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To recovering the moments from the spherical mean Radon transform [☆]

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ABSTRACT

This article deals with characterizations of a function in terms of its circular mean Radon transform. We present a new approach (the consistency method) showing how to describe the class of real-valued, planar functions f which have the given circular mean Radon transform $\mathcal{M}f$ over circles centered on the unit circle. Also, expressions are derived for the geometric moments of an unknown function in terms of its circular mean Radon transform.

1. Introduction and formulation of the problem

The problems studied in this article are related to Computed Tomography. X-ray Tomography is based on the classical Radon transform that maps a function to its integrals over straight lines. Recently, new methods for computed tomography have been developed. Of these, thermoacoustic tomography (TAT) is the most successful method, as described in [7–9,16–18,22–24,26,27]. In TAT, one effectively measures the integrals of the energy-absorption distribution function f over all spheres centered at the detector locations. Thus, to recover f, one needs to invert the spherical mean Radon transform of f.

Consider the Euclidean n-dimensional space $\mathbf{R}^n$ ($n \geq 2$). By $\mathcal{C}^\infty$ we denote the class of real-valued functions for which the derivatives of all orders exist, and by $S(P, t)$, we denote the sphere of radius $t > 0$ centered at $P \in \mathbf{R}^n$. The main mathematical problem is to recover a real-valued function f, supported on a compact region $G \subset \mathbf{R}^n$, from the mean value $\mathcal{M}f$ of f over spheres, centered on some set L, i.e., to invert the spherical mean Radon transform.

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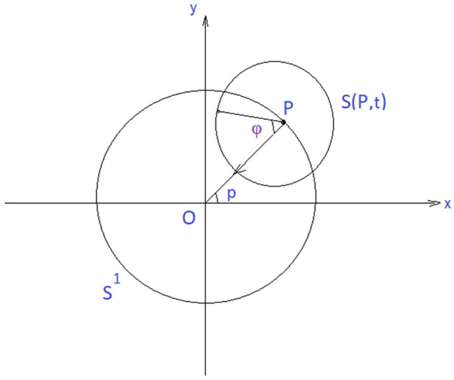

Fig. 1. The scheme of modeling the spherical mean Radon transform.

Agranovsky and Quinto in [2] gave a complete characterization of sets of uniqueness (sets of centers) for the circular mean Radon transform on compactly supported functions in the plane. Articles [4] and [1] describe the complete range of the spherical mean Radon transform for different geometries of detectors.

In $\mathbf{R}^n$, several inversion formulas are derived for the spherical mean Radon transform for different geometries of detectors (including the cases with incomplete data) (see [3,6,8,9,14,18,23,24,26]). Recently, in [11], the 2D 'local reconstruction formula' was obtained for detectors on a line.

It should be noted that the problem of recovering functions from the values of the Radon transform is related to the bivariate moment problem, which has a rich history (see [5,15,25]).

Goncharov [13] and Milanfar [19] showed that the Radon transform of a function can be converted into its moments. Mnatsakanov [20] suggested a new explicit moment-recovered formula, which gives an algorithm to recover a positive function via its exponential moments. Using the moment-recovered formula in [21], the rate of approximation of a positive function, via the values of a modified Radon transform, was also derived (see [12]).

The purpose of this article is to describe the class of real-valued functions defined in $\mathbf{R}^2$ (not necessarily with compact support), which have the given circular mean Radon transform $\mathcal{M}f$ defined over the circles with the centers on $\mathbf{S}^1$, using the consistency method suggested by Aramyan in [10].

Also, in this paper we study the relationship between the moments (also known as the multi-indexed moments) of a real-valued function f and the values of its circular mean Radon transforms in $\mathbf{R}^2$. The results obtained in this article are formulated for the two-dimensional case, but they can be extended to higher dimensions.

We now introduce the circular mean Radon transform $\mathcal{M}f$ that integrates a function f defined on $\mathbf{R}^2$ over circles. Let $S(P,t)$ be the circle with center $P = (\cos p, \sin p) \in \mathbf{S}^1$ and radius $t > 0$. Note that the point $P \in \mathbf{S}^1$ is uniquely determined by the corresponding angle $p \in [0, 2\pi)$ (see Fig. 1). In the sequel, the point $P \in \mathbf{S}^1$ is identified with p. We define $\mathcal{M}f(p,t)$ as the integral of f over $S(P,t)$, i.e.,

$$\mathcal{M}f(p,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(P + t\omega)\, d\varphi, \quad \textit{for} \;\; (p,t) \in [0, 2\pi) \times [0, \infty). \tag{1}$$

