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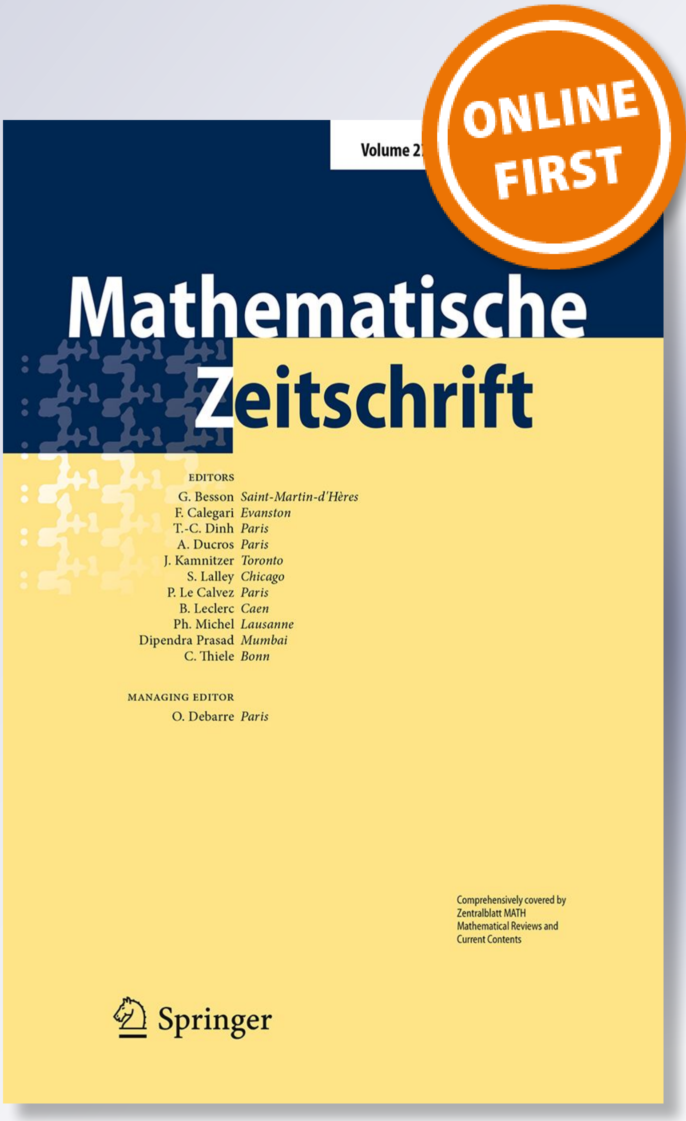

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On logarithmic bounds of maximal sparse operators

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Abstract
Given sparse collections of measurable sets $\mathcal{S}_k$, $k = 1, 2, \dots, N$, in a general measure space $(X, \mathfrak{M}, \mu)$, let $\Lambda_{\mathcal{S}_k}$ be the sparse operator, corresponding to $\mathcal{S}_k$. We show that the maximal sparse function $\Lambda f = \max_{1\le k\le N} \Lambda_{\mathcal{S}_k} f$ satisfies

$$\|\Lambda\|_{L^p(X)\mapsto L^{p,\infty}(X)} \lesssim \log N \cdot \|M_{\mathcal{S}}\|_{L^p(X)\mapsto L^{p,\infty}(X)},\ 1 \le p < \infty,$$
$$\|\Lambda\|_{L^p(X)\mapsto L^p(X)} \lesssim (\log N)^{\max\{1,1/(p-1)\}} \cdot \|M_{\mathcal{S}}\|_{L^p(X)\mapsto L^p(X)},\ 1 < p < \infty,$$

where $M_{\mathcal{S}}$ is the maximal function corresponding to the collection of sets $\mathcal{S} = \cup_k \mathcal{S}_k$. As a consequence, one can derive norm bounds for maximal functions formed from taking measurable selections of one-dimensional Calderón–Zygmund operators in the plane. Prior results of this type had a fixed choice of Calderón–Zygmund operator for each direction.

Keywords Calderón–Zygmund operator · Sparse operator · Derectional maximal function · Logarithmic bound

Mathematics Subject Classification 42B20 · 42B25

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1 Introduction

Let $H_v f(x) = \int_{\mathbb{R}} f(x - tv)\frac{dt}{t}$ be the Hilbert transform performed in direction v in $\mathbb{R}^2$. Here and throughout we take v to be a unit vector. Given finite set of unit vectors V define the operator

$$H_V f(x) = \max_{v\in V} |H_v f(x)|$$

It is a well known consequence of the Rademacher–Menshov theorem that we have

Theorem A *For any finite set of unit vectors V we have*

$$\|H_V\|_{L^2\to L^2} \lesssim \log_+ \#V.$$

Here and below $\#V$ denotes the cardinality of V and $\log_+ n = \max\{1, \log_2 n\}$. Many different extensions of this result have been studied. One of us [11] showed that the norm bound is necessarily logarithmic in $\#V$, in strong contrast to the classical result on the maximal function in a lacunary set of directions of Nagel, Stein and Wainger [16]. Namely, we have

