*, *semigroup* on $\mathfrak{D}$ for a *linear*, or *nonlinear* on $D \subseteq X$ function $F : D \ni u \mapsto F(u) \in X$ in the ACP.

• **Remark 1.1**.[*advantages* of linearity vs *pitfales* of nonlinearity]
(L) The *linear* ACP corresponds to a *linear* function: $F(u) = Au$, for a *linear* operator A on domain $\mathrm{dom}(A) = D \subseteq X$. Solution of linear ACP is defined by a *bounded linear* strongly *continuous*, or C_0-semigroup $\{T(t)\}_{t\geq 0}$, satisfying the following conditions:
(**1**) $T(t = 0) = \mathbb{I}$;
(**2**) if $0 \leq t_1, t_2 < \infty$, then $T(t_1)T(t_2) = T(t_1 + t_2)$;
(**3**) mappings $\mathbb{R}_0^+ \ni t \to T(t)x \in X$ are *continuos* for every $x \in X$.
(a) By (**1**)-(**3**) there exists limit $A\,x := \lim_{t\to +0} t^{-1}(\mathbb{I} - T(t))x$, $x \in \mathrm{dom}(A) \subset X$, called *generator* of semigroup $\{T(t)\}_{t\geq 0}$.
(b) Operator A is *closed* and *commutes* with $T(t)$ on $\mathrm{dom}(A)$, which is *dense* in X and $T(t)(\mathrm{dom}(A)) \subseteq \mathrm{dom}(A)$ (*invariance*).
(c) If linear operator A has $\overline{\mathrm{dom}(A)} = X$, $\mathrm{ran}(\mathbb{I} + \lambda A) = X$ and $\|(\mathbb{I} + \lambda A)^{-1}\| \leq 1$, for $\lambda \geq 0$, then *Euler formula*: $T_A(t)x := \lim_{n\to\infty}(\mathbb{I} + tA/n)^{-n}x$, defines on X a *contraction* C_0-semigroup ($\|T_A(t \geq 0)\| \leq 1$) with *generator* A. **[Hille-Phillips (1948-61)]**

(NL) The *nonlinear* ACP corresponds to a *nonlinear* function: $F(u) = Au$, where (unbounded) *nonlinear* operator A is defined on domain $\mathrm{dom}(A) = D \subseteq X$.

• In general, a **lack** of *linearity* makes the set of *initial conditions* for ACP $\mathfrak{D} \subset X$ much *smaller* than in (**3**) for **linear continuity**. As a consequence solution of *nonlinear* ACP is defined by a *non-linear* strongly continuous (C_0)-semigroup $\{S(t)\}_{t\geq 0}$ satisfying two conditions (**1**), (**2**) as for the *linear* C_0-semigroup $\{T(t)\}_{t\geq 0}$ but only on a subset $\mathfrak{D} \subset X$.

• The **lack** of *linearity* modifies jointly continuos (**3**) into *two* conditions. The strong continuity of $\{S(t)x\}_{t\geq 0}$ in $t \geq 0$ on subset $\mathfrak{D}$ and the *Lipschitz continuity* in $x \in \mathfrak{D}$ for any $t \geq 0$:

- For every $x \in \mathfrak{D}$ the function $t \to S(t)x$ is continuos on $\mathbb{R}_0^+$.
- For every $t \geq 0$ and $x_1, x_2 \in \mathfrak{D}$ the *nonlinear* semigroup $\{S(t)\}_{t\geq 0}$ satisfies condition $\|S(t)x_1 - S(t)x_2\| \leq e^{\omega t}\|x_1 - x_2\|$, where $\omega = 0$ for *contraction* nonlinear semigroups.