Here $\varphi \in [-\pi, \pi]$ is the angular coordinate of a point on $S(P,t)$ (we measure φ from the direction $\overrightarrow{PO}$ (see Fig. 1)); $\omega \in \mathbf{S}^1$ is the unit direction corresponds to φ. Consider $\mathcal{M}f$ as a function on the unit cylinder

$$\mathrm{C}^1 = \{(p,t) : p \in [0, 2\pi), t \in [0, \infty)\}.$$

For a fixed $P = (\cos p, \sin p) \in \mathbf{S}^1$, one can use the usual polar system of coordinates (t, φ) on the plane with respect to P. Thus, we have $(x, y) = (P, t, \varphi)$.

Here, and in the sequel, for a fixed $(P,t) \in \mathbf{S}^1 \times [0,\infty)$, the restriction of f to the circle $S(P,t)$ is written in the form

$$f_{p,t}(\varphi), \quad \varphi \in [-\pi,\pi]. \tag{2}$$

It is known that a 2π-periodic, differentiable function f with continuous derivative can be written as its Fourier series expansion. For any $(p,t) \in C^1$, the Fourier series expansion of the restricted f is

$$f_{p,t}(\varphi) = \sum_{k=0}^{\infty} \left(a_k(p,t)\cos k\,\varphi + b_k(p,t)\sin k\,\varphi\right). \tag{3}$$

Taking into account (1) we have

$$f_{p,t}(\varphi) = \mathcal{M}f(p,t) + \sum_{k=1}^{\infty} \left(a_k(p,t)\cos k\,\varphi + b_k(p,t)\sin k\,\varphi\right), \tag{4}$$

where for $k \geq 1$,

$$a_k(p,t) = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{p,t}(\varphi)\cos k\varphi\, d\varphi \quad \textit{and} \quad b_k(p,t) = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{p,t}(\varphi)\sin k\varphi\, d\varphi. \tag{5}$$

It is obvious that the restrictions $f_{p,t}(\varphi)$, of the function f defined on $\mathbf{R}^2$, are consistent in the following sense: for a point $(x,y) \in \mathbf{R}^2$ and for the bundle of circles $S(P,t)$ $(P \in \mathbf{S}^1)$ containing (x,y), we have

$$f_{p,t}(\varphi) = f(x,y) \;\; \textit{for all} \;\; (P,t,\varphi) = (x,y), \tag{6}$$

i.e., there is no dependence on a circle from the bundle containing (x,y).

The opposite statement is also true. Let $G_{p,t}(\varphi)$ be a family of functions defined on $S(P,t)$ $((P,t) \in \mathbf{S}^1 \times [0,\infty))$ that are consistent. Then $G_{p,t}(\varphi)$ represents the restrictions of a function f defined on $\mathbf{R}^2$. Indeed, one can produce f via the definition: for $(x,y) \in \mathbf{R}^2$,

$$f(x,y) = G_{p,t}(\varphi) \;\; \textit{for} \;\; (P,t,\varphi) = (x,y). \tag{7}$$

The principle of consistency defined above was introduced and applied in other models as well (cf. with [10] and [11]).

In this article we apply the consistency method: we consider equation (1) as an integral equation on the circle $S(P,t)$ for every $(P,t) \in \mathbf{S}^1 \times [0,\infty)$; we write the general solution of the integral equation in terms of a Fourier series expansion with unknown coefficients; then, we seek the unknown coefficients to find a family of consistent solutions. Thus, we reduce the problem of recovering a real valued function f from the mean value $\mathcal{M}f$ over circles, centered on $\mathbf{S}^1$, to finding consistent solutions of integral equations (1).

Now we present the main results. Let $f \in \mathcal{C}^\infty$ be a real valued function defined on $\mathbf{R}^2$ and $\mathcal{M}f$ be the circular mean Radon transform of f over circles with the centers on $\mathbf{S}^1$. Lemma 1 (see below) shows that the Fourier coefficients, $a_k(p,t)$ and $b_k(p,t)$, $k = 1,2,\ldots$, of the restrictions $f_{p,t}(\varphi)$, of f onto $S(P,t)$, satisfy the system of differential equations (15) and (16) with boundary conditions (17). Using Lemma 1 we get the following theorem:

