High Contrast Periodic Diffusion

and

Sticky Brownian Motion

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95th Anniversary!

The goal of my talk is to discuss how

the sticky Brownian motion appears

in **the problem of high contrast periodic homogenization**

for diffusion operators

Plan of my talk:

- We start our discussion with the 1-d Neumann problem on $[0, \infty)$, the probabilistic approach (Skorokhod problem for reflecting Brownian motion), the local time
- Sticky reflected Brownian motion (SRBM) on $[0, \infty)$, changing of time, the generator
- High contrast (HC) diffusions, the generator of corresponding Markov process
- Relation between SRBM and HCD

1-d Neumann problem

1) Reflected Brownian motion (RBM) $X(t)$ on $[0,\infty)$ is a 1-d Brownian motion (BM) with reflection at 0. The generator is given by

$$Lf(x) = \frac{1}{2} f^{''}(x),\ x > 0, \quad D(L) = \{f \in C^2(R_+),\ f^{'}(0) = 0\}$$

2) Probabilistic approach (The Skorokhod' problem)

The problem is to construct a pair of processes $(X(t), L(t))$ such that
1) $X(t) \geq 0\ \forall\, t$ and has continuous trajectories,
2) $L(t) \geq 0,\ L(0) = 0$, is a non-negative non-decreasing process with continuous trajectories, and

$$\int_0^t \mathbf{1}_{X(s)>0} dL(s) = 0$$

3) $X(t) = x_0 + B(t) + L(t)$

About local time = local time for BM $B(t)$

Thus, the RBM $X(t)$ is constructed together with another process $L(t)$ called local time process, and

$$\mathbf{dX(t) = 1_{(0,\infty)}(X(t))dW(t) + 1_{\{0\}}(X(t))dL(t)}$$

Local time $L_a(t)$ is a positive continuous additive functional (PCAF) of the process $B(t)$. One can consider other stochastic processes $X(t)$ and can define the corresponding local time as PCAF for $X(t)$.

$L_a(t)$ is a density (w.r.t. a Lebesgue measure dx) of the occupation measure for any t = a "measure" of time the process spends near position a:

$$L_a(t) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{B(s)\in[a-\varepsilon,a+\varepsilon]\}} ds$$

Properties of $L_0(t)$

The process $L(t) = L_0(t)$:

1) continuous trajectories,

2) non-decreasing and $L(0) = 0$;

3) $L(t) = \int_0^t 1_{B(s)=0}\, dL(s)$, i.e. $supp\, dL(s) \subset \{s :\ B(s) = 0\}$

$L(t)$ increases only when $B(t) = 0$ which is a set of 0 Lebesgue measure

3') $L(t) \le t$

Only one-dimensional $B(t)$!

Changing of time

There is a way of slowing down the process at 0. Let

$$r(s) = \varrho L(s) + s, \quad \varrho > 0 \text{ is a parameter.}$$

Define increasing family of stopping times τ_t by

$$r(\tau_t) = t, \quad \varrho L(\tau_t) + \tau_t = t, \qquad \text{i. e.} \quad \tau_t = r^{-1}(t).$$

There exists a unique solution which is strictly increasing and continuous in t.

Define a new process

$$\mathbf{Y}(\mathbf{t}) = \mathbf{X}(\tau_{\mathbf{t}}), \qquad X(s) \text{ is the RBM.}$$

SRBM $Y(t)$

Then we get for the new process $Y(t)$ the following SDE

$$dY(t) = 1_{(0,\infty)}(Y(t))\, dW(\tau_t) + 1_{\{0\}}(Y(t))\, dL(\tau_t)$$

Process $Y(t)$ is a Markov process with generator

$$Af(y) = \lim_{t\to 0} \tfrac{1}{t} \big(\mathbb{E}_y(f(y(t)) - f(y)\big) = \lim_{t\to 0} \tfrac{1}{t}\, \mathbb{E}_y \big(f(y(t)) - f(y(0))\big)$$

$$= \lim_{t\to 0} \tfrac{1}{t} \Big(f'(0)\, \varrho^{-1} \int_0^t 1_{\{0\}}(y(0))\, ds + \tfrac{1}{2} f''(y) \int_0^t 1_{(0,\infty)}(y(0))\, ds\Big)$$

$$= \begin{cases} \varrho^{-1} f'(0), & y = 0, \\ \tfrac{1}{2} f''(y), & y > 0 \end{cases}$$

$$Af(x) = \frac{1}{2} f^{''}(x),\ x > 0, \quad D(L) = \{f \in C^2(R_+),\ f^{'}(0) = \frac{\varrho}{2} f''(0)\}$$

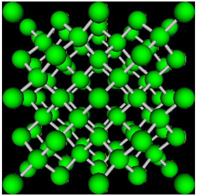
Problem setup: High-contrast models at microscale

Consider a symmetric diffusion operator in divergence form

$$A_\varepsilon f(x) = \operatorname{div}\left(a_\varepsilon(x)\nabla f(x)\right), \quad a_\varepsilon(x) = \{a_\varepsilon^{ij}(x)\}_{i,j=1}^{d}, \quad x \in \mathbb{R}^d,$$

where $a_\varepsilon^{ij}(x)$ are periodic for all $i, j = 1, \dots, d$ with period $[0,1)^d$.
In what follows we identify $[0,1)^d$-periodic functions with functions on the unit torus $\mathbb{T}^d$.

