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A. V. Poghosyan

Asymptotic Behavior of Hermite-Eckhoff Interpolation

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1. A sequence of trigonometric Hermite interpolation polynomials with equidistant interpolation nodes and uniform multiplicities is investigated. Convergence acceleration based on the Eckhoff approach is considered. Corresponding asymptotic error is derived.

2. In this paper we continue investigations started in [1], where the sequence $T_{p,N}(f)(x)$, $p \geq 1$, $N \geq 1$ of trigonometric Hermite interpolation polynomials is considered. This method of construction of the trigonometric Hermite interpolants may be considered as a continuation of the method of Berrut and Welcher [2]. Convergence acceleration is achieved by application of the Krylov-Lanczos approach ([3], [4]) that uses the idea of subtracting a polynomial which represents the discontinuities in the function and some of its first derivatives (jumps). In [1] we assume that the exact values of the jumps are known. Here we consider the problem of jumps approximation based on the Eckhoff approach ([5]).

Let $f \in C^{p-1}[-1,1]$, $p \geq 1$. Let $\check{f}_m^{(j)}$ denote the discrete Fourier coefficients of $f^{(j)}$

$$\check{f}_m^{(j)} := \frac{1}{2N+1} \sum_{k=-N}^{N} f^{(j)}(x_k) e^{-i\pi m x_k},\ x_k = \frac{2k}{2N+1},\ j = 0, \cdots, p-1,\ |m| \leq N.$$

We set $\check{f}_n := \check{f}_n^{(0)}$.

In [1] we derive relatively compact formula for the trigonometric Hermite interpolation that gives the interpolants as functions of the coefficients in the discrete Fourier coefficients of the derivative values

$$T_{p,N}(f)(x) := \sum_{m=-N(1-\sigma)}^{N(1+\sigma)} \sum_{j=0}^{p-1} \check{f}_m^{(j)} \sum_{k=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} c_{k,j}(m) e^{i\pi(m+k(2N+1))x}.$$

Here $\sigma = 0$ for odd values of parameter p, $\sigma = 1$ for even values and $[x]$ denotes the greatest integer less than or equal to x. Numbers $c_{k,j}(m)$ are defined by the formula

$$c_{k,j}(m) := \frac{1}{(i\pi(2N+1))^j} \beta_{k,j}\left(\frac{m}{2N+1}\right),$$

where

$$\beta_{k,j}(x) := \frac{1}{\prod_{\substack{\ell=-[\frac{p}{2}] \\ \ell \neq k}}^{[\frac{p-1}{2}]} (k-\ell)} \sum_{s=j+1}^{p} (-1)^{p-s} \rho_s(x)(x+k)^{s-j-1} \tag{1}$$

and the $\rho_j(x)$ are coefficients of the polynomial

$$\prod_{s=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} (y + (x+s)) = \sum_{s=0}^{p} \rho_s(x) y^s. \tag{2}$$

The accelerating convergence of $T_{p,N}(f)$ is achieved ([1]) by the Krylov-Lanczos approach. Let $A_k(f)$ be the jump of the k-th derivative of function f

$$A_k(f) := f^{(k)}(1) - f^{(k)}(-1), \; k = 0, \cdots, q.$$

The following expansion is crucial for the Krylov-Lanczos approach

$$f(x) = F(x) + \sum_{k=0}^{q-1} A_k(f) B_k(x), \tag{3}$$

where the B_k are 2-periodic extensions of the Bernoulli polynomials with the Fourier coefficients

$$B_{k,n} := \begin{cases} 0, & n = 0, \\ \frac{(-1)^{n+1}}{2(i\pi n)^{k+1}}, & n \neq 0, \end{cases}$$

and F is a 2-periodic and smooth function ($F \in C^{q-1}(R)$) on the real line.