Theorem 1. *Let $\mathcal{M}f$ be the circular mean Radon transform of a function over circles with the centers on $\mathbf{S}^1$ and let $a_1(p,t)$, which has continuous partial derivatives and $a_1(p,0) = 0$, be a function defined on the* *unit cylinder* C^1*. Let* $a_k(p,t)$*,* $k = 2, 3, \ldots,$ *and* $b_k(p,t)$*,* $k = 1, 2, \ldots,$ *be the unique solutions of the system of differential equations* (15) *and* (16) *with boundary conditions* (17)*. If for any* $(p,t) \in \mathrm{C}^1$ *the series*

$$\mathcal{M}f(p,t) + \sum_{k=1}^{\infty} \left(a_k(p,t)\cos k\,\varphi + b_k(p,t)\sin k\,\varphi\right), \tag{8}$$

converges to $f_{p,t}(\varphi)$*, then the family of functions* $f_{p,t}$ *is consistent and produces a solution of* (1) *via definition* (7)*.*

As a consequence of Theorem 1, we have

Theorem 2. *Let* $\mathcal{M}f$ *be the circular mean Radon transform of a function* f *over circles with the centers on* $\mathbf{S}^1$*. The function* f *is uniquely determined by its circular mean Radon transform* $\mathcal{M}f$*, i.e., equation* (1) *has a unique solution, if and only if the first Fourier coefficient of the restriction of* f *onto* $S(P,t)$*,*

$$a_1(p,t) = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{p,t}(\varphi)\cos\varphi\, d\varphi,$$

is uniquely determined by $\mathcal{M}f$*.*

In section 3, we establish a new linear relationship between the moments of an unknown function f and the values of its circular mean Radon transform $\mathcal{M}f$ (see Theorem 3). Using this relationship one can recover the moments of f from the values of $\mathcal{M}f$, and then approximate f from its moments.

2. The consistency condition and proof of Theorem 1

For a fixed point $(x,y) \in \mathbf{R}^2$, we consider the bundle of circles $S(P,t)$ $(P \in \mathbf{S}^1)$ containing (x,y). Any circle from the bundle is uniquely determined by its center $P = (\cos p, \sin p) \in \mathbf{S}^1$. For the polar coordinates (t,φ) of the point (x,y) on the plane with respect to P, we have (see Fig. 1)

$$\begin{cases} x = \cos p - t\cos(p+\varphi) \\ y = \sin p - t\sin(p+\varphi). \end{cases} \tag{9}$$

We need to calculate the derivatives of the polar coordinates (t,φ) of the point (x,y) with respect to p. We denote the (partial) derivative of a function f with respect to a variable, say v, by f'_v. Taking the derivative of both sides of the equations of (9) with respect to p, we obtain

$$\begin{cases} -\sin p - t'_p\cos(p+\varphi) + t\sin(p+\varphi)(1+\varphi'_p) = 0 \\ \cos p - t'_p\sin(p+\varphi) - t\cos(p+\varphi)(1+\varphi'_p) = 0. \end{cases} \tag{10}$$

From (10) we get:

$$\varphi'_p = \frac{\cos\varphi}{t} - 1, \quad t'_p = \sin\varphi. \tag{11}$$

We now find the coefficients $a_k(p,t)$, $b_k(p,t)$ $(k = 1, 2, \ldots)$ in (4) as functions of $(p,t) \in C^1$ from the consistency condition. For a fixed point $(x,y) \in \mathbf{R}^2$ we write f in polar coordinates and require that the right-hand side of (4) should not depend on p:

$$
\begin{gathered}
(f(x,y))'_p = (f_{p,t}(\varphi))'_p = \\
(\mathcal{M}f(p,t) + \sum_{k=1}^{\infty}(a_k(p,t)\cos k\,\varphi + b_k(p,t)\sin k\,\varphi))'_p = 0.
\end{gathered}
\tag{12}
$$

Termwise differentiation using the expressions in (11) yields

$$
\begin{gathered}
-t\,(\mathcal{M}f(p,t))'_p - t\,(\mathcal{M}f(p,t))'_t\,\sin\varphi = \\
\sum_{k=1}^{\infty}\big[t((a_k(p,t))'_p + \;(a_k(p,t))'_t\,\sin\varphi)\cos k\varphi - k\,a_k(p,t)\sin k\varphi(\cos\varphi - t) + \\
t(\,(b_k(p,t))'_p + \;(b_k(p,t))'_t\,\sin\varphi)\sin k\varphi + k\,b_k(p,t)\cos k\varphi(\cos\varphi - t)\big]\,.
\end{gathered}
\tag{13}
$$