2. Nonlinearity and accretive operators

• To establish a relation between *nonlinearities* of $\{S(t)\}_{t\geq 0}$ and function F we use representation $F(u) =: Au$, for nonlinear operator A on domain $D \subseteq X$, and the *Euler* (*step lines*) approximation scheme for ACP on bounded interval: $0 < t_1 < \ldots < t_N < T$,

$$\frac{x(t_k) - x(t_{k-1})}{t_k - t_{k-1}} + Ax(t_k) = 0\,, \quad k = 1, \ldots, N\,.$$

For *nontrivial* solutions: $x(t_k) = (\mathbb{I} + (t_k - t_{k-1})A)^{-1} x(t_{k-1})$ the operator $(\mathbb{I} + \lambda A)$ must be invertible for $\lambda \geq 0$ on the ran$(\mathbb{I} + \lambda A)$.

• **Definition 2.1** [Kato (1967)] A (*nonlinear*) operator A in X is called (*monotonous*) accretive if for $\lambda \geq 0$ one has

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(Ax_1 - Ax_2)\| \quad \text{for } x_1, x_2 \in \mathrm{dom}(A)\,.$$

N.B. If A is accretive, then operator $(\mathbb{I} + \lambda A)$ is invertible.

• **Remark 2.1** If nonlinear A is *accretive*, then the mapping $(\mathbb{I}+\lambda A)^{-1}$: ran$(\mathbb{I}+\lambda A) \to$ dom(A) is *non-expansive* (contractive):

$$\|(\mathbb{I}+\lambda A)^{-1}x_1 - (\mathbb{I}+\lambda A)^{-1}x_2\| \leq \|x_1 - x_2\| \quad \text{for } \lambda \geq 0 \,. \qquad (*)$$

• **Remark 2.2 If** besides the *Euler* (*step lines*) approximation scheme, the operator $(\mathbb{I}+\lambda A)^{-1}$ ensures solution of ACP via *Euler product* approximation scheme for *exponential*:
$x_1(\tau) = (\mathbb{I}+\tau A)^{-1}x, x_2(\tau) = (\mathbb{I}+\tau A)^{-2}x \ldots, x_n(\tau) = (\mathbb{I}+\tau A)^{-n}x$
Then the chain of contractive inequalities $(*)$ yields that

$$\|(\mathbb{I}+\frac{t}{n}A)^{-n}x_1 - (\mathbb{I}+\frac{t}{n}A)^{-n}x_2\| \leq \|x_1 - x_2\| \quad \text{for } n \in \mathbb{N}\,,\ \ t \geq 0\,. \qquad (**)$$

• **Theorem 2.1** [Kömura (1967)] Let $\mathcal{H}$ be a Hilbert space. If ran$(\mathbb{I}+\lambda A) = \mathcal{H}$, $\lambda > 0$ (A is m-accretive), then $\overline{\mathrm{dom}(A)} \subset \mathcal{H}$ is *closed, convex* and such that $\overline{\mathrm{dom}(A)} = \mathfrak{D}$ is *domain* of convergence of *Euler approximants*: $S_A(t)x = \lim_{n\to\infty}(\mathbb{I}+tA/n)^{-n}x$.

These comments motivate definition of *nonlinear* C_0-semigroups.

• **Definition 2.1** [Kömura (1967)] Let X be a vector space (Banach or Hilbert). A one-parametric family of (nonlinear) operators $\{S(t)\}_{t\geq 0}$ acting on a subset $\mathfrak{D} \subseteq X$ and possessing the following properties:

(**1**) $S(t=0)x = x$ for $x \in \mathfrak{D}$;

(**2**) if $0 \leq t_1, t_2 < \infty$, then $S(t_1)S(t_2)x = S(t_1 + t_2)x$ for $x \in \mathfrak{D}$;

(**3**) for every $x \in \mathfrak{D}$ the function $t \to S(t)x$ is continuos on $\mathbb{R}_0^+$;

(4) $\|S(t)x_1 - S(t)x_2\| \leq \|x_1 - x_2\|$ for $x_1, x_2 \in \mathfrak{D}$ and $t \geq 0$,

is called *nonlinear* contraction semigroup, strongly continuos on the semigroup domain $\mathfrak{D}$.

