Some remarks on quantum gases

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underlying space: bounded measure space:

$$(X, \mathcal{B}, \mathcal{B}_0, \varrho)$$

X lcscH topological space
$\mathcal{B}$ Borel sets
$\mathcal{B}_0$ bounded (relatively compact) Borel sets
ϱ Radon, i.e. locally finite measure on X (Radon measure)

next level: simple (point) processes in X

$$(\mathcal{M}^{\cdot}, P)$$

pair potential in X:

$$\Phi(x,y), \qquad x,y \in X$$

local desciption of a Gibbs process: $K \in \mathcal{B}_0$:

$$Q_K(f) = \frac{1}{\Xi_K} \cdot \int_{\mathfrak{X}_K} f(\xi)\, \mathrm{e}^{-E_\Phi(\xi)}\; \Lambda_\varrho(\mathrm{d}\,\xi), \quad f \in F$$

(empty boundary conditions!)

global description of a Gibbs process: (Lake Sevan 1976)

P of first order, solution of the equation (write $P \in \mathcal{G}(\varrho,\Phi)$)

$$\mathcal{C}_P(h) = \int_{\mathcal{M}^\cdot} \int_X h(x,\mu+\varepsilon_x)\, \mathrm{e}^{-W_\Phi(x,\mu)}\; \varrho(\mathrm{d}\,x) P(\mathrm{d}\,\mu), \qquad h \in F.$$

c-stability: there exists $\mathsf{c} : X \to [0, +\infty)$ mb such that

$$E_\Phi(\xi) \geq -\xi(\mathsf{c}), \qquad \xi \in \mathcal{X}.$$

$\mathcal{P}$-stability for c: there exists c mb such that for all x and ξ visible

$$W_\Phi(x, \xi) \geq -\mathsf{c}(x), \qquad \xi \in \mathcal{V}_\Phi = \{E_\Phi < +\infty\}.$$

(Penrose 1964)

under c-stability or $\mathcal{P}$-stability for c assume a regularity condition:

strong c-regularity: there exist $\mathrm{a} \geq \varepsilon$ and $0 < p < 1$ with

$$\int_X \left| \mathrm{e}^{-\Phi(x,y)} - 1 \right| \cdot \mathrm{e}^{(\mathrm{c}+\mathrm{a})(y)} \; \varrho(\mathrm{d}\, y) \leq p \cdot \mathrm{a}(x), \qquad x \in X;$$

c-regularity: $\varepsilon = 0, p = 1$, where always

$$\varrho_{\mathrm{c},\mathrm{a}}(\mathrm{d}\, y) := \mathrm{e}^{(\mathrm{c}+\mathrm{a})(y)} \; \varrho(\mathrm{d}\, y) \quad \text{locally finite}$$

two groups of assumptions on Φ:

$(\wp_\varrho)$: c-stable, 2c-regular for a;

$(\wp_\varrho^*)$: $\wp$-stable for c, strongly c-regular for a.

Theorem

$(\wp_\varrho)$ $\mathsf{G} = \lim_N \ Q_{K_n}$, *the limiting Gibbs process with empty boundary conditions, exists;* G *is of infinite order;* $\mathsf{G} \in \mathcal{G}(\varrho, \Phi)$.

$(\wp_\varrho^*)$ $\mathcal{G}(\varrho, \Phi)$ *is a singleton consisting of* G.

Polymers inspired by Ginibre's quantum gas (Les Houches 1970 [2, 3, 1])

$E=\mathbb{R}^d, \lambda(\mathrm{d}\,a)=\mathrm{d}\,a$
φ pp in E, $\wp$-stable for some constant b;
example: LJ-potential with hard core R_0: there exists some $0<R_0<R$ and positive constants C, C' such that

$$\begin{cases} \varphi(a)=+\infty, & 0\leq |a|\leq R_0, \\ \varphi(a)\geq \frac{C}{\left(|a|-R_0\right)^{d+\varepsilon}}, & R_0\leq |a|<R; \\ |\varphi|(a)\leq \frac{C'}{|a|^{d+\varepsilon}}, & R<|a|. \end{cases}$$

($\varphi=\varphi_1+\varphi_2$: $\varphi_2(a)\equiv 0, |a|\leq R$; integrable otherwise.
φ_1 arbitrary repulsion at the core.)