For simplicity,



$$a_\varepsilon(x) = \begin{cases} 1, & \text{if } x \in Y^\sharp = \mathbb{R}^d \backslash G^\sharp \\ \varepsilon^2, & \text{if } x \in G^\sharp \end{cases}$$

where $G \subset (0,1)^d$ is a smooth bounded simply connected domain such that $\overline{G} \subset (0,1)^d$, and $G^\sharp$ is a periodic extension of G in $\mathbb{R}^d$.

Homogenization: from micro to macro

We are interested in the macroscopic description of this model.

Scaling transformation with the coefficient ε: $x \to \varepsilon x,\ t \to \varepsilon^2 t$

$$A_\varepsilon f(x) = \mathrm{div}\left(a_\varepsilon(\frac{x}{\varepsilon})\nabla f(x)\right).$$

In $L^2(\mathbb{R}^d)$ we introduce a domain of A_ε by

$$D(A_\varepsilon) = \Big\{ f \in H^1(\mathbb{R}^d), \quad f \in H^2(G_\varepsilon^\sharp) \cap H^2(\mathbb{R}^d \setminus G_\varepsilon^\sharp),$$

$$\varepsilon^2 a_2(\tfrac{x}{\varepsilon})\nabla f(x)\big|_{\partial G_\varepsilon^\sharp} \cdot n^+ = -a_1(\tfrac{x}{\varepsilon})\nabla f(x)\big|_{\partial G_\varepsilon^\sharp} \cdot n^- \Big\}$$

The last condition on $f \in D(A_\varepsilon)$ is the condition of equality of the flows $a_\varepsilon \nabla f$ through the boundary $\partial G_\varepsilon^\sharp$.

Then $(A_\varepsilon, D(A_\varepsilon))$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$

Motivation: a model of oil percolation in porous media

T. Arbogast, J. Douglas, and U. Hornung, Derivation of the double porosity model of single phase flow via homogenization theory, SIAM J. Math. Anal., **21***(1990), pp. 823–836.*

$$\begin{cases} \partial_t u^\varepsilon = div(a^\varepsilon(x)\nabla u^\varepsilon) = (A_\varepsilon u^\varepsilon)(x) \\ u^\varepsilon|_{t=0} = u_0(x), \end{cases}$$

Under diffusive scaling $\varepsilon \to 0$ the limit evolution for $u(t,x)$ has a memory:

$$\partial_t u(x,t) = div(a^{eff}\nabla u(x,t)) + \int_0^t \mathfrak{D}(t-s)u(s,x)ds,$$

where $\mathfrak{D}(\cdot)$ is an exponentially decaying kernel.

Here we can raise a question: What is a mechanism of this phenomenon?

To answer this question we considered the homogenization problems using the "language" of stochastic processes and related Markov semigroups.

A. Piatnitski, S. Pirogov, E. Zhizhina, Limit behaviour of diffusion in high-contrast periodic media and related Markov semigroups, Applicable Analysis, 2019.

Homogenization of high-contrast elliptic operators

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Scaling limit of Markov processes in high-contrast periodic media

Homogenization = scaling limit

Homogenization of $A_\varepsilon \implies$

The corresponding semigroup $T_\varepsilon(t) = e^{tA_\varepsilon}$ is associated with the Markov process $\mathfrak{X}_\varepsilon(t) = \varepsilon X_\varepsilon(t/\varepsilon^2)$:

$$u_\varepsilon(x,t) = T_\varepsilon(t)u_0(x) \equiv \mathbb{E}_x u_0(\mathfrak{X}_\varepsilon(t)) \quad \Leftrightarrow \quad \begin{cases} \frac{\partial}{\partial t} u_\varepsilon(x,t) = A_\varepsilon u_\varepsilon(x,t) \\ u_\varepsilon(x,0) = u_0(x) \end{cases}$$

Question:

$T_\varepsilon(t) \to ???$ and $\mathfrak{X}_\varepsilon(t) \to ???$ as $\varepsilon \to 0$?

What process will we get in the limit?