• **Theorem 2.2** [Kömura (1969)] (*converse* to Theorem 2.1 in Hilbert space $\mathcal{H}$) Given a *closed, convex* $\mathfrak{D} \subseteq \mathcal{H}$ and nonlinear contraction semigroup $\{S(t)\}_{t\geq 0}$ with $\overline{\mathrm{dom}(S(t>0)) = \mathfrak{D}}$, there exists a nonlinear m-accretive operator A such that $\mathrm{dom}(A) = \mathfrak{D}$ and that $\{S_A(t)\}_{t\geq 0}$ *generated* by A *coincides* with $\{S(t)\}_{t\geq 0}$.

• **Remark 2.3** Problem of *analogue* the Hille-Phillips-Yosida theory for nonlinear C_0-semigroups in Hilbert/Banach spaces.
- Theorems 2.1, 2.2 are the *analogue* of the Hille-Phillips theory for nonlinear C_0-semigroups in Hilbert space $\mathcal{H}$ (Remark 1.1).
- Because of an "*amorphous*" geometry of Banach spaces there is no such *analogue* in X, except some partial results.

• **Theorem 2.3** [Crandall-Leggette *"one-way"* theorem (1971)] Let operator A in Banach space X be *accretive* and satisfy for $\lambda > 0$ the *range condition*: $\text{ran}(\mathbb{I} + \lambda A) \supset \overline{\text{dom}(A)}$. Then by *Euler formula* it generates a *nonlinear* (contraction) semigroup:

$$S_A(t)x = \lim_{n\to\infty} (\mathbb{I} + tA/n)^{-n}x\,, \quad x \in \overline{\text{dom}(A)}.$$

N.B. Crandall-Leggette (1971): Given a *nonlinear* (contraction) semigroup $\{S(t)\}_{t\geq 0}$ we do not know whether the *Euler formula* generates this semigroup for *some* infinitesimal A because $\overline{\text{dom}(A)}$ may be *not unique*, or *not invariant*, or even *empty*.

• **Example 2.1** (inviscid Burgers' equation)
- We consider nonlinear ACP in Hilbert space $\mathcal{H} = L^2(a,b)$:

$$\begin{cases} \partial_t u + u\,\partial_\sigma u = 0\,, \text{ here } u : \mathbb{R}_0^+ \times (a,b) \to L^2(a,b)\,, \\ u|_{t=0} = u(0,\sigma)|_{\sigma\in(a,b)} \in \mathfrak{D}_0 \subseteq L^2(a,b)\,, \end{cases}$$

where $\mathfrak{D}_0$ is the set of *initial* conditions for *orbits* $\{u\}_{t\geq 0} \subseteq \mathcal{H}$.
- Operator $Au := u\,\partial_\sigma u$ satisfies: $(Au_1 - Au_2, u_1 - u_2)_{\mathcal{H}} \geq 0$, for $u_1, u_2 \in \mathrm{dom}(A)$, that is, operator A is (*monotonous*) accretive on a domain $\mathrm{dom}(A) \subset \mathcal{H}$.
- Moreover, for any $\lambda > 0$ the *range* of operator A has the property: $\mathrm{ran}(\mathbb{I} + \lambda A) \supset \overline{\mathrm{dom}(A)}$.
- As a consequence (**Theorem 2.3** [Crandall-Leggette]), by *Euler formula* the operator A generates a *nonlinear* C_0-semigroup $\{S_A(t)\}_{t\geq 0}$ of *contractions* on a *subset* of Hilbert space $\mathcal{H}$:

$$S_A(t)u = \lim_{n\to\infty} (\mathbb{I} + tA/n)^{-n} u\,, \quad u \in \overline{\mathrm{dom}(A)}.$$

3. Product formulæ for (non)linear semigroups on $\mathcal{H}$

Lie Product Formula [Sophus Lie (1875)]

$$\|(e^{-tA/n}e^{-tB/n})^n - e^{-tC}\| = O(n^{-1}),$$

$A, B \in \mathcal{M}_{m\times m}$, $C = A+B$, and the matrix *operator-norm*: $\|A\| := \{\sup_{\|u\|=1} \|Au\| : u \in$ Hilbert space $\mathcal{H}^{(m)} = \mathbb{R}^m\}$ (or for $\mathcal{H}$).

