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REAL AND COMPLEX ANALYSIS

On an Interpolation by the Modified Trigonometric System

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Abstract—We consider interpolations by the modified trigonometric system and explore convergence in different frameworks. We prove better convergence of such interpolations for odd functions compared to the interpolations with the classical trigonometric system.

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1. INTRODUCTION

In this paper, we consider interpolations by the modified trigonometric system

$$\mathcal{H} = \{\cos \pi n x : n \in \mathbb{Z}_+\} \cup \{\sin \pi (n - \tfrac{1}{2})x : n \in \mathbb{N}\}. \tag{1.1}$$

The set $\mathcal{H}$ is an orthonormal basis of $L_2[-1,1]$, as it consists of eigenfunctions of the Sturm-Liouville operator $\mathcal{L} = -d^2/dx^2$ with Neumann boundary conditions $u'(1) = u'(-1) = 0$. Both, the orthogonality and density in $L_2[-1,1]$ follow from the classical spectral theory ([1]).

The basis was originally proposed by Krein [2] without investigation of its properties. Expansions by the modified trigonometric system were investigated in a series of papers [3] − [12]. We denote by $\mathcal{M}_N(f,x)$ the truncated modified Fourier series

$$\mathcal{M}_N(f,x) = \frac{1}{2} f_0^c + \sum_{n=1}^{N} [f_n^c \cos \pi n x + f_n^s \sin \pi (n - \tfrac{1}{2})x], \tag{1.2}$$

where

$$f_n^c = \int_{-1}^{1} f(x) \cos \pi n x dx, \;\; f_n^s = \int_{-1}^{1} f(x) \sin \pi (n - \tfrac{1}{2}) x dx. \tag{1.3}$$

It is easy to verify that the modified trigonometric system can be written also in the form $\mathcal{H} = \{\varphi_n(x) : n \in \mathbb{Z}_+\}$, where

$$\varphi_0(x) = \frac{1}{\sqrt{2}}, \;\; \varphi_n(x) = \frac{1}{2}\left((-1)^n e^{\frac{i\pi n x}{2}} + e^{-\frac{i\pi n x}{2}}\right), \;\; n \in \mathbb{N}. \tag{1.4}$$

The truncated modified Fourier series can be rewritten more compactly

$$\mathcal{M}_N(f,x) = \sum_{n=0}^{2N} f_n^m \varphi_n(x), \tag{1.5}$$

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where

$$f_n^m = \int_{-1}^{1} f(x)\overline{\varphi_n(x)}dx. \tag{1.6}$$

It follows from (1.2) and (1.3) that expansions by the modified trigonometric system for even functions on $[-1, 1]$ coincide with the expansions by the classical trigonometric system

$$\mathcal{H}_{class} = \{\cos \pi nx : n \in \mathbb{Z}_+\} \cup \{\sin \pi nx : n \in \mathbb{N}\},\ x \in [-1, 1]. \tag{1.7}$$

Moreover, the modified trigonometric system can be derived from the other classical basis $\mathcal{H}^*$ on $[0, 1]$

$$\mathcal{H}^* = \{\cos \pi nx : n \in \mathbb{Z}_+\},\ x \in [0, 1] \tag{1.8}$$

by means of a change of a variable.

Expansions by the modified trigonometric system have better convergence properties for smooth odd functions on $[-1, 1]$ compared to the classical expansions, as coefficients f_n^s tend to zero with higher order $O(n^{-2})$, $n \to \infty$ than the classical coefficients corresponding to $\sin \pi nx$.

Theorem 1.1. *[6, 7] Assume $f \in C^{2q+2}(-1, 1)$, $f^{(2q+2)} \in BV[-1, 1]$, $q \geq 0$ and f obeys the first q odd derivative conditions $f^{(2r+1)}(\pm 1) = 0$, $r = 0, \ldots, q-1$. Then, if $|x| < 1$,*

$$f(x) - \mathcal{M}_N(f, x) = O(N^{-2q-2}), \quad N \to \infty,$$

otherwise,

$$f(\pm 1) - \mathcal{M}_N(f, \pm 1) = O(N^{-2q-1}), \quad N \to \infty.$$

Theorem 1.2. *[6, 11] Assume $f \in C^{2q+1}(-1, 1)$, $f^{(2q+1)} \in BV[-1, 1]$, $q \geq 0$ and f obeys the first q odd derivative conditions $f^{(2r+1)}(\pm 1) = 0$, $r = 0, \ldots, q-1$. Then,*

$$||f(x) - \mathcal{M}_N(f, x)||_{L_2} = O(N^{-2q-\frac{3}{2}}), \quad N \to \infty.$$