Theorem B [11] *For any finite set V of unit vectors it holds*

$$\|H_V\|_{L^2\to L^2} \gtrsim \sqrt{\log_+ \#V}. \tag{1.1}$$

The maximal function variant in the strong and weak-type estimates was first established by Nets Katz [12,13]. Namely, set

$$M_v f(x) = \sup_{t>0}(2t)^{-1}\int_{-t}^{t} |f(x - tv)|\, dt,$$

for unit vectors v, and for a finite set of unit directions V, let $M_V f = \max_{v\in V} M_v f$.

Theorem C [12,13] *For any set of unit vectors V, we have*

$$\|M_V\|_{L^2\to L^{2,\infty}} \lesssim \sqrt{\log_+ \#V}, \qquad \|M_V\|_{L^2\to L^2} \lesssim \log_+ \#V.$$

Many extensions of these results have been considered, and we will cite several of these extensions. Herein, we prove results, which allow for much rougher examples than singular integrals in a choice of directions. Let $K_a(x)$, $a \in \mathbb{R}$, be a family of Calderón–Zygmund kernels with uniformly bounded Fourier transforms, $\|\hat{K}_a\|_\infty < M$, such that $K_a(x)$ as a function in two variables a and x is measurable on $\mathbb{R}^2$. For a unit vector v in $\mathbb{R}^2$ with a perpendicular vector $v^\perp$ we consider an operator T_v written by

$$T_v f(x) = \int_{\mathbb{R}} K_{x\cdot v^\perp}(t) f(x - tv)dt, \quad x \in \mathbb{R}^2,$$

for compactly supported smooth functions f on $\mathbb{R}^2$. Notice that on the v-directer lines $x \cdot v^\perp = l$ the operator T_v defines one dimensional Calderón–Zygmund operators, and those can be different as the line varies. For a finite collection of unit vectors V denote

$$T_V f(x) = \max_{v\in V} |T_v f(x)|.$$

Among the others below, as a corollary to our main result we derive the following.

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Corollary 1.2 *If the family of Calderón–Zygmund kernels $K_a(x)$ satisfies the above conditions, then for any finite collection of unit vectors V, we have*

$$\|T_V\|_{L^2\to L^{2,\infty}} \lesssim (\log_+|V|)^{3/2},$$
$$\|T_V\|_{L^2\to L^2} \lesssim (\log_+|V|)^2.$$

No prior result we are aware of has permitted a variable choice of operator, as the line varies. The method of proof is by way of *sparse operators*. Namely we use the recent pointwise domination of singular integrals by a positive operator [2,14,15] to reduce the corollary above to a setting, where the operators are *positive*. These positive operators, called sparse operators are 'bigger than the maximal function by logarithmic terms', and so the proofs of the sparse operator bounds imply the corollary above.

2 Sparse operators

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Given collection of measurable sets $\mathfrak{B} \subset \mathfrak{M}$ defines the maximal function

$$\mathcal{M}_{\mathfrak{B}} f(x) = \sup_{B\in\mathfrak{B}} \langle f\rangle_B \cdot \mathbf{1}_B(x),$$

where $\langle f\rangle_B = \mu(B)^{-1}\int_B |f|$. By a *sparse operator* we mean an operator

$$\Lambda_{\mathcal{S}} f(x) = \sum_{S\in\mathcal{S}} \langle f\rangle_S \mathbf{1}_S(x),$$

where $\mathcal{S} \subset \mathfrak{M}$ is a sparse collection of measurable sets, that means there is a constant $0 < \gamma < 1$ so that any set $S \in \mathcal{S}$ has a portion $E_S \subset S$ with $\mu(E_S) \geq \gamma\mu(S)$ and those are pairwise disjoint.

Without recalling the exact definition of a bounded Calderón–Zygmund operator, the main result we need from [2,14,15] is this.

Theorem D *For any bounded Calderón–Zygmund operator T, and compactly supported function f on $\mathbb{R}^n$, there is a sparse collection $\mathcal{S} = \mathcal{S}_{T,f}$ of n-dimensional balls so that*

$$|Tf(x)| \lesssim \Lambda_{\mathcal{S}} f(x).$$

This inequality contains many deep results about Calderón–Zygmund operators, for which we refer the reader to the referenced papers. Sparse bounds hold for other functionals of Calderón–Zygmund operators, like variational estimates [5]. The result above has been extended in a number of interesting ways. Among many we could point to, the reader can consult [1,3,4,10].