Trotter Product Formula (1959)

- Let A, B be generators of contraction C_0-semigroups on $\mathcal{H}$. If closure $C := \overline{A+B}$ is also generator, then strong operator limit:

$$\lim_{n\to\infty} \left(e^{-tA/n}e^{-tB/n}\right)^n x = e^{-tC}x\,, \quad x \in \mathcal{H}\,. \quad t \geq 0\,.$$

Rogava Operator-Norm Convergence Theorem (1993)

- For **self-adjoint** generators $A \geq 0$, $B \geq 0$ the product formula

$$\left\|\left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-t(A+B)}\right\| \leq O(\ln(n)/\sqrt{n}), \quad n \in \mathbb{N}.$$

in operator-norm topology if operator $(A+B) \geq 0$ is **self-adjoint**.

Trotter product formulæ for nonlinear semigroups.

• **Theorem 3.1** Let A and B be m-accretive operators in a Hilbert space $\mathcal{H}$ and the closed convex set $\mathcal{C} \subset \overline{\text{dom}(A)} \bigcap \overline{\text{dom}(B)}$ be invariant under mappings by $(\mathbb{I} + \lambda A)^{-1}$ and $(\mathbb{I} + \lambda B)^{-1}$ for $\lambda > 0$. Then one gets

$$S_{A+B}(t)u = \lim_{n\to\infty} \left(S_A(\frac{t}{n})\, S_B(\frac{t}{n})\right)^n u\,, \qquad (*)$$

for $t \geq$ and any $u \in \mathcal{C} \bigcap \overline{\text{dom}(A) \bigcap \text{dom}(B)} \subset \mathcal{H}$.

• Then (**our aim !**) we shall continue by description on $\mathcal{H}$ a class of nonlinear C_0-semigroups such that:

- convergence in (*) will be for all $u \in \mathcal{H}$;
- and it holds even in the operator-norm topology on $\mathcal{L}(\mathcal{H})$.

4. Affine semigroups [Crandall-Pazy (1969)]

• Let $\{T(t)\}_{t\geq 0}$ be a *linear* strongly continuous *contraction* semigroup on Hilbert space $\mathcal{H}$. We define on $\mathrm{dom}(T(t)) = \mathcal{H}$ a family of mappings $\{S(t)\}_{t\geq 0}$ as a "*scalar*" shift of the C_0-semigroup $\{T(t)\}_{t\geq 0}$:

$$S(t) : u \mapsto S(t)u := T(t)u + f(t)\,, \quad u \in \mathcal{H}\,, \quad t \geq 0\,,$$

by a *continuous* vector-valued function $t \mapsto f(t) \in \mathcal{H}$.

Proposition 4.1 If for $t \geq 0$ the *shifting* function $t \mapsto f(t) \in \mathcal{H}$, satisfies equation:

$$T(t)f(\tau) = f(t+\tau) - f(t)\,, \quad t, \tau \geq 0\,,$$

then family $\{S(t)\}_{t\geq 0}$ is a *strongly* continuous *semigroup* on $\mathcal{H}$.

N.B. By definition of $S(t)$ and equation for $f(t)$ one gets that $f(0) = 0$, $S(0) = \mathbb{I}$ and $S(\tau)(S(t)u) = T(\tau)(S(t)u) + f(\tau) = T(\tau+t)u + T(\tau)f(t) + f(\tau) = T(\tau+t)u + f(\tau+t) = S(\tau+t)u$.