C_0 setting, $\varepsilon > 0$

We consider A_ε as an unbounded operator in $C_0(\mathbb{R}^d)$, where $C_0(\mathbb{R}^d)$ is the Banach space of continuous functions on $\mathbb{R}^d$ vanishing at infinity with the norm $\|f\| = \max_x \|f(x)\|$.

The domain $D(A_\varepsilon)$ of the operator A_ε is the closure in the graph norm of the set of functions

$$\Big\{ f \in C_0^b(\mathbb{R}^d),\ f^+ = f\big|_{G_\varepsilon^\sharp} \in C^\infty(\overline{G_\varepsilon^\sharp}),\ f^- = f\big|_{\mathbb{R}^d\setminus G_\varepsilon^\sharp} \in C^\infty(\overline{\mathbb{R}^d \setminus G_\varepsilon^\sharp}),$$

$$\varepsilon^2 a_2(\frac{x}{\varepsilon})\nabla f^+(x)\big|_{\partial G_\varepsilon^\sharp} \cdot n^+ = -a_1(\frac{x}{\varepsilon})\nabla f^-(x)\big|_{\partial G_\varepsilon^\sharp} \cdot n^-,$$

$$\operatorname{div}\Big(\varepsilon^2 a_2(\frac{x}{\varepsilon})\nabla f^+(x)\Big)\big|_{\partial G_\varepsilon^\sharp} = \operatorname{div}\Big(a_1(\frac{x}{\varepsilon})\nabla f^-(x)\Big)\big|_{\partial G_\varepsilon^\sharp}\Big\}.$$

Then the closure of A_ε is the generator of a strongly continuous contraction semigroup $T_\varepsilon(t)$ in $C_0(\mathbb{R}^d)$ associated with diffusion process $\mathfrak{X}_\varepsilon(t)$ in $\mathbb{R}^d$:

$$T_\varepsilon(t)f(x) = \mathbb{E}_x f(\mathfrak{X}_\varepsilon(t)).$$

Construction of the extended process

Let us supply the process $\mathfrak{X}_\varepsilon(t)$ in $\mathbb{R}^d$ with an additional component $k_\varepsilon(t) = k_\varepsilon(\mathfrak{X}_\varepsilon(t))$, and for the extended process $\{\mathfrak{X}_\varepsilon(t), k_\varepsilon(t)\}$ we prove the convergence of the corresponding semigroups.

The component $k_\varepsilon(x)$ indicates the position in the cell, it takes values in $G^\star = G \cup \{\star\}$, and is defined by

$$k_\varepsilon(x) \;=\; \begin{cases} \star, & \text{if } \; x \in \varepsilon Y^\sharp, \\ \left\{\frac{x}{\varepsilon}\right\} \in G, & \text{if } \; x \in \varepsilon G^\sharp, \end{cases} \qquad \mathbb{R}^d = \varepsilon Y^\sharp \cup \varepsilon G^\sharp.$$

Then the extended process $\{\mathfrak{X}_\varepsilon(t), k_\varepsilon(t)\}$ lives in

$$E_\varepsilon \;=\; \left\{(x, k_\varepsilon(x)),\; x \in \mathbb{R}^d,\; k_\varepsilon(x) \in G^\star\right\}, \quad E_\varepsilon \subset E = \mathbb{R}^d \times G^\star.$$

Question: What is the limit $(\mathfrak{X}_\varepsilon(t), k_\varepsilon(t)) \to (B(t), Y(t))$ as $\varepsilon \to 0$?

The limit semigroup $T(t)$

The limit process lives on

$$E = \mathbb{R}^d \times G^*, \qquad G^* = G \cup \{*\}, \;\; * = \partial G,$$

where all points of the boundary ∂G are identified with a single point, denoted $*$.

Denote by $C_0^G(E)$ a linear space of functions $f(x,y) \in C_0^G(E)$ on the extended space E:

$$f(x,y), \qquad x \in \mathbb{R}^d; \quad y \in G^* = G \cup \{*\}.$$

$$f(x,y) = (f_0(x), f_1(x,y)), \quad f_0(x) = f(x,*), \quad f_1(x,y) = f(x,y), \text{ as } \; y \in G.$$

Here

$$f_0 \in C_0(\mathbb{R}^d), \; f_1 \in C_0(\mathbb{R}^d, C(G^*))$$

$$f_1(x,y)|_{y\in\partial G} := \lim_{dist(y,\partial D)\to 0} f_1(x,y) = f_0(x) \quad \forall x \in \mathbb{R}^d.$$