We see that the conditions

$$f^{(2r+1)}(\pm 1) = 0,\ \ r = 0, \ldots, q-1 \tag{1.9}$$

are crucial for convergence properties of the expansions by the modified trigonometric system. If a function doesn't obey those derivative conditions, then, application of a well-known polynomial subtraction method (see [13]-[16]) will correct the derivatives at the endpoints $x = \pm 1$. More specifically, let $f \in C^{2q-1}[-1, 1]$ and denote

$$B_{2k+1}(f) = \left(f^{(2k+1)}(1) + f^{(2k+1)}(-1)\right)(-1)^k, \quad k = 0, \ldots, q-1, \tag{1.10}$$

$$A_{2k+1}(f) = \left(f^{(2k+1)}(1) - f^{(2k+1)}(-1)\right)(-1)^k, \quad k = 0, \ldots, q-1. \tag{1.11}$$

Assume that even polynomials $P_k(x)$ and odd polynomials $Q_k(x)$, $k = 0, \ldots, q-1$ satisfy the following conditions (see [16])

$$A_{2k+1}(P_j(x)) = \delta_{k,j}, \quad B_{2k+1}(Q_j(x)) = \delta_{k,j}, \quad 0 \leq k, j \leq q-1. \tag{1.12}$$

The first few polynomials are

$$P_0(x) = \frac{1}{4}x^2, \quad P_1(x) = \frac{1}{48}x^2(x^2 - 2), \quad Q_0(x) = \frac{1}{2}x, \quad Q_1(x) = \frac{1}{12}x(x^2 - 3).$$

Let F be defined as follows

$$F(x) = f(x) - \sum_{k=0}^{q-1} A_{2k+1}(f)P_k(x) - \sum_{k=0}^{q-1} B_{2k+1}(f)Q_k(x). \tag{1.13}$$

Then, F obeys the first q odd derivative conditions. Now, if we denote

$$\mathcal{M}_{N,q}(f,x) = \mathcal{M}_N(F,x) + \sum_{k=0}^{q-1} A_{2k+1}(f)P_k(x) + \sum_{k=0}^{q-1} B_{2k+1}(f)Q_k(x), \tag{1.14}$$

Theorems 1.1 and 1.2 will be valid for approximation $\mathcal{M}_{N,q}(f,x)$ without derivative conditions if the exact values of A_{2k+1} and B_{2k+1}, $k = 0, \ldots, q-1$ are known. Otherwise, they can be approximated by a solution of system of linear equations (see [16]).

In this paper, we consider interpolation by the modified trigonometric system and explore its convergence in different frameworks. We derive exact constants of the asymptotic errors for the L_2-convergence, pointwise convergence on $|x| < 1$ and at the endpoints $x = \pm 1$. Comparison to the interpolation with the classical trigonometric system shows better convergence of the modified interpolation for odd functions in all frameworks.

2. MODIFIED INTERPOLATION

Eigenfunctions defined by (1.4) have important discrete orthogonality properties. Let $x_k = \frac{2k}{2N+1}$, $|k| \le N$ be the uniform grid on $[-1,1]$. It is easy to verify that

$$\frac{2}{2N+1} \sum_{n=0}^{2N} \varphi_n(x_k)\overline{\varphi_n}(x_s) = \delta_{k,s}, \quad |k|, |s| \le N, \tag{2.1}$$

and

$$\frac{2}{2N+1} \sum_{n=-N}^{N} \varphi_n(x_k)\overline{\varphi_m}(x_k) = \delta_{n,m}, \quad 0 \le m, n \le 2N. \tag{2.2}$$

Hence

$$\mathcal{I}_N(f,x) = \sum_{n=0}^{2N} \check{f}_n^m \varphi_n(x), \tag{2.3}$$

where

$$\check{f}_n^m = \frac{2}{2N+1} \sum_{k=-N}^{N} f(x_k)\overline{\varphi_n}(x_k) \tag{2.4}$$

is exact on the orthonormal basis $\mathcal{H}$ and interpolates $f \in C[-1,1]$ on the grid x_k, $|k| \le N$.