• **Theorem 4.1** $\{S(t)\}_{t\geq 0}$ is a *nonlinear contraction* semigroup strongly continuos on the semigroup domain $\mathfrak{D} = \mathcal{H}$.

N.B. Semigroup $\{S(t)\}_{t\geq 0}$ is *nonlinear*, since $S(t)\lambda u = T(t)\lambda u + f(t) \neq \lambda S(t)u$ (cf. $\lambda = 0$), and it verifies the properties **(1)**-**(3)** and **(4)** in Definition 2.1. This semigroup is a *contraction*: $\|S(t)u_1 - S(t)u_2\| = \|T(t)(u_1 - u_2)\| \leq \|u_1 - u_2\|$, because the linear semigroup $\{T(t)\}_{t\geq 0}$ is a *contraction*: $\|T(t)\| \leq 1$.

• The next statement makes analysis of the affine semigroups almost *explicit*.

Proposition 4.2 The solution of equation for family $\{f(t)\}_{t\geq 0}$ in Proposition 3.1 has the following *canonical* form:

$$f(t) = (\mathbb{I} - T(t)g + \int_0^t ds\, T(s)h\,, \quad (**)$$

where g and h are *any* two vectors in Hilbert space $\mathcal{H}$.

5. Trotter product formulæ for affine semigroups [VZ (2026)]

- Let affine semigroups $\{S_\alpha(t) := S_{(A_\alpha, g_\alpha, h_\alpha)}(t)\}_{\alpha,\, t\geq 0}$ correspond to the triplets $(A_\alpha, g_\alpha, h_\alpha)$ for $\alpha = 1, 2, 3$. Since for these nonlinear affine semigroups *domain* $\mathfrak{D} = \mathrm{dom}(S_\alpha(t)) = \mathcal{H}$, the Trotter product formulæ for couples of them *may* converge in the *strong operator topology* in the Hilbert space $\mathcal{H}$:

$$\lim_{n\to\infty} (S_1(t/n)\, S_2(t/n))^n\, u = S_3(t)u\,, \quad \text{for} \quad u \in \mathcal{H}\,.$$

- **Remark 5.1** Consider a couple of affine semigroups for *triples*: $(A, g_1, h_1 = 0)$, $(A, g_2, h_2 = 0)$ and suppose that the linear semigroup $\{T_A(t)\}_{t\geq 0}$ is contractive and $\|T_A(t > 0)\| < 1$. Then

$$\Big(S_{A,g_1}(t/n)\, S_{A,g_2}(t/n)\Big)^n u = \sum_{k=0}^{n-1} T_A(2t/n)^k W_A(t/n, g_{1,2}) + T_A(2t)u\,,$$

here $W_A(t/n, g_{1,2}) := (\mathbb{I} - T_A(t/n))(g_1 + g_2) - (\mathbb{I} - T_A(t/n))^2 g_2$ for $S_{A,g}(t/n)u = T_A(t/n)u + (\mathbb{I} - T_A(t/n))g$ and $u \in \mathcal{H}$, $n \geq 1$, $t \geq 0$.

As a consequence we obtain the limit:

$$\lim_{n\to\infty} \Big(S_{A,g_1}(t/n)\, S_{A,g_2}(t/n)\Big)^n u =$$
$$\lim_{n\to\infty} \left\{\frac{\mathbb{I}-(T_A(2t/n))^n}{\mathbb{I}-T_A(2t/n)} W_A(t/n, g_{1,2})\right\} + T_A(2t) u =$$
$$T_{2A}(t)u + (\mathbb{I} - T_{2A}(t))(g_1+g_2)/2 = S_{2A,(g_1+g_2)/2}(t)u.$$

• **Remark 5.2** This limit also implies the *operator-norm* convergence of the Trotter product formula on $\mathcal{H}$:

$$\lim_{n\to\infty} \| \Big(S_{A,g_1}(t/n)\, S_{A,g_2}(t/n)\Big)^n - S_{2A,(g_1+g_2)/2}(t)\| = 0.$$