The generator of the semigroup $T(t)$

Let us consider in $C_0^G(E)$ unbounded operator A of the following form

$$(Af)(x,y) = \begin{cases} \Theta\nabla\nabla f_0(x) + \frac{1}{|G^c|}\int\limits_{\partial G} \frac{\partial f_1(x,y)}{\partial n_y^-} d\sigma(y), \quad y = *, \\ \\ \triangle_y f_1(x,y), \quad y \in G \end{cases}$$

The domain $D(A)$ of the operator A is a closure (in the graph norm) of

$$D_A = \Big\{ f_0 \in C_0^\infty(\mathbb{R}^d),\ f_1 \in C_0^\infty(\mathbb{R}^d;\ C^\infty(\overline{G})),\ f_1(x,y)|_{y\in\partial G} = f_0(x),$$
$$\triangle_y f_1(x,y)\Big|_{y\in\partial G} = \Theta\nabla\nabla f_0(x) + \frac{1}{|G^c|}\int\limits_{\partial G} \frac{\partial f_1(x,y)}{\partial n_y^-} d\sigma(y) \Big\}.$$

Lemma (PPZh 19)

The closure of the operator A is **a generator of a strongly continuous, positive, contraction semigroup $T(t)$ on $C_0^G(E)$**.

Let us go back to our generator for the limit high contrast diffusion:

$$(Lf)(x,y) = \begin{cases} \Theta\cdot\nabla_x\nabla_x f(x,*) + \frac{1}{|G^c|}\int\limits_{\partial G} \frac{\partial f(x,y)}{\partial n_y^-} d\sigma(y), & y = *, \\ \triangle_y f(x,y), \quad y \in G \end{cases}$$

The domain $D(L)$ of the operator L is a closure (in the graph norm) of

$$D_L = \Big\{ f(\cdot,*) \in C_0^\infty(\mathbb{R}^d),\ f \in C_0^\infty(\mathbb{R}^d;\ C^\infty(\overline{G})),\ f(x,y)|_{y\in\partial G} = f(x,*),$$
$$\triangle_y f(x,y)\Big|_{y\in\partial G} = \Theta\cdot\nabla_x\nabla_x f(x,*) + \frac{1}{|G^c|}\int\limits_{\partial G} \frac{\partial f(x,y)}{\partial n_y^-} d\sigma(y)\Big\}.$$

Let us rewrite L as follows

$$Lf(x,y) = \Theta\cdot\nabla_x\nabla_x f(x,y)\mathbf{1}_{\{y=*\}} + Mf(x,y),$$

where....

$$(Mf)(x,y) = \begin{cases} \frac{1}{|G^c|} \int\limits_{\partial D} \frac{\partial f(x,y)}{\partial n_y^-} d\sigma(y), & y \in \partial G = *, \\ \\ \triangle_y f(x,y), & y \in G. \end{cases}$$

Thus, the operator M acts only on y-variable: $M = M_G \otimes I_x$ (with identity operator I_x), where

$$(M_G g)(y) = \begin{cases} \frac{1}{|G^c|} \int\limits_{\partial D} \frac{\partial g(y)}{\partial n_y^-} d\sigma(y), & y \in \partial G = *, \\ \triangle_y g(y), & y \in G, \end{cases}$$

$g \in C^2(G^*)$. Here G^* is the domain G with boundary ∂G, where all boundary points are identified with a single point, denoted $*$: $\partial G = \{*\}$.

Question: Is M_G a generator of a Markov process?

We found the answer in the paper by M. Freidlin, L. Koralov and A. Wentzell, "On diffusions in media with pockets of large diffusivity", *Probability Theory and Related Fields* (2017). They consider the operator

$$(M_G g)(y) = \begin{cases} \frac{1}{|G^c|} \int\limits_{\partial D} \frac{\partial g(y)}{\partial n_y^-} d\sigma(y), \quad y \in \partial G = *, \\ \triangle_y g(y), \quad y \in G \end{cases}$$

Theorem (Freidlin, Koralov, Wentzell 2017)

Operator M_G is the generator of a **sticky reflected Brownian motion in G^* with the sticky parameter** $|G^c|$.

Together with M we define the operator B_y, which depends on the value of the base point $y \in G^*$:

$$(B_y f)(x,y) = \begin{cases} \Theta \nabla_x \nabla_x f(x,y), & \text{if} \quad y = *, \\ 0, & \text{if} \quad y \in G \end{cases}$$

Thus, the operator (generator) L has a "skew" structure:

$$Lf(x,y) = \big(B_y f(\cdot,y)\big)(x) + \big(Mf(x,\cdot)\big)(y).$$

and we can describe the whole process generated by L:

M generates evolution in G^*, while B_y generates diffusion in $\mathbb{R}^d$, controlled by the first process in G^*: the process in $\mathbb{R}^d$ evolves only when the diffusion in G^* is glued to the boundary.

Thank you for your attention!