Approximation of F in (3) by $T_{p,N}(f)$, for $q \geq p$, leads to the *Hermite-Krylov-Lanczos interpolation* ([1])

$$T_{q,p,N}(f)(x) := \sum_{m=-N(1-\sigma)}^{N(1+\sigma)} \sum_{j=0}^{p-1} \check{F}_m^{(j)} \sum_{k=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} c_{k,j}(m) e^{i\pi(m+k(2N+1))x} + \sum_{k=0}^{q-1} A_k(f) B_k(x),$$

where the coefficients $\check{F}_m^{(j)}$ can be calculated from (3).

We put

$$R_{q,p,N}(f)(x) := f(x) - T_{q,p,N}(f)(x)$$

and by $||f||$ we denote the standard norm in the space $L_2(-1,1)$

$$||f|| := \left(\int_{-1}^{1} |f(x)|^2 dx \right)^{1/2}.$$

The next theorem reveals the asymptotic behavior of the Hermite-Krylov-Lanczos interpolation.

Theorem 1. *[1] Let $f \in C^q[-1,1]$, $q \geq 1$ be such that $f^{(q)}$ is absolutely continuous on $[-1,1]$. Then the following estimate holds for $q \geq p$*

$$\lim_{N\to\infty} (2N+1)^{q+\frac{1}{2}} \|R_{q,p,N}(f)\| = |A_q(f)| t(q,p),$$

$$t(q,p) := \frac{1}{\sqrt{2}\,\pi^{q+1}} \left(\int_{-\frac{1-\sigma}{2}}^{\frac{1+\sigma}{2}} \sum_{k=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} \left| \sum_{j=0}^{p-1} \beta_{k,j}(x) \sum_s{}^{*} \frac{(-1)^s}{(x+s)^{q-j+1}} \right|^2 dx + \frac{2^{2q+2}}{(2q+1)p^{2q+1}} \right)^{\frac{1}{2}},$$

where

$$\sum_s{}^{*} := \sum_{s=-\infty}^{-[\frac{p}{2}]-1} + \sum_{s=[\frac{p-1}{2}]+1}^{\infty}$$

and the $\beta_{k,j}$ are defined by (1).

Numerical values of $t(q,p)$ are presented in Table 1.

Table 1: **Numerical values of $t(q,p)$.**

$q \setminus p$	1	2	3	4	5
1	0.24	–	–	–	–
2	0.11	0.025	–	–	–
3	0.063	$9.4 \cdot 10^{-3}$	$3.6 \cdot 10^{-3}$	–	–
4	0.034	$2.4 \cdot 10^{-3}$	$5.8 \cdot 10^{-4}$	$2.2 \cdot 10^{-4}$	–
5	0.020	$7.7 \cdot 10^{-4}$	$1.5 \cdot 10^{-4}$	$5.0 \cdot 10^{-5}$	$2.3 \cdot 10^{-5}$

3. First we consider the problem of the jumps approximation by the Fourier coefficients $f_n^{(j)}$. In view of (3) we write

$$\check{F}_n^{(j)} = \check{f}_n^{(j)} - \sum_{k=j}^{q-1} A_k(f) \check{B}_{k-j,n},\ n \neq 0,\ j = 0, \cdots, p-1. \tag{4}$$

Taking into account that $\check{F}_n^{(j)}$ asymptotically ($n \to \infty$) decay faster than the coefficients $\check{f}_n^{(j)}$ we get the following system of equations with unknowns $\widetilde{A}_k(f, N)$

$$\check{f}_n^{(j)} = \sum_{k=j}^{q-1} \widetilde{A}_k(f, N) \check{B}_{k-j,n}, \ n = n_1, n_2, \ldots, n_q, \ j = 0, \cdots, p-1. \tag{5}$$

In the remainder of this paper we will assume for simplicity that q is even. Odd values of q can be handled similarly. We are interested in the case when $p = q$.