We call $\mathcal{I}_N(f,x)$ as interpolation by the modified trigonometric system, or simply, as modified interpolation. When f is a real-valued function, then the modified interpolation could be rewritten as follows

$$\mathcal{I}_N(f,x) = \sum_{n=0}^{N} \check{f}_n^c \cos \pi n x + \sum_{n=1}^{N} \check{f}_n^s \sin \pi (n - \tfrac{1}{2})x, \tag{2.5}$$

where

$$\check{f}_0^c = \frac{1}{2N+1} \sum_{k=-N}^{N} f(x_k), \quad \check{f}_n^c = \frac{2}{2N+1} \sum_{k=-N}^{N} f(x_k) \cos \pi n x_k, \tag{2.6}$$

and

$$\check{f}_n^s = \frac{2}{2N+1} \sum_{k=-N}^{N} f(x_k) \sin \pi (n - \tfrac{1}{2}) x_k. \tag{2.7}$$

This form should be more convenient for analysis when f is either odd or even on $[-1,1]$. Recall that the modified interpolation for even f coincide with the classical interpolation. Below, we explore the properties of the modified interpolation only for odd f on $[-1, 1]$. Furthermore, the polynomial correction approach is valid also for the modified interpolation. We write,

$$\mathcal{I}_{N,q}(f,x) = \sum_{n=0}^{N} \check{F}_n^c \cos \pi n x + \sum_{n=1}^{N} \check{F}_n^s \sin \pi (n - \tfrac{1}{2})x + \sum_{k=0}^{q-1} A_{2k+1}(f) P_k(x) + \sum_{k=0}^{q-1} B_{2k+1}(f) Q_k(x),$$

where

$$\check{F}_n^c = \check{f}_n^c - \sum_{k=0}^{q-1} A_{2k+1}(f) \check{P}_n^c(k), \quad \check{F}_n^s = \check{f}_n^s - \sum_{k=0}^{q-1} B_{2k+1}(f) \check{Q}_n^s(k).$$

Here, $\check{P}_n^c(k)$ and $\check{Q}_n^s(k)$ are the discrete modified Fourier coefficients of $P_k(x)$ and $Q_k(x)$, respectively. Let

$$R_{N,q}(f,x) = f(x) - \mathcal{I}_{N,q}(f,x). \tag{2.8}$$

The main goal of the paper is investigation of $R_{N,q}(f,x)$ in different frameworks. Section 3 explores convergence in the L_2-norm and Section 4, the pointwise convergence on $[-1, 1]$. In each case, we perform comparison with the corresponding results of the interpolations by the classical trigonometric system

$$\mathcal{I}_N^{classic}(f,x) = \sum_{n=-N}^{N} \check{f}_n e^{i\pi n x}, \tag{2.9}$$

where

$$\check{f}_n = \frac{1}{2N+1} \sum_{k=-N}^{N} f(x_k) e^{-i\pi n x_k}.$$

3. CONVERGENCE IN THE L_2-NORM

In this section, we explore the convergence of the modified interpolation in the L_2-norm. The next lemma establishes connection between the modified discrete and continuous coefficients.

Lemma 3.1. *Assume that $f \in C^2[-1, 1]$ and $f'' \in BV[-1, 1]$. Then, the following identity holds*

$$\check{f}_n^m = f_n^m + \sum_{j=1}^{\infty} f_{n+(2N+1)2j}^m + (-1)^n \sum_{j=1}^{\infty} f_{-n+(2N+1)2j}^m, \; n = 1, \ldots 2N. \tag{3.1}$$

Proof. From the pointwise convergence of the modified Fourier expansion (see Theorem 1.1 with $q = 0$), we have

$$f(x) = \sum_{j=0}^{\infty} f_j^m \varphi_j(x) = \sum_{r=0}^{\infty} \sum_{j=0}^{4N+1} f_{j+2r(2N+1)}^m \varphi_{j+2r(2N+1)}(x). \tag{3.2}$$

Taking into account that $\varphi_{j+2r(2N+1)}(x_k) = \varphi_j(x_k)$, we write

$$\check{f}_n^m = \sum_{r=0}^{\infty} \sum_{j=0}^{4N+1} f_{j+2r(2N+1)}^m \frac{2}{2N+1} \sum_{k=-N}^{N} \varphi_j(x_k) \overline{\varphi_n}(