Definition 2.1 Let $(X, \mathfrak{M}, \mu)$ be a measure space. A family of measurable sets $\mathfrak{B} \subset \mathfrak{M}$ is said to be martingale collection if for any two elements $A, B \in \mathfrak{B}$ we have either

$$A \subset B, \quad B \subset A \text{ or } A \cap B = \varnothing.$$

We say that $\mathfrak{B}$ is a finite-martingale collection if there are finite number of martingale collections

$$\mathfrak{B}_1, \ldots, \mathfrak{B}_d \tag{2.2}$$

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such that for any $B \in \mathfrak{B}$ there is a set $B' \in \cup_k \mathfrak{B}_k$ with

$$B \subset B', \quad \mu(B') \leq C\mu(B).$$

It is well known that any family of balls in $\mathbb{R}^n$ forms a finite-martingale collection. Moreover, the corresponding martingale collections (2.2) can be taken to be dyadic grids. Such dyadization is a key point in many applications of sparse operators.

We turn to the statement of the main theorem. Let $\mathcal{S}_k$, $k = 1, 2, \ldots, N$ be a finite-martingale sparse collections in a measure space $(X, \mathfrak{M}, \mu)$, and suppose $\mathcal{S} = \cup_{k=1}^N \mathcal{S}_k$. The family $\mathfrak{G} = \{\mathcal{S}_1, \ldots, \mathcal{S}_N\}$ defines the operator

$$\Lambda_{\mathfrak{G}} f(x) = \max_{1\leq k\leq N} \Lambda_{\mathcal{S}_k} f(x). \tag{2.3}$$

Theorem 2.4 *With the notations above we have the inequalities*

$$\|\Lambda_{\mathfrak{G}}\|_{L^p\to L^{p,\infty}} \lesssim \log_+ N \cdot \|\mathcal{M}_{\mathcal{S}}\|_{L^p\to L^{p,\infty}}, \ 1 \leq p < \infty, \tag{2.5}$$

$$\|\Lambda_{\mathfrak{G}}\|_{L^p\to L^p} \lesssim (\log_+ N)^{\max\{1,1/(p-1)\}} \cdot \|\mathcal{M}_{\mathcal{S}}\|_{L^p\to L^p}, \ 1 < p < \infty. \tag{2.6}$$

Here and below the notation $a \lesssim b$ will stand for the inequality $a \leq c\cdot b$, where the constant $c > 0$ may depend only on p and on the constants from the above definitions of different type of set collections. As we said, a sparse operator is logarithmically larger than a maximal function, as indicated after Corollary 1.2. Our inequalities above match this heuristic. In fact, Corollary 1.2 as well as Corollary 2.7 below may be analogously formulated in $\mathbb{R}^n$ for any $n \geq 2$, taking instead of parallel lines parallel hyperplanes of dimension $m < n$ and consider different m-dimensional Calderón–Zygmund operator on each hyperplane.

For a direction v, and a smooth compactly supported function f, we let

$$\mathcal{S}_v f(x) = \Lambda_{\mathcal{S}(v_\perp\cdot x)} f(x),$$

where $v_\perp$ is orthogonal to v, and $y \mapsto \Lambda_{\mathcal{S}(y)}$ is a measurable choice of sparse operators. Given a finite set of unit vectors V, we set $\mathcal{S}_V f = \max_{v\in V} \mathcal{S}_v f$.

Corollary 2.7 *With the notation above, for any finite set of unit vectors V we have the inequalities*

$$\|\mathcal{S}_V\|_{L^2\to L^{2,\infty}} \lesssim (\log_+ V)^{3/2}, \tag{2.8}$$

$$\|\mathcal{S}_V\|_{L^2\to L^2} \lesssim (\log_+ V)^2, \tag{2.9}$$

$$\|\mathcal{S}_V\|_{L^p\to L^p} \lesssim (\log_+ V)^{1+1/p}, \quad p > 2. \tag{2.10}$$

Corollary 2.7 immediately follows from Theorem 2.4. Indeed, there is no need to consider the measurable choice of sparse operators directly. By standard arguments, it suffices to consider a simplified discrete situation described here. For any pair of orthogonal vectors $(v, v^\perp)$, let $\mathcal{R}_v$ be the collection of dyadic rectangles in the plane, in the coordinates $(v, v^\perp)$, whose lengths in the direction $v^\perp$ is one (see Fig. 1).

Let $V = \{v_1, \ldots, v_N\}$ be a finite collection of unit vectors and $\mathcal{S}_k \subset \mathcal{R}_{v_k}, k = 1, 2, \ldots, N$, be a sparse collections of rectangles. One can easily see that the operator (2.3) generated by those collections is a discrete version of $\mathcal{S}_V$ from (2.8), (2.9) and (2.10). On the other hand for the maximal function $\mathcal{M}_{\mathcal{S}} f$ corresponding to the family of sets $\mathcal{S} = \cup_k \mathcal{S}_k$ we have the bound

$$\mathcal{M}_{\mathcal{S}} f \leq M_V f = \max_{v\in V} \mathcal{M}_{\mathcal{R}_v} f,$$