For $n = \pm N$ from (5) we derive

$$\check{f}_N^{(2j)} = \sum_{k=j}^{\frac{q}{2}-1} \widetilde{A}_{2k} \check{B}_{2k-2j,N} + \sum_{k=j}^{\frac{q}{2}-1} \widetilde{A}_{2k+1} \check{B}_{2k-2j+1,N}, \ 0 \le j \le \frac{q}{2} - 1,$$

$$\check{f}_{-N}^{(2j)} = \sum_{k=j}^{\frac{q}{2}-1} \widetilde{A}_{2k} \check{B}_{2k-2j,-N} + \sum_{k=j}^{\frac{q}{2}-1} \widetilde{A}_{2k+1} \check{B}_{2k-2j+1,-N}, \ 0 \le j \le \frac{q}{2} - 1.$$

Taking into account the obvious relations

$$\check{B}_{2k,-n} = -\check{B}_{2k,n}, \ \ \check{B}_{2k+1,-n} = \check{B}_{2k+1,n} \tag{6}$$

we get

$$\frac{\check{f}_N^{(2j)} - \check{f}_{-N}^{(2j)}}{2} = \sum_{k=j}^{\frac{q}{2}-1} \widetilde{A}_{2k} \check{B}_{2k-2j,N}, \ 0 \le j \le \frac{q}{2} - 1. \tag{7}$$

Similarly, we obtain

$$\frac{\check{f}_N^{(2j+1)} - \check{f}_{-N}^{(2j+1)}}{2} = \sum_{k=j}^{\frac{q}{2}-1} \widetilde{A}_{2k+1} \check{B}_{2k-2j,N}, \ 0 \le j \le \frac{q}{2} - 1. \tag{8}$$

First we need the following lemma.

Lemma 1. *The following estimate is true*

$$\check{B}_{k,N} = \frac{(-1)^{N+1} \varphi_k}{2(i\pi(2N+1))^{k+1}} + O(N^{-k-2}), \ N \to \infty,$$

where

$$\varphi_k := \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(s + \frac{1}{2})^{k+1}}.$$

Proof. We have

$$\begin{aligned} \check{B}_{k,N} &= \sum_{s=-\infty}^{\infty} B_{k,N+s(2N+1)} = \frac{(-1)^{N+1}}{2(i\pi)^{k+1}} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(N(2s+1)+s)^{k+1}} \\ &= \frac{(-1)^{N+1}}{2(i\pi)^{k+1}} \sum_{m=0}^{\infty} \begin{pmatrix} m+k \\ k \end{pmatrix} \frac{(-1)^m}{N^{m+k+1}} \sum_{s=-\infty}^{\infty} \frac{(-1)^s s^m}{(2s+1)^{m+k+1}}. \end{aligned}$$

This ends the proof.

The next theorem reveals the accuracy of the jumps approximation by systems (7) and (8).

Theorem 2. *Let $q \geq 2$ be an even number and $f \in C^q[-1,1]$ with absolutely continuous $f^{(q)}$ on [-1,1]. Then the following estimates hold for the solutions of systems* (7) *and* (8) *as $N \to \infty$ and $k = 0, \cdots, \frac{q}{2}-1$*

$$\widetilde{A}_{2k}(f,N) - A_{2k}(f) = A_q(f)\frac{\nu_k}{(i\pi(2N+1))^{q-2k}} + o(N^{-q+2k}), \qquad (9)$$

$$\widetilde{A}_{2k+1}(f,N) - A_{2k+1}(f) = A_{q+1}(f)\frac{\nu_k}{(i\pi(2N+1))^{q-2k}} + o(N^{-q+2k}), \qquad (10)$$

where the ν_k are defined by the recurrent relation

$$\sum_{k=j}^{\frac{q}{2}-1} \nu_k \, \varphi_{2k-2j} = \varphi_{q-2j}, \; j = 0, \cdots, \frac{q}{2}-1. \qquad (11)$$

Proof. We start with the proof of estimate (9). Lemma 1, Equations (4) and (5) imply as $N \to \infty$ for $j = 0, \cdots, \frac{q}{2}-1$

$$\check{f}^{(2j)}_{\pm N} = \sum_{k=2j}^{q-1} A_k(f)\check{B}_{k-2j,\pm N} + A_q(f)\frac{(-1)^{N+1}\varphi_{q-2j}}{2(i\pi(2N+1))^{q-2j+1}} + o(N^{-q+2j-1}). \qquad (12)$$