Using the trigonometric formulas and grouping the summands in (13), we obtain

$$
\begin{gathered}
-t\,(\mathcal{M}f(p,t))'_p - t\,(\mathcal{M}f(p,t))'_t\,\sin\varphi = \\
\sum_{k=1}^{\infty}\Big([t(a_k(p,t))'_p - kt\,b_k(p,t)]\cos k\varphi - [\frac{t}{2}(b_k(p,t))'_t - \frac{k}{2}\,b_k(p,t)]\cos(k+1)\varphi + \\
[\frac{t}{2}(b_k(p,t))'_t + \frac{k}{2}\,b_k(p,t)]\cos(k-1)\varphi + [t(b_k(p,t))'_p + kt\,a_k(p,t))]\sin k\varphi + \\
[\frac{t}{2}(a_k(p,t))'_t - \frac{k}{2}\,a_k(p,t))]\sin(k+1)\varphi - [\frac{t}{2}(a_k(p,t))'_t + \frac{k}{2}\,a_k(p,t))]\sin(k-1)\varphi\Big)\,.
\end{gathered}
\tag{14}
$$

By uniqueness of the Fourier coefficients, we obtain the following system of differential equations for the unknown coefficients a_k, b_k $(k \geq 1)$:

$$
\begin{cases}
t(b_1(p,t))'_t + b_1(p,t) = -2t\,(\mathcal{M}f(p,t))'_p \\
t(b_2(p,t))'_t + 2\,b_2(p,t) + 2\,t(a_1(p,t))'_p - 2t\,b_1(p,t) = 0 \\
t(a_2(p,t))'_t + 2\,a_2(p,t) - 2\,t(b_1(p,t))'_p - 2t\,a_1(p,t) = 2t\,(\mathcal{M}f(p,t))'_t,
\end{cases}
\tag{15}
$$

for $k = 1$, $k = 2$, and

$$
\begin{cases}
t(b_k(p,t))'_t + k\,b_k(p,t) - t(b_{k-2}(p,t))'_t + (k-2)b_{k-2}(p,t) + \\
2\,t(a_{k-1}(p,t))'_p - 2(k-1)t\,b_{k-1}(p,t) = 0 \\
t(a_k(p,t))'_t + k\,a_k(p,t) - t(a_{k-2}(p,t))'_t + (k-2)a_{k-2}(p,t) - \\
2\,t(b_{k-1}(p,t))'_p - 2(k-1)t\,a_{k-1}(p,t) = 0
\end{cases}
\tag{16}
$$

for $k > 2$.

We now find the boundary conditions for the differential equations. From (5) for $k \geq 1$, taking into account that $f_{p,0}$ does not depend on φ, we get the following boundary conditions:

$$
a_k(p,0) = b_k(p,0) = 0 \quad \textit{for} \quad k = 1, 2, \ldots. \tag{17}
$$

Thus, we obtain the following lemma:

Lemma 1. *Let $f \in \mathcal{C}^{\infty}$ be a real valued function defined on $\mathbf{R}^2$ and let $\mathcal{M}f$ be the circular mean Radon transform of f over circles with the centers on $\mathbf{S}^1$. Then the Fourier coefficients $a_k(p,t)$ and $b_k(p,t)$, $k = 1, 2, \ldots,$ of the restrictions $f_{p,t}(\varphi)$ satisfy the system of differential equations (15) and (16) with boundary conditions (17).*

It follows from (15), (16) and (17) that knowing $\mathcal{M}f$, the circular mean Radon transform of f, and the first Fourier coefficient $a_1(p,t)$ of $f_{p,t}(\varphi)$, one can calculate step-by-step the unknown coefficients $a_k(p,t)$, $k = 2,3,\ldots$, and $b_k(p,t)$, $k = 1,2,\ldots$, (at first we find b_1, next we find a_2, next b_2, next a_3, etc.), and hence we reconstruct f. In general, equation (1), where $\mathcal{M}f$ is the circular mean Radon transform of a function over circles with the centers on $\mathbf{S}^1$, can have many solutions.