φ induces a pair potential Φ in $\mathcal{X}' = \mathcal{X} \setminus \{o\}$ with selfinteraction:
If $x = \varepsilon_{a_1} + \cdots + \varepsilon_{a_k}$ and $y = \varepsilon_{b_1} + \cdots + \varepsilon_{b_\ell}$

$$\Phi(x, y) = x \otimes y\,(\varphi) = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \varphi(a_i, b_j)$$

$$\Phi(x) = E_\varphi(x) = \sum_{1 \le i < j \le k} \varphi(a_i, a_j)$$

now underlying space $\mathcal{X}'$, not $E = \mathbb{R}^d$!

$$E_\Phi(\varkappa) = \sum_{x \in \kappa} \Phi(x) + \sum_{1 \le i < j \le n} \Phi(x_i, x_j), \quad \varkappa = \varepsilon_{x_1} + \cdots + \varepsilon_{x_n}$$

$$\mathcal{V}_\Phi = \{E_\Phi < +\infty\} = \{\varkappa : x_1 + \cdots + x_n \text{ hard core configuration}\}$$

Questions:

(A) Φ stable?

(B) $\mathcal{X}'$ lcscH?

(C) $\mathcal{B}_0(\mathcal{X}')$?

(D) ϱ?

first answers (straightforward; without proof):

(A) Φ $\mathcal{P}$-stable for $\mathrm{c}(x) = 3/2 \cdot \mathrm{b} \cdot |x|, x \in \mathcal{X}'$. $(W_\Phi(x, \varkappa) \geq -\mathrm{c}(x), \text{all } \varkappa)$;
(B) $\mathcal{X}'$ subspace of $\mathfrak{F}'$, the space of all closed subspaces $F \neq \emptyset$ of E.
$\mathfrak{F}'$ is lcscH for the Fell topology; $\mathcal{X}'$ mb subset of $\mathfrak{F}'$.
(C) bounded sets generated by

$$\mathfrak{F}_K = \{F \in \mathfrak{F}' : F \cap K \neq \emptyset\}, \qquad K \in \mathcal{B}_0.$$

bounded sets in $\mathcal{X}'$ generated by $\mathcal{X}'_K := \{x \in \mathcal{X}' : x(K) \geq 1\}$.

ad (D) $\varrho(\mathrm{d}\,y) = \mathrm{e}^{-E_\varphi(y)}\,.\tau_z(\mathrm{d}\,y)$, where

$$\tau_z(f) = \sum_{m=1}^{\infty} \frac{z^m}{m} \int_E \int_{E^{m-1}} f(\varepsilon_a + \varepsilon_{a_2} + \cdots + \varepsilon_{a_m})\, B_m^a(\mathrm{d}\,a_2 \ldots \mathrm{d}\,a_m)\,\mathrm{d}\,a,$$

$$B_m^a(\mathrm{d}\,a_2 \ldots \mathrm{d}\,a_m) = g(a, a_2) \cdots g(a_m, a)\ \mathrm{d}\,a_2 \ldots \mathrm{d}\,a_m,$$

$$g(a,b) = \frac{1}{(2\pi\beta)^{d/2}} \cdot \exp\Big(-\frac{|a-b|^2}{2\beta}\Big), \quad \beta > 0.$$

Lemma

If z small enough,

(a) *ϱ locally finite, i.e. $\varrho(\zeta_K \geq 1) < \infty, K \in \mathcal{B}_0$.*

(b) *Φ satisfies both regularity conditions for z small enough.*

Corollary

If z small enough, $\mathcal{G}(\Phi, \varrho) = \{\mathrm{G}\}$

(Lemma not evident, proof indicated): calculate Campbell measure of τ_z

$$\mathcal{C}_{\tau_z}(h) = \int_{\mathcal{X}'} \int_E h(a, y)\, y(\mathrm{d}\, a) \tau_z(\mathrm{d}\, y)$$

$$= \cdots$$

$$= \sum_{m=1}^{\infty} z^m \int_{E^m} h(a, \varepsilon_a + \varepsilon_{a_2} + \cdots + \varepsilon_{a_m}) g(a, a_2) \cdots g(a_m, a) \ \mathrm{d}\, a_2 \ldots \mathrm{d}\, a_m \,\mathrm{d}\, a$$

This implies that τ_z is of first order and thereby τ_z locally finite for z small:

$$\tau_z(\zeta_K \geq 1) \leq \nu^1_{\tau_z}(K) = \mathcal{C}_{\tau_z}(1_K \otimes 1)$$
$$= \sum_{m=1}^{\infty} z^m \cdot |K| g^{*m}(0,0) = \frac{|K|}{(2\pi\beta)^{d/2}} \cdot \sum_{m=1}^{\infty} \frac{z^m}{m^{d/2}}.$$

c-regularity of Φ for $\mathrm{a}(x) = \mathrm{v} \cdot |x|$:

v is the volume of a closed ball of diameter R_0.
$|S(x)| \leq \mathrm{a}(x)$, where $S(x)$ is the sausage of balls around the particles of x.
to show finiteness of