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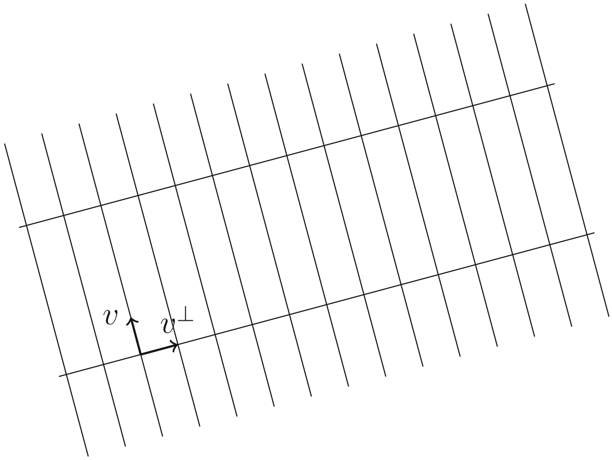

Fig. 1 The rectangles in $\mathcal{R}_v$

so it satisfies to apply inequalities (2.5) and (2.6) combined with estimates (1.1). For (2.10) we will additionally need the bound

$$\|\mathcal{M}_{\mathcal{S}}\|_{L^p\to L^p}\lesssim(\log_+ V)^{1/p},\quad p>2,$$

which is obtained from (1.1) by the Marcinkiewicz interpolation theorem.

In light of the pointwise sparse bound in Theorem D, one can easily see that Corollary 1.2 in turn follows from (2.8) and (2.9).

Since the maximal function corresponding to the n-dimensional canonical rectangles (with sides parallel to axes) in $\mathbb{R}^n$ is bounded on $L^p(\mathbb{R}^n)$, $1<p\le\infty$, applying the Marcinkiewicz interpolation theorem, from (2.5) we can immediately deduce the following result.

Corollary 2.11 *If $\mathcal{S}_k$, $k=1,2,\dots,N$, are sparse collections of canonical rectangles in $\mathbb{R}^n$, then for the maximal sparse operator* (2.3) *it holds the inequality*

$$\|\Lambda_{\mathfrak{S}}\|_{L^p\to L^p}\lesssim\log_+ N,\quad 1<p<\infty, \tag{2.12}$$

Applying the weak-L^1 estimate of the maximal function corresponding to n-dimensional balls in $\mathbb{R}^n$, from (2.5) we also obtain

Corollary 2.13 *If $\mathcal{S}_k$, $k=1,2,\dots,N$, are sparse collections of balls in $\mathbb{R}^n$, then for operator* (2.3) *we have*

$$\|\Lambda_{\mathfrak{S}}\|_{L^1\to L^{1,\infty}}\lesssim\log_+ N \tag{2.14}$$

Combining sparse domination Theorem D with Corollary 2.11, one can easily get

Corollary 2.15 *Let T be a Calderón–Zygmund operator on $\mathbb{R}^n$. Then for any sequence of measurable functions f_k, $k=1,2,\dots,N$, satisfying $|f_k(x)|\le f(x)$, $x\in\mathbb{R}^n$, it hold the inequalities*

$$\left\|\sup_{1\le k\le N} Tf_k\right\|_{L^p}\lesssim\log_+ N\cdot\|f\|_{L^p},\ 1<p<\infty, \tag{2.16}$$

$$\left\|\sup_{1\le k\le N} Tf_k\right\|_{L^{1,\infty}}\lesssim\log_+ N\cdot\|f\|_{L^1}. \tag{2.17}$$

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Indeed, applying Theorem D, we get

$$|Tf_k| \lesssim \Lambda_{\mathcal{S}_k} f_k \le \Lambda_{\mathcal{S}_k} f,$$

for some sparse collections of balls $\mathcal{S}_k$, and then the estimates in Corollary 2.15 can be deduced from (2.12) and (2.14) respectively.

3 Proof of Theorem 2.4

From Definition 2.1 it easily follows that any sparse operator, corresponding to a finite-martingale sparse collection of sets, can be dominated by a sum of bounded number of martingale sparse operators. So we can consider only martingale collections $\mathcal{S}_k$ in Theorem 2.4.

The basic key to the proofs are the following properties of a sparse collection. Let $\mathcal{S}$ be a martingale sparse collection. For $R \in \mathcal{S}$ denote by $\mathcal{S}_j(R)$ the j generation of R. That is $\mathcal{S}_0(R) = \{R\}$ and inductively set $\mathcal{S}_{j+1}(R)$ to be the maximal elements in

$$\{R' \in \mathcal{S} \ : \ R' \subset R\} \setminus \bigcup_{i=0}^{j} \mathcal{S}_i(R). \tag{3.1}$$