Now let $a_1(p,t)$ be a function defined on the unit cylinder C^1. We assume that $a_1(p,t)$ has continuous partial derivatives and $a_1(p,0) = 0$. We substitute $a_1(p,t)$ into (15) and find the unique solutions $a_k(p,t)$, $k = 2,3\ldots$, and $b_k(p,t)$, $k = 1,2,\ldots$, of the system of differential equations (15) and (16) with boundary conditions (17). Now for $(p,t) \in C^1$ using $a_k(p,t)$ and $b_k(p,t)$, $k = 1,2,\ldots$, we compose series (8) and assume that the series converges to $f_{p,t}(\varphi)$. The family of functions $f_{p,t}$ will be consistent since the coefficients $a_k(p,t)$, $k = 1,2,3,\ldots$, and $b_k(p,t)$, $k = 1,2,\ldots$, satisfy the system of differential equations (15) and (16). Hence, this family produces a solution of (1) via definition (7). Thus, Theorem 1 is proved.

3. A formula for the moments of an unknown function f

In this section, assuming that $f \in \mathcal{S}(\mathbf{R}^2)$ (the Schwartz space), and given the values of spherical mean Radon transform $\mathcal{M}f$, we will show how to evaluate the moments of f. Namely, consider the ordinary (geometric) moments of f. By definition, the multi-indexed moment of order (m,j) is defined as follows:

$$\mu_{m,j}(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^m\, y^j\, f(x,y)\, dx\, dy, \quad m,j \in N = \{0,1,\ldots\}. \tag{18}$$

It is known that under the certain conditions (see, e.g., [5,15,25]) the sequence of multi-indexed moments $\{\mu_{m,j}(f),\, m,j \in N\}$ determines f uniquely.

For a fixed $P \in \mathbf{S}^1$, we return to the formula (4) and write

$$f(x,y) = \mathcal{M}f(p,t) + \sum_{k=1}^{\infty} (a_k(p,t)\cos k\,\varphi + b_k(p,t)\sin k\,\varphi) \tag{19}$$

for every (x,y) with $(P,t,\varphi) = (x,y)$. A substitution of (19) into (18) yields

$$\mu_{m,j}(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^m\, y^j\, f(x,y)\, dx\, dy = \tag{20}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^m\, y^j [\mathcal{M}f(p,t) + \sum_{k=1}^{\infty} (a_k(p,t)\cos k\,\varphi + b_k(p,t)\sin k\,\varphi)]\, dx\, dy. \tag{21}$$

Taking into account (9) and the fact $dx\,dy = tdt\,d\varphi$, we obtain

$$\begin{aligned}\mu_{m,j}(f) = &\int_{-\pi}^{\pi}\int_{0}^{\infty} (\cos p - t\cos(p+\varphi))^m (\sin p - t\sin(p+\varphi))^j \mathcal{M}f(p,t) tdt\, d\varphi + \\ &\sum_{k=1}^{\infty}\Bigg(\int_{-\pi}^{\pi}\int_{0}^{\infty} a_k(p,t)(\cos p - t\cos(p+\varphi))^m (\sin p - t\sin(p+\varphi))^j \cos k\,\varphi\ tdt\, d\varphi + \\ &\int_{-\pi}^{\pi}\int_{0}^{\infty} b_k(p,t)(\cos p - t\cos(p+\varphi))^m (\sin p - t\sin(p+\varphi))^j \sin k\,\varphi\ tdt\, d\varphi\Bigg).\end{aligned} \tag{22}$$

Using Fubini's theorem, we get

$$\mu_{m,j}(f) = \int_0^\infty \mathcal{M}f(p,t)\,t[\int_{-\pi}^{\pi} (\cos p - t\cos(p+\varphi))^m (\sin p - t\sin(p+\varphi))^j\, d\varphi]dt + \tag{23}$$
$$\sum_{k=1}^{\infty}\left(\int_0^\infty a_k(p,t)\,t[\int_{-\pi}^{\pi} (\cos p - t\cos(p+\varphi))^m (\sin p - t\sin(p+\varphi))^j \cos k\,\varphi\, d\varphi]dt + \right.$$
$$\left.\int_0^\infty b_k(p,t)\,t[\int_{-\pi}^{\pi} (\cos p - t\cos(p+\varphi))^m (\sin p - t\sin(p+\varphi))^j \sin k\,\varphi\, d\varphi]dt\right).$$

Consider two sequences of *standard* functions $C_{k,m,j}$ and $S_{k,m,j}$ defined on the unique cylinder, where k, m and j are nonnegative integers. We call them standard functions because their constructions do not depend on f:

$$C_{k,m,j}(p,t) = \int_{-\pi}^{\pi} (\cos p - t\cos(p+\varphi))^m (\sin p - t\sin(p+\varphi))^j \cos k\,\varphi\, d\varphi \tag{24}$$

and

$$S_{k,m,j}(p,t) = \int_{-\pi}^{\pi} (\cos p - t\cos(p+\varphi))^m (\sin p - t\sin(p+\varphi))^j \sin k\,\varphi\, d\varphi. \tag{25}$$