(§) $\tau_z(|\Phi_x| = \infty) \quad \Big(\leq \nu^1_{\tau_z}(S(x)) \Big)$;

(§§) $\int_{|\Phi_x|<\infty} |\Phi_x|(y)\, \tau_z(\mathrm{d}\, y) \quad \Big(\leq \sum_{u \in x} \mathcal{C}_{\tau_z}(h_u) \Big)$,

where

$$h_u(b, y) = \frac{1}{|u - b|^{d+\epsilon}} \cdot 1_{\{|\Phi_x|<\infty\}}(y).$$

Strauss' model for clustering [5]

slightly nearer to the quantum gas of Ginibre;
differs only in the choice of the underlying one-particle space

$$E_\imath = \{x = (x)_n = (a_1, \ldots, a_n) | a_j \in E, n \geq 1\} = \bigcup_{n \geq 1} E^n \times \{n\};$$

particles x have an inner structure given by polygonal trajectories in Euclidean space E.
$x(1) = a_1$ starting point of x.

$E_\imath$ lcscH topological space

B bounded subset in $E_\imath$ if

$$\Lambda = \{x(1) |\ x \in B\} \in \mathcal{B}_0(E);$$

(other choices for the collection of bounded sets are possible)

$$B \subseteq \mathcal{F}_\Lambda = \{x \in E_\imath |\ x(1) \in \Lambda\};$$

system $\mathcal{F}_\Lambda, \Lambda \in \mathcal{B}_0(E)$, generates a collection of bounded sets in $E_\imath$.

proceed as in the last example: Given φ, define in $E_\imath$ the pair potential Φ with self potential and consider the measure $\widetilde{\varrho}(\mathrm{d}\, y) = \mathrm{e}^{-E_\varphi(y)} .\widetilde{\tau}_z$ on $E_\imath$:

$$\widetilde{\tau}_z(f) = \sum_{m=1}^{\infty} \frac{z^m}{m} \int_E \int_{E^{m-1}} f(a, a_2, \ldots, a_m)\; B_m^a(\mathrm{d}\, a_2 \ldots \mathrm{d}\, a_m)\; \mathrm{d}\, a, \quad f \in F.$$

Corollary

z small enough

$$\mathcal{G}(\widetilde{\varrho}, \Phi) = \{\widetilde{\mathrm{G}}\}$$

(same proof with the obvious modifications)

Quantum gas?

§ In Ginibre's model the underlying space X is the space of winding loops in E of lengths $m\beta, m \in \mathbb{N}, \beta > 0$;
X Polish for the topology of uniform convergence on compacta; bounded sets characterized by the Arzela-Ascoli theorem;
(Nehring [4] known:existence G for small z for stable, integrable φ; Gibbsianess and unicity remained open.)

§§ if considered as a classical gas (Ginibre),
characteristic feature: particles have complicated inner structure; particles interact in a stable and regular way (e.g. Lennard-Jones).
Thus the underlying 1-particle space is no longer Euclidean;
for some abstract theory of quantum gases it should by at least Polish;
The most general assumption for the 1-particle space would be a bounded, mb measure space. There remains a lot to do!

further comments

(1) G, the limiting Gibbs process with empty boundary conditions, is an extremal element of $\mathcal{G}(\varrho, \Phi)$ and as such has nice mixing properties, which imply weak laws of large numbers and central limit theorems for the particle numbers in bounded reagons.

(2) If $\varphi \equiv 0$ then $\mathsf{G} = P_\varrho$, the Poisson process in $\mathcal{X}'$ with intensity ϱ. This process realizes clusters of interacting particles which are independent of one another.

(3) Consider the image of P_ϱ under the cluster dissolution mapping

$$\chi : \varepsilon_{x_1} + \cdots + \varepsilon_{x_n} \mapsto x_1 + \cdots + x_n.$$

It is well known (cf. [4]) that this is a permanental process, i.e. its correlation functions are permanents defined by g? Question: What is $\chi\mathsf{G}$ if there is an interaction between the clusters?

Some references

- Poghosyan, S., Ueltschi, D.: Abstract cluster expansion with applications, J. Math. Phys. 50 (2009)
- Ginibre, J.: Reduced density matrices of quantum gases. III. Hard-Core Potentials, J. Math. Phys. 6, 1432 (1965).
- Ginibre, J.: Some applications of functional integration in statistical mechanics, in: Statistical mechanics and field theory, C. de Witt and R. Stora eds., Gordon Breach (1971).
- Nehring, B.: Construction of classical and quantum gases, thesis (2013)
- Strauss, D.J.: A model for clustering, Biometrika 62, 467 - 475 (1975)