Observe that for a fixed $j \ge 0$ the collection of sets

$$G_j(R) = \bigcup_{R' \in \mathcal{S}_j(R)} R', \quad R \in \mathcal{S},$$

is itself martingale sparse collection. Besides, from the definition of martingale sparse collection it follows that

$$\mu(G_j(R)) \le \gamma^j \mu(B). \tag{3.2}$$

This implies the exponential estimate

$$\mu\left\{\sum_{\substack{S\in\mathcal{S}\\ S\subset R_0}} \mathbf{1}_S > \lambda\right\} \lesssim |R_0| \cdot \gamma^{\lambda}. \tag{3.3}$$

Proof of (2.5) Take $f \in L^p(X)$, $p \ge 1$, of norm one. For a $\lambda > 0$ and a small constant $\delta > 0$ we denote

$$\mathcal{S}_{k,0} = \left\{R \in \mathcal{S}_k : \langle f\rangle_R > \frac{\delta\lambda}{\log N}\right\},$$
$$\mathcal{S}_{k,s} = \left\{R \in \mathcal{S}_k : \frac{\delta\lambda}{\log N}\cdot 2^{-s+1} \ge \langle f\rangle_R > \frac{\delta\lambda}{\log N}\cdot 2^{-s}\right\}, \ s = 1, 2, \ldots.$$

Observe that for a fixed k the families $\mathcal{S}_{k,s}$, $s = 0, 1, 2, \ldots$, form a partition for the sparse collection $\mathcal{S}_k$. Besides, we have

$$\begin{aligned}\mu\left(\bigcup_{k=1}^{N}\bigcup_{R\in\mathcal{S}_{k,s}} R\right) &\le \mu\left\{\mathcal{M}_{\mathcal{S}} f > \frac{\delta\lambda}{\log N}\cdot 2^{-s}\right\}\\ &\le \left(\frac{\log N}{\delta\lambda}\right)^p \cdot 2^{sp}\cdot \|M_{\mathcal{S}}\|^p_{L^p\to L^{p,\infty}}.\end{aligned} \tag{3.4}$$

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Hence, using the definition of $\mathcal{S}_{k,s}$, we get

$$\begin{aligned} E_\lambda = \{\Lambda_{\mathfrak{G}} f > \lambda\} &= \bigcup_{k=1}^{N} \{\Lambda_{\mathcal{S}_k} f > \lambda\} \\ &\subset \bigcup_{k=1}^{N} \bigcup_{s\geq 0} \Big\{ \sum_{R\in\mathcal{S}_{k,s}} \langle f\rangle_R \mathbf{1}_R > c2^{-s/2}\lambda \Big\} \\ &\subset \left(\bigcup_{k=1}^{N} \bigcup_{R\in\mathcal{S}_{k,0}} R \right) \bigcup \left(\bigcup_{k=1}^{N} \bigcup_{s\geq 1} \Big\{ \sum_{R\in\mathcal{S}_{k,s}} \langle f\rangle_R \mathbf{1}_R > c2^{-s/2}\lambda \Big\} \right) \\ &\subset \left(\bigcup_{k=1}^{N} \bigcup_{R\in\mathcal{S}_{k,0}} R \right) \bigcup \left(\bigcup_{k=1}^{N} \bigcup_{s\geq 1} \Big\{ \sum_{R\in\mathcal{S}_{k,s}} \mathbf{1}_R > c2^{s/2-1} \cdot \frac{\log N}{\delta} \Big\} \right), \end{aligned} \tag{3.5}$$

where $c > 0$ is an absolute constant. From (3.4) we deduce

$$\mu \left(\bigcup_{k=1}^{N} \bigcup_{R\in\mathcal{S}_{k,0}} R \right) \lesssim \left(\frac{\log N}{\delta\lambda} \right)^p \|M_{\mathcal{S}}\|^p_{L^p\to L^{p,\infty}}. \tag{3.6}$$

Applying exponential estimate (3.3) and (3.4) again, we see that

$$\begin{aligned} \mu|\Big\{ \sum_{R\in\mathcal{S}_{k,s}} \mathbf{1}_R > c2^{s/2-1} \cdot \frac{\log N}{\delta} \Big\} &\lesssim (\gamma^{c/(2\delta)})^{2^{s/2}\log N} \mu\Big(\bigcup_{R\in\mathcal{S}_{k,s}} R \Big) \\ &\lesssim (\gamma^{c/(2\delta)})^{2^{s/2}\log N} \cdot 2^{sp} \cdot \left(\frac{\log N}{\delta\lambda} \right)^p \|M_{\mathcal{S}}\|^p_{L^p\to L^{p,\infty}} \\ &\leq \frac{1}{N} \cdot 2^{-s} \cdot \left(\frac{\log N}{\delta\lambda} \right)^p \|M_{\mathcal{S}}\|^p_{L^p\to L^{p,\infty}}, \end{aligned} \tag{3.7}$$

where the last inequality is obtained by a small enough choice of δ. From (3.5), (3.6) and (3.7) we immediately get