In view of relations (6) we get from (7) and (12) as $N \to \infty$

$$\sum_{k=j}^{\frac{q}{2}-1}(\widetilde{A}_{2k} - A_{2k})\check{B}_{2k-2j,N} = A_q(f)\frac{(-1)^{N+1}\varphi_{q-2j}}{2(i\pi(2N+1))^{q-2j+1}} + o(N^{-q+2j-1}). \qquad (13)$$

We proceed by the help of mathematical induction. For $j = (q-2)/2$ Equation (13) implies

$$(\widetilde{A}_{q-2} - A_{q-2})\check{B}_{0,N} = A_q\frac{(-1)^{N+1}}{2(i\pi(2N+1))^3}\varphi_2 + o(N^{-3})$$

which coincides with (9) when $k = (q-2)/2$. Suppose that (9) is valid for $k = j_0, \cdots, (q-2)/2$. For $j = j_0 - 1$ we have from (13)

$$\begin{aligned}(\widetilde{A}_{2j_0-2} &- A_{2j_0-2})\check{B}_{0,N} + \sum_{k=j_0}^{\frac{q}{2}-1}(\widetilde{A}_{2k} - A_{2k})\check{B}_{2k-2j_0+2,N} \\ &= A_q(f)\frac{(-1)^{N+1}\varphi_{q-2j_0+2}}{2(i\pi(2N+1))^{q-2j_0+3}} + o(N^{-q+2j_0-3}).\end{aligned}$$

Taking into account that (9) is valid for $k = j_0, \cdots, (q-2)/2$ we get (in view of (11))

$$\begin{aligned}\widetilde{A}_{2j_0-2} &- A_{2j_0-2} = \frac{A_q}{2(i\pi(2N+1))^{q-2j_0+2}}\frac{1}{\varphi_0}\left(\sum_{k=j_0}^{\frac{q}{2}-1}\nu_k\varphi_{2k-2j_0+2} - \varphi_{q-2j_0+2}\right) \\ &+ o(N^{-q+2j_0-2}) = \frac{A_q}{2(i\pi(2N+1))^{q-2j_0+2}}\nu_{j_0-1} + o(N^{-q+2j_0-2}).\end{aligned}$$

This ends the proof of the first estimate. The second can be proved similarly.

Suppose that $\widetilde{A}_k$ are determined from system (5). We expand the interpolated function in the form

$$f(x) = G(x) + \sum_{k=0}^{q-1} \widetilde{A}_k(f,N) B_k(x). \tag{14}$$

Approximation of G, for $q \geq p$, by the trigonometric Hermite interpolation $T_{p,N}(f)$ leads to the *Hermite-Eckhoff interpolation*

$$\widetilde{T}_{q,p,N}(f)(x) := \sum_{m=-N(1-\sigma)}^{N(1+\sigma)} \sum_{k=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} \sum_{j=0}^{p-1} \check{G}_m^{(j)} c_{k,j}(m) e^{i\pi(m+k(2N+1))x} + \sum_{k=0}^{q-1} \widetilde{A}_k(f,N) B_k(x),$$

where the $\check{G}_m^{(j)}$ can be calculated from (14).

Denote

$$\widetilde{R}_{q,p,N}(f)(x) := f(x) - \widetilde{T}_{q,p,N}(f)(x). \tag{15}$$

We are interested in asymptotic behavior of $\widetilde{R}_{q,p,N}$. First we will proof some auxiliary lemmas.

Let $f_n^{(j)}$ be the Fourier coefficients of the j-th derivative of f

$$f_n^{(j)} := \frac{1}{2} \int_{-1}^{1} f^{(j)}(x) e^{-i\pi n x} dx, \; j \geq 0.$$

We set $f_n := f_n^{(0)}$.