The first few functions are:

$$\begin{cases} C_{0,1,0}(p,t) = 2\pi\cos p, \;\; C_{0,0,1}(p,t) = 2\pi\sin p, \;\; C_{0,1,1}(p,t) = 2\pi\sin p\cos p, \\ C_{1,1,0}(p,t) = -t\pi\cos p, \;\; C_{1,0,1}(p,t) = -t\pi\sin p, \;\; C_{1,1,1}(p,t) = -t\pi\sin 2p, \\ S_{1,1,0}(p,t) = t\pi\sin p, \;\; S_{1,0,1}(p,t) = -t\pi\cos p, \;\; S_{1,1,1}(p,t) = -t\pi\cos 2p, \\ \dots \end{cases}$$

By substituting (24) and (25) into (23), we obtain

$$\mu_{m,j}(f) = \int_0^\infty C_{0,m,j}(p,t)\,\mathcal{M}f(p,t)\,t\,dt + \tag{26}$$
$$\sum_{k=1}^{\infty}\left(\int_0^\infty C_{k,m,j}(p,t)\,a_k(p,t)\,t\,dt + \int_0^\infty S_{k,m,j}(p,t)\,b_k(p,t)\,t\,dt\right).$$

The following lemma is valid.

Lemma 2. *For $m + i < k$, we have*

$$C_{k,m,j}(p,t) = S_{k,m,j}(p,t) \equiv 0. \tag{27}$$

Proof. It is obvious that for any (p,t), the function

$$\pi(\cos p - t\cos(p+\varphi))^m (\sin p - t\sin(p+\varphi))^j$$

is a polynomial of $\sin\varphi$ and $\cos\varphi$, functions of degree $r \le m+j$. Hence, it is orthogonal to $\sin k\varphi$ and $\cos k\varphi$ for $m+j<k$. Lemma 2 is proved.

Finally, we have the following theorem:

Theorem 3. *Let $\mathcal{M}f$ be the circular mean Radon transform of a function over circles with the centers on $\mathbf{S}^1$, and $a_1(p,t)$ be a function defined on the unit cylinder C^1. Let $a_k(p,t)$, $k=2,3,\dots$, and $b_k(p,t)$, $k=1,2,\dots$, be the unique solutions of the system of differential equations (15) and (16) with boundary conditions (17). If for any (p,t) the series (8) converges to $f_{p,t}(\varphi)$, then the family of functions $f_{p,t}$ is consistent and produces a solution f of (1) via definition (7), and*

$$\mu_{m,j}(f) = \int_0^\infty C_{0,m,j}(p,t)\,\mathcal{M}f(p,t)t\,dt + \tag{28}$$

$$\sum_{k=1}^{m+j}\left(\int_0^\infty C_{k,m,j}(p,t)\,a_k(p,t)\,t\,dt + \int_0^\infty S_{k,m,j}(p,t)\,b_k(p,t)\,t\,dt\right).$$

In other words, if $a_1(p,t)$ is uniquely determined by $\mathcal{M}f$, then one can recover the moments of the function f from the values of $\mathcal{M}f$, and afterwards reconstruct f itself via the sequence of its multi-indexed moments $\{\mu_{m,j}(f), m,j \in N\}$ (cf. with [12]).

Remark 1. Suppose the support of f is a unit cube $[0,1]^2$. Given the moments $\mu_{m,j}(f)$ $(m,j=0,1,\dots,n)$ of f up to order (n,n), one can derive the upper bound of order $1/n$ for the rate of approximation f_n in a sup-norm, as $n\to\infty$ (see Theorem 5.2 in [12]). The suggested approximation of f has the following form:

$$f_n(x) = C_{m,n}(x)\sum_{l=0}^{m-[mx_1]}\sum_{k=0}^{n-[nx_2]}\frac{(-1)^{l+k}\mu_{l+[mx_1],k+[nx_2]}(f)}{l!k!(m-[mx_1]-l)!(n-[nx_2]-k)!}, \tag{29}$$

defined for $x=(x_1,x_2)\in[0,1]^2$, and

$$C_{m,n}(x) = \frac{\Gamma(m+2)\Gamma(n+2)}{\Gamma([mx_1]+1)\Gamma([nx_2]+1)}.$$

In general, one can assume that m and n in (29) are different and both are tending to infinity.

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