• **Remark 5.3** Let $\{T_A(t)\}_{t\geq 0}$ be linear (*exponentially stable*: $\|T_A(t)\| \leq e^{\beta t}$, $\beta < 0$) *contraction* semigroup. Then A is *invertible* and similarly to **Remark 5.1** we can perform explicit calculations in the case of *triplets*: $(A, g_1 = 0, h_1)$, $(A, g_2 = 0, h_2)$, when

$$S_{A,h}(t)u := T_A(t)u + \int_0^t ds\, T_A(s)h \{= T_A(t)u + A^{-1}(\mathbb{I} - T_A(t))h\}.$$

- As a result: $\Big(S_{A,h_1}(t)\,S_{A,h_2}(t)\Big)\,u = T_A(2t)u + Z_A(t,h_{1,2})$, where

$$Z_A(t,h_{1,2}) := T_A(t)\,\int_0^t ds\,T_A(s)h_2 + \int_0^t ds\,T_A(s)h_1\,.$$

$\Big(S_{A,h_1}(t/n)\,S_{A,h_2}(t/n)\Big)^n u = \sum_{k=0}^{n-1} T_A(2t/n)^k Z_A(t/n,h_{1,2}) + T_A(2t)u$.
Then for contractive semigroup and $\|T_A(t>0)\| < 1$:

$$\lim_{n\to\infty}\Big(S_{A,h_1}(t/n)\,S_{A,h_2}(t/n)\Big)^n u =$$
$$\lim_{n\to\infty}\left\{\frac{\mathbb{I}-(T_A(2t/n))^n}{\mathbb{I}-T_A(2t/n)} Z_A(t/n,h_{1,2})\right\} + T_A(2t)u =$$
$$T_{2A}(t)u + \int_0^t ds\,T_{2A}(s)(h_1+h_2) = S_{2A,(h_1+h_2)}(t)u.$$

- **Remark 5.4** In fact, this limit yields the *operator-norm* convergence for the Trotter product formula:
$\sup_{\|u\|=1}\|\Big(S_{A,h_1}(t/n)\,S_{A,h_2}(t/n)\Big)^n u - S_{2A,(h_1+h_2)}(t)u\|\Big|_{n\to\infty} = 0$.

• **Remark 5.5** Now we consider a couple of affine semigroups for *triples*: $(A_1, g, h_1 = 0)$, $(A_2, g, h_2 = 0)$. We suppose that the linear C_0-semigroups $\{T_{A_1}(t)\}_{t\geq 0}$ and $\{T_{A_2}(t)\}_{t\geq 0}$ are *exponentially stable* and *self-adjoint* generators A_1 and A_2 satisfy the *Rogava condition*: the operator $A_{1,2} := A_1 + A_2$, is also *self-adjoint*.

$$
\begin{aligned}
& S_{A_k,g}(t)u = T_{A_k}(t)u + (\mathbb{I} - T_{A_k}(t))g\,, \;\; k = 1,2 \\
& \Big(S_{A_1,g}(t)\, S_{A_2,g}(t)\Big)\, u = T_{A_1}(t)T_{A_2}(t)u + V_{A_1,A_2}(t,g)\,, \\
& V_{A_1,A_2}(t,g) := [(\mathbb{I} - T_{A_1}(t)) + (\mathbb{I} - T_{A_2}(t))]g - \\
& (\mathbb{I} - T_{A_1}(t))(\mathbb{I} - T_{A_2}(t))g\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
& \Big(S_{A_1,g}(t/n)\, S_{A_2,g}(t/n)\Big)^n u = (T_{A_1}(t/n)T_{A_2}(t/n))^n u + \\
& \sum_{k=0}^{n-1} (T_{A_1}(t/n)T_{A_2}(t/n))^k \; V_{A_1,A_2}(t/n,g)\,,
\end{aligned}
$$