$$\mu(E_\lambda) \lesssim \left(1 + \sum_{k=1}^{N} \sum_{s\geq 1} \frac{2^{-s}}{N} \right) \left(\frac{\log N}{\delta\lambda} \right)^p \|M_{\mathcal{S}}\|^p_{L^p\to L^{p,\infty}} \lesssim \left(\frac{\log N}{\lambda} \right)^p \|M_{\mathcal{S}}\|^p_{L^p\to L^{p,\infty}},$$

that implies (2.5). □

To prove (2.6) we will need a simple lemma below. Let $\mathcal{S}$ be a martingale sparse collection with a constant γ. Attach to each $R \in \mathcal{S}$ a measurable set $G(R) \subset R$ such that $\mu(G(R)) < \delta\mu(R)$, $0 < \delta < 1$ and suppose that $\mathcal{S}' = \{G(R) : R \in \mathcal{S}\}$ is itself a martingale sparse collection with the same constant γ. For $\alpha > 0$ consider the sparse like operator

$$\Lambda^\alpha_{\mathcal{S},\mathcal{S}'} f(x) = \left(\sum_{R\subset\mathcal{S}} \langle f\rangle^\alpha_R \mathbf{1}_{G(B)}(x) \right)^{1/\alpha}. \tag{3.8}$$

Notice that in the case $\alpha = 1$ and $G(R) = R$ it gives the ordinary sparse operator. The proof of the following lemma is based on a well-known argument.

Lemma 3.9 *The operator (3.8) is bounded on $L^p(X)$ for $1 < p < \infty$. Moreover, we have*

$$\|\Lambda^\alpha_{S,S'}\|_{L^p(X)\to L^p(X)} \leq c\delta^{1/p}.$$

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where $c > 0$ is a constant depended on α and on the constants from the above definitions.

Proof For $R \in \mathcal{S}$ we have

$$\mu(G(R)) \le \delta\mu(R) \le \delta \cdot \gamma^{-1}\mu(E_R), \quad \mu(G(R)) \le \gamma^{-1}\mu(E_{G(R)}),$$

where E_R and $E_{G(R)}$ denote the disjoint portions of the members of $\mathcal{S}$ and $\mathcal{S}'$ respectively. Suppose $\|f\|_p = 1$. For some positive function $g \in L^{p/(p-\alpha)}(X)$ of norm one, we have

$$\begin{aligned}
\|\Lambda^\alpha_{\mathcal{S},\mathcal{S}'}(f)\|_p^\alpha &= \left\| \sum_{R\in\mathcal{S}_k} \langle f\rangle_R^\alpha \mathbf{1}_{G(R)} \right\|_{p/\alpha} \\
&= \left\langle \sum_{R\in\mathcal{S}_k} \langle f\rangle_R^\alpha \mathbf{1}_{G(R)}, g \right\rangle \\
&= \sum_{R\in\mathcal{S}_k} \langle f\rangle_R^\alpha \langle g\rangle_{G(R)} \mu(G(R)) \\
&= \sum_{R\in\mathcal{S}_k} \langle f\rangle_R^\alpha \Big(\mu(G(R))\Big)^{\alpha/p} \cdot \langle g\rangle_{G(R)} \Big(\mu(G(R))\Big)^{(p-\alpha)/p} \\
&\le \left(\sum_{R\in\mathcal{S}_k} \langle f\rangle_R^p \cdot \mu(G(R)) \right)^{\alpha/p} \left(\sum_{R\in\mathcal{S}_k} \langle g\rangle_{G(R)}^{p/(p-\alpha)} \cdot \mu(G(R)) \right)^{(p-\alpha)/p} \\
&\le \gamma^{-1}\delta^{\alpha/p} \left(\sum_{R\in\mathcal{S}_k} \langle f\rangle_R^p \mu(E_R)) \right)^{\alpha/p} \left(\sum_{R\in\mathcal{S}_k} \langle g\rangle_{G(R)}^{p/(p-\alpha)} \mu(E_{G(R)}) \right)^{(p-\alpha)/p} \\
&\le \gamma^{-1}\delta^{\alpha/p} \|\mathcal{M}_{\mathcal{S}}(f)\|_p^\alpha \|\mathcal{M}_{\mathcal{S}'}(g)\|_{p/(p-\alpha)} \\
&\lesssim \gamma^{-1}\delta^{\alpha/p} \|f\|_p^\alpha \|g\|_{p/(p-\alpha)} = \gamma^{-1}\delta^{\alpha/p}.
\end{aligned}$$

In the last inequality we use the boundedness of maximal functions $\mathcal{M}_{\mathcal{S}}$ and $\mathcal{M}_{\mathcal{S}'}$ corresponding to the martingale sparse collections $\mathcal{S}$ and $\mathcal{S}'$. □

Proof of (2.6) Let E_k, $k = 1, 2, \ldots, N$ be a measurable partition of X. Linearizing the supremum in the definition of Λ, we can redefine

$$\Lambda_{\mathfrak{G}} f(x) = \sum_{k=1}^N \sum_{R\in\mathcal{S}_k} \langle f\rangle_R \mathbf{1}_{E_{k,R}}(x), \qquad E_{k,R} = E_k \cap R.$$