Lemma 2. *Let $f \in C^q[-1,1]$ for some $q \geq 1$ and $f^{(q)}$ be absolutely continuous on $[-1,1]$. Then for $0 \leq j \leq q-1$ the following is true*

$$G_m^{(j)} = (i\pi m)^j G_m - \sum_{k=0}^{j-1} (A_k(f) - \widetilde{A}_k(f,N)) \frac{(-1)^{m+1}}{2(i\pi m)^{k-j+1}}.$$

Proof. Due to integration by parts we get

$$G_m^{(j)} = (i\pi m)^j G_m - \sum_{k=0}^{j-1} A_k(G) \frac{(-1)^{m+1}}{2(i\pi m)^{k-j+1}}.$$

This completes the proof as $A_k(G) = A_k(f) - \widetilde{A}_k(f)$, $k = 0, \cdots, q-1$.

Lemma 3. *Let $q \geq 2$ be an even number and $f \in C^q[-1,1]$ with absolutely continuous $f^{(q)}$ on [-1,1]. Suppose that approximate values of the jumps are calculated from systems* (7) *and* (8)*. Then the following estimate holds for $0 \leq j \leq q-1$ as $N \to \infty$*

$$G_m^{(j)} = \frac{(-1)^m A_q(f)}{2(i\pi(2N+1))^{q-j+1}} \sum_{k=[\frac{j+1}{2}]}^{\frac{q}{2}} \frac{\nu_k}{(\frac{m}{2N+1})^{2k-j+1}} + o(N^{-q+j-1}) \frac{(-1)^m}{\frac{m}{2N+1}}, \; m \neq 0,$$

where $\nu_q := -1$.

Proof. Expansion (14) implies for $m \neq 0$

$$G_m^{(j)} = \sum_{k=j}^{q-1} (A_k - \widetilde{A}_k) B_{k-j,m} + A_q B_{q-j,m} + o(m^{-q+j-1}),\ m \to \infty.$$

Application of Theorem 2 leads to the required estimate.

The next investigates the Hermite-Eckhoff interpolation for even values of q when the approximate values of the jumps are calculated from systems (7) and (8).

Theorem 3. *Let* $q \geq 2$ *be an even number and* $f \in C^q[-1,1]$ *with absolutely continuous* $f^{(q)}$ *on [-1,1]. Suppose that approximate values of the jumps are calculated from systems* (7) *and* (8)*. Then the following estimate holds*

$$\lim_{N\to\infty} (2N+1)^{q+\frac{1}{2}} \|\widetilde{R}_{q,q,N}\| = \frac{|A_q(f)|}{\sqrt{2}\pi^{q+1}} \widetilde{t}(q)$$

$$\widetilde{t}(q) := \left(\sum_{k=-\frac{q}{2}}^{\frac{q}{2}-1} \int_0^1 \left| \sum_{\ell=0}^{\frac{q}{2}-1} \nu_\ell \sum_{j=2\ell+1}^{q-1} \beta_{k,j}(x) \sum_{s=-\frac{q}{2}}^{\frac{q}{2}-1} \frac{(-1)^s}{(x+s)^{2\ell-j+1}} \right.\right.$$

$$\left.\left. - \sum_{j=0}^{q-1} \beta_{k,j}(x) \sum_{\ell=[\frac{j+1}{2}]}^{\frac{q}{2}} \nu_\ell {\sum_s}^* \frac{(-1)^s}{(x+s)^{2\ell-j+1}} \right|^2 dx + {\sum_k}^* \int_0^1 \left| \sum_{\ell=0}^{\frac{q}{2}} \frac{\nu_\ell}{(x+k)^{2\ell+1}} \right|^2 dx \right)^{\frac{1}{2}},$$

where

$${\sum_k}^* = \sum_{k=\frac{q}{2}}^{\infty} + \sum_{k=-\infty}^{-\frac{q}{2}-1},$$

$\nu_q := -1$ *and the* $\beta_{k,j}$ *and the* ν_ℓ *are defined by* (1) *and* (11)*, respectively.*