- **Theorem 5.1** Since due to the *Rogava condition* one has

$$\|\cdot\| - \lim_{n\to\infty}(T_{A_1}(t/n)T_{A_2}(t/n))^n = T_{A_1+A_2}(t)\,,\ t \geq 0\,,$$

on $\mathcal{L}(\mathcal{H})$ in the *operator norm*, we can deduce the operator-norm convergent Trotter product formula for the corresponding *nonlinear affin semigroups*. To this aim we use the identity

$$\Big(S_{A_1,g}(t/n)\,S_{A_2,g}(t/n)\Big)^n u = (T_{A_1}(t/n)T_{A_2}(t/n))^n u$$
$$- \frac{\mathbb{I} - (T_{A_1}(t/n)T_{A_2}(t/n))^n}{\mathbb{I} - T_{A_1}(t/n)T_{A_2}(t/n)}\ V_{A_1,A_2}(t/n,g)\,, \quad u, g \in \mathcal{H}\,,$$

and the *operator-norm* limit (due to the *Rogava condition*):

$$\lim_{n\to\infty}\sup_{\|u\|=1}\Big\|[T_{A_1+A_2}(t)u + (\mathbb{I} - T_{A_1+A_2}(t))g] -$$
$$\{(T_{A_1}(t/n)T_{A_2}(t/n))^n u + \frac{\mathbb{I} - (T_{A_1}(t/n)T_{A_2}(t/n))^n}{\mathbb{I} - T_{A_1}(t/n)T_{A_2}(t/n)}V_{A_1,A_2}(t/n,g)\}\Big\|\,.$$

Summarising these items we obtain the final result:

$$\lim_{n\to\infty} \sup_{\|u\|=1} \left\| \left(S_{A_1,g}(t/n)\, S_{A_2,g}(t/n)\right)^n u - \right.$$
$$\left. [T_{A_1+A_2}(t)u + (\mathbb{I} - T_{A_1+A_2}(t))g] \right\| = 0\,.$$

• **Remark 5.6** Combining the arguments in **Remark 5.3** and in **Remark 5.5** one can prove the *operator-norm* convergent Trotter product formula for *triplets*: $(A_1, g_1 = 0, h)$, $(A_2, g_2 = 0, h)$, when

$$S_{A_k,h}(t)u := T_{A_k}(t)u + \int_0^t ds\, T_{A_k}(s)\, h\,, \;\; k = 1, 2\,.$$
$$\left(S_{A_1,h}(t)\, S_{A_2,h}(t)\right) u = T_{A_1}(t)T_{A_2}(t)u + Y_{A_1,A_2}(t,h)\,,$$
$$Y_{A_1,A_2}(t,h) := T_{A_1}(t) \int_0^t ds\, T_{A_2}(s)h + \int_0^t ds\, T_{A_1}(s)h\,.$$

Now taking into account definitions in **Remark 5.6** one gets the following assertion about operator-norm convergence of the Trotter product formula.

• **Theorem 5.2** Under same conditions as in **Theorem 5.1** we obtain for *triplets*: $(A_1, 0, h)$, $(A_2, 0, h)$ the *operator-norm* convergent Trotter product formula:

$$\lim_{n\to\infty} \sup_{\|u\|=1} \Big\| \Big(S_{A_1,h}(t/n)\, S_{A_2,h}(t/n) \Big)^n u -$$
$$[T_{A_1+A_2}(t)u + \int_0^t ds\, T_{A_1+A_2}(s)\, 2h] \Big\| = 0\,.$$

6. Comments about general triplets: (A_1, g_1, h_1), (A_2, g_2, h_2)

• **Remark 6.1** As a first extension to a general case we consider a couple of affine semigroups for *triples*: $(A_1, g_1, 0)$, $(A_2, g_2, 0)$. Semigroups $\{T_{A_1}(t)\}_{t\geq 0}$ and $\{T_{A_2}(t)\}_{t\geq 0}$ are *exponentially stable* and *self-adjoint* generators satisfy the *Rogava condition*.