Denote $\alpha = \min\{1, p-1\} \le 1$. Let $\mathcal{S}_{k,j}(R)$ be the j generation of $R \in \mathcal{S}_k$ (see the definition in (3.1)). For a function $f \in L^p(X)$ of norm one we denote

$$A_j^{(1)} = \int_X \sum_{k=1}^N \sum_{R\in\mathcal{S}_k} \langle f\rangle_R \mathbf{1}_{E_{k,R}} \left(\sum_{k=1}^N \sum_{R'\in\mathcal{S}_{k,j}(R)} \langle f\rangle_{R'} \mathbf{1}_{E_{k,R'}} \right)^\alpha (\Lambda_{\mathfrak{G}} f)^{p-\alpha-1},$$

$$A_j^{(2)} = \int_X \sum_{k=1}^N \sum_{R\in\mathcal{S}_k} \langle f\rangle_R \mathbf{1}_{E_{k,R}} \left(\sum_{k=1}^N \sum_{R' :\, R\in\mathcal{S}_{k,j}(R')} \langle f\rangle_{R'} \mathbf{1}_{E_{k,R'}} \right)^\alpha (\Lambda_{\mathfrak{G}} f)^{p-\alpha-1}.$$

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Then, using the inequality $(\sum_k x_k)^\alpha \le \sum_k x_k^\alpha$, we get

$$\begin{aligned}\|\Lambda_{\mathfrak{G}} f\|_p^p &= \int_X \sum_{k=1}^N \sum_{R\in\mathcal{S}_k} \langle f\rangle_R \mathbf{1}_{E_{k,R}} \left(\sum_{k=1}^N \sum_{R\in\mathcal{S}_k} \langle f\rangle_R \mathbf{1}_{E_{k,R}}\right)^\alpha \cdot (\Lambda_{\mathfrak{G}} f)^{p-\alpha-1} \\ &\le \sum_{j=0}^\infty (A_j^{(1)} + A_j^{(2)}). \end{aligned} \tag{3.10}$$

Since $\langle f\rangle_{R'}\mathbf{1}_{E_{k,R'}} \le \mathcal{M}_{\mathcal{S}}(f)$, for any $i = 1, 2$ it holds the inequality

$$\begin{aligned} A_j^{(i)} &\le \int_X (\mathcal{M}_{\mathcal{S}}(f))^\alpha \left(\sum_{k=1}^N \sum_{R\in\mathcal{S}_k} \langle f\rangle_R \mathbf{1}_{E_{k,R}}\right) \cdot (\Lambda_{\mathfrak{G}} f)^{p-\alpha-1} \\ &= \int_X (\mathcal{M}_{\mathcal{S}}(f))^\alpha \cdot (\Lambda_{\mathfrak{G}} f)^{p-\alpha} \\ &\le \|\mathcal{M}_{\mathcal{S}}\|_p^\alpha \|\Lambda_{\mathfrak{G}}\|_p^{p-\alpha}. \end{aligned} \tag{3.11}$$

For a fixed j and $R \in \mathcal{S}_k$ denote $G_j(R) = \cup_{R'\in\mathcal{S}_{k,j}(R)} R'$. Observe that the family $\mathcal{S}_{k,j} = \{G_j(R) : R \in \mathcal{S}_k\}$ forms a martingale-system and (3.2) implies $\mu(G_j(R)) \le \gamma^j \mu(R)$. So the family $\mathcal{S}_k$ together with $\mathcal{S}_{k,j}$ satisfies the conditions of Lemma 3.9. Thus we have

$$\begin{aligned} A_j^{(1)} &\le \sum_{k=1}^N \int_X \sum_{R\in\mathcal{S}_k} \langle f\rangle_R \mathbf{1}_{G(R)} \cdot (\Lambda_{\mathfrak{G}} f)^\alpha \cdot (\Lambda_{\mathfrak{G}} f)^{p-\alpha-1} \\ &= \sum_{k=1}^N \int_X \sum_{R\in\mathcal{S}_k} \langle f\rangle_R \mathbf{1}_{G(R)} \, (\Lambda_{\mathfrak{G}} f)^{p-1} \\ &\le \|\Lambda_{\mathfrak{G}}\|_{L^p\to L^p}^{p-1} \sum_{k=1}^N \left\| \Lambda^1_{\mathcal{S}_k,\mathcal{S}_{k,j}}(f)\right\|_p \\ &\le 2^{-cj} N \|\Lambda_{\mathfrak{G}}\|_{L^p\to L^p}^{p-1} \\ &\le 2^{-cj} N \|\Lambda_{\mathfrak{G}}\|_{L^p\to L^p}^{p-\alpha}, \end{aligned} \tag{3.12}$$