Proof. Taking into account that

$$\check{G}_m^{(j)} = \sum_{s=-\infty}^{\infty} G_{m+s(2n+1)}^{(j)} \tag{16}$$

it is easy to verify that

$$\begin{aligned} \|\widetilde{R}_{q,q,N}(f)\|^2 &= 2\sum_{m=0}^{2N} \sum_{k=-\frac{q}{2}}^{\frac{q}{2}-1} \left| G_{m+k(2N+1)} - \sum_{j=0}^{q-1} c_{k,j}(m) \check{G}_m^{(j)} \right|^2 \\ &+ 2\sum_{m=0}^{2N} {\sum_k}^* \left| G_{m+k(2N+1)} \right|^2 . \end{aligned} \tag{17}$$

Application of Lemmas' 2, 3 and Theorem 2 yields

$$G_{m+k(2N+1)} - \sum_{j=0}^{q-1} c_{k,j}(m)\check{G}_m^{(j)} = \frac{(-1)^{m+1}A_q(f)}{2(i\pi(2N+1))^{q+1}}$$

$$\times \left(- \sum_{\ell=0}^{\frac{q}{2}-1} \nu_\ell \sum_{j=2\ell+1}^{q-1} \beta_{k,j}\left(\frac{m}{2N+1}\right) \sum_{s=-\frac{q}{2}}^{\frac{q}{2}-1} \frac{(-1)^s}{\left(\frac{m}{2N+1}+s\right)^{2\ell-j+1}} \right.$$

$$\left. + \sum_{j=0}^{q-1} \beta_{k,j}(t_m) \sum_{\ell=[\frac{j+1}{2}]}^{\frac{q}{2}-1} \nu_\ell {\sum_s}^{*} \frac{(-1)^s}{\left(\frac{m}{2N+1}+s\right)^{2\ell-j+1}} \right) + o(N^{-q-1}).$$

Replacing the last into (17), taking into account Lemma 3, and then, tending N to infinity by replacing the Riemann's sums with the corresponding integrals we get the required estimate.

In Table 2 we present the numerical values of $\widetilde{t}(q)$ for different values of q (also for odd values).

Table 2: **Numerical values of $\widetilde{t}(q)$.**

$q=1$	$q=2$	$q=3$	$q=4$	$q=5$
0.24	0.2	0.04	0.01	$2 \cdot 10^{-3}$

Comparison of Theorems 1 and 3 shows that approximation of the jumps by systems (7) and (8) doesn't degrade the rate of convergence, it influences only on the constant $t(p,q)$ (see Theorem 1).

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Institute of Mathematics of NAS RA

A. V. Poghosyan

Asymptotic Behavior of Hermite-Eckhoff Interpolation

A sequence of Hermite trigonometric interpolation polynomials with equidistant interpolation nodes and uniform multiplicities is investigated. We derive relatively compact formula that gives the interpolants as functions of the coefficients in the DFTs of the derivative values. The coefficients can be calculated by the FFT algorithm. Convergence acceleration based on the Eckhoff method is considered. Approximation of jumps is explored and the corresponding exact constant of the asymptotic error is obtained.

А. В. Погосян

Асимптотика интерполяции Эрмита-Экгофа

Изучена эрмитова тригонометрическая равноотстоящая интерполяция и представлена явная формула, которая реализуется посредством дискретного преобразования Фурье значений функции и ее производных. Рассматриваются задача ускорения сходимости с применением метода Экгофа и проблема аппроксимации скачков. Получена асимптотически точная оценка ошибки.

Ա. Վ. Պողոսյան

Հերմիտ-Էկհոֆի ինտերպոլյացիայի ասիմպտոտական վարքը

Ուսումնասիրվում է Հերմիտի եռանկյունաչափական հավասարահեռ հանգույցներով ինտերպոլյացիան, եւ վերջինիս համար ներկայացվում է բացահայտ բանաձեւ՝ ֆունկցիայի եւ նրա ածանցյալների Ֆուրիեի դիսկրետ ձեւափոխությունների տերմիններով: Ուսումնասիրվում է ինտերպոլյացիայի զուգամիտության արագացման խնդիրը, Էկհոֆի հայտնի մեթոդի կիրառմամբ: Դիտարկվում է թռիչքների մոտարկման խնդիրը եւ ստացվում են ինտերպոլյացիայի սխալանքի ասիմպտոտորեն ճշգրիտ գնահատականները:

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