$$S_{A_k,g_k}(t)u = T_{A_k}(t)u + (\mathbb{I} - T_{A_k}(t))g_k\,, \;\; k = 1,2$$

$$\Big(S_{A_1,g_1}(t)\, S_{A_2,g_2}(t)\Big)\, u = T_{A_1}(t)T_{A_2}(t)u + V_{A_1,A_2}(t, g_{1,2})\,,$$

$$V_{A_1,A_2}(t, g_{1,2}) := (\mathbb{I} - T_{A_1}(t))g_1 + (\mathbb{I} - T_{A_2}(t))g_2 -$$

$$(\mathbb{I} - T_{A_1}(t))(\mathbb{I} - T_{A_2}(t))g_2\,.$$

• **Theorem 6.1** For these triplets we obtain the *operator-norm* convergent Trotter product formula to **affine semigroup** $(**)$:

$$\lim_{n\to\infty} \sup_{\|u\|=1} \Big\| \Big(S_{A_1,g_1}(t/n)\, S_{A_2,g_2}(t/n)\Big)^n u - \{T_{A_1+A_2}(t)u +$$

$$(\mathbb{I} - T_{A_1+A_2}(t))[A_1(A_1 + A_2)^{-1}g_1 + A_2(A_1 + A_2)^{-1}g_2]\}\Big\| = 0\,.$$

• **Remark 6.2** Under same conditions as above we consider a couple of affine semigroups for *triples*: $(A_1, 0, h_1)$, $(A_2, 0, h_2)$.

$$S_{A_k,h_k}(t)u := T_{A_k}(t)u + \int_0^t ds\, T_{A_k}(s)\, h\,,\;\; k = 1, 2\,.$$

$$\Big(S_{A_1,h_1}(t)\, S_{A_2,h_2}(t)\Big)\, u = T_{A_1}(t)T_{A_2}(t)u + Y_{A_1,A_2}(t, h_{1,2})\,,$$

$$Y_{A_1,A_2}(t, h_{1,2}) := T_{A_1}(t) \int_0^t ds\, T_{A_2}(s)h_2 + \int_0^t ds\, T_{A_1}(s)h_1\,.$$

• **Theorem 6.2** For triplets $(A_1, 0, h_1)$, $(A_2, 0, h_2)$ one gets the *operator-norm* convergent Trotter product formula:

$$\lim_{n\to\infty} \sup_{\|u\|=1} \Big\| \Big(S_{A_1,h_1}(t/n)\, S_{A_2,h_2}(t/n)\Big)^n u -$$

$$\{T_{A_1+A_2}(t)u + \int_0^t ds\, T_{A_1+A_2}(s)(h_1 + h_2)\}\Big\| = 0\,,$$

(again) to **affine semigroup** $(**)$.

• **Theorem 6.3** For general triplets (A_1, g_1, h_1), (A_2, g_2, h_2) and by virtue of the Rogava condition on *self-adjoint* operators A_1 and A_2, one obtains for corresponding *exponentially stable* semi-groups the *operator-norm* convergent Trotter product formula:

$$\lim_{n\to\infty} \sup_{\|u\|=1} \Big\| \Big(S_{A_1,h_1}(t/n)\, S_{A_2,h_2}(t/n) \Big)^n u -$$

$$\{T_{A_1+A_2}(t)u + (\mathbb{I} - T_{A_1+A_2}(t))g + \int_0^t ds\, T_{A_1+A_2}(s)h\} \Big\| = 0\,,$$

where

$$g = [A_1(A_1 + A_2)^{-1} g_1 + A_2(A_1 + A_2)^{-1} g_2]\,,$$

and

$$h = (h_1 + h_2)\,,$$

to **affine semigroup** $(**)$.

Merci!