where the last inequality follows from $\|\Lambda_{\mathfrak{G}}\|_{L^p\to L^p} \ge 1$. Likewise, again applying $(\sum_k x_k)^\alpha \le \sum_k x_k^\alpha$, we get

$$\begin{aligned} A_j^{(2)} &\le \int_X \sum_{k=1}^N \sum_{R\in\mathcal{S}_k} \langle f\rangle_R \mathbf{1}_{E_{k,R}} \left(\sum_{k=1}^N \sum_{R':\, R\in\mathcal{S}_{k,j}(R')} \langle f\rangle_{R'}^\alpha \mathbf{1}_{E_{k,R'}}\right) (\Lambda_{\mathfrak{G}} f)^{p-\alpha-1} \\ &= \sum_{k=1}^N \int_X \sum_{R'\in\mathcal{S}_k} \langle f\rangle_{R'}^\alpha \mathbf{1}_{E_{k,R'}} \left(\sum_{R\in\mathcal{S}_{k,j}(R')} \langle f\rangle_R \mathbf{1}_{E_{k,R}}\right) (\Lambda_{\mathfrak{G}} f)^{p-\alpha-1} \\ &\le \sum_{k=1}^N \int_X \sum_{R'\in\mathcal{S}_k} \langle f\rangle_{R'}^\alpha \mathbf{1}_{G(R')} \cdot \Lambda_{\mathfrak{G}} f \cdot (\Lambda_{\mathfrak{G}} f)^{p-\alpha-1} \\ &= \sum_{k=1}^N \int_X (\Lambda^\alpha_{\mathcal{S}_k,\mathcal{S}'_k}(f))^\alpha \, (\Lambda_{\mathfrak{G}} f)^{p-\alpha} \end{aligned}$$

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$$\leq \|\Lambda_{\mathfrak{G}}\|_{L^p\to L^p}^{p-\alpha} \sum_{k=1}^{N} \left\| \Lambda^{\alpha}_{\mathcal{S}_k,\mathcal{S}_{k,j}}(f) \right\|_p^{\alpha}$$

$$\leq 2^{-cj} N \|\Lambda_{\mathfrak{G}}\|_{L^p\to L^p}^{p-\alpha}. \tag{3.13}$$

Combining (3.10), (3.11), (3.12) and (3.13), we will get

$$\|\Lambda_{\mathfrak{G}}\|_{L^p\to L^p}^{\alpha} \lesssim \sum_{j=0}^{\infty} \min\{\|M_{\mathcal{S}}\|_{L^p\to L^p}^{\alpha}, N2^{-cj}\}.$$

Since $\|M_{\mathcal{S}}\|_{L^p\to L^p} \geq 1$, for an appropriate choice of a constant $c' > 0$ we obtain

$$\begin{aligned}\|\Lambda_{\mathfrak{G}}\|_{L^p\to L^p}^{\alpha} &\leq c' \log N \|M_{\mathcal{S}}\|_{L^p\to L^p}^{\alpha} + \sum_{j=c'\log N}^{\infty} N2^{-cj}\\ &\leq c' \log N \|M_{\mathcal{S}}\|_{L^p\to L^p}^{\alpha} + 1\\ &\lesssim \log N \|M_{\mathcal{S}}\|_{L^p\to L^p}^{\alpha}.\end{aligned}$$

Taking into account the definition of α, this completes the proof of (2.6). □

4 Extensions

The logarithmic gains in the main theorem are sharp, in general. Indeed, it is enough to show the optimality of logarithm in (2.12). The function f is taken to be identically one on a large cube $Q \subset \mathbb{R}^n$. For each $k = 1, 2, \ldots, N$, it is very easy to construct a sparse operator $\Lambda_{\mathcal{S}_k}$ based on a sparse collection of cubes $\mathcal{S}_k$ so that $\{\Lambda_{\mathcal{S}_k} f > c \log N\}$ will have measure at least $|Q|/N$. These sets can be made to be essentially statistically independent, so that one sees that the logarithmic bound is sharp in (2.12). A careful examination of the same argument can show also the sharpness of the estimates (2.16) and (2.17), in general.

The papers [6,7] prove a variety of results for T_V defined as a maximum of a fixed Hormander-Mihklin multiplier computed in directions $v \in V$. Their estimates are slightly better than ours in Corollary 1.2. This raises two questions:

Question 4.1 *First, if one fixes the specific sparse operator computed in every direction, can bounds be proved that match those of say [7]?*

This paper [7] proves results for the maximal truncations of the Hilbert transform computed in different directions. Again, their bounds are better than ours.

Question 4.2 *Can one formulate a maximal sparse operator which is less general than ours, but still general enough to capture these results for maximal truncations of the Hilbert transform?*

Recent papers [17,18] have established variants of these results in higher dimensions. Other papers [8,9] consider certain Lipschitz versions. It would be interesting to study the analogous questions for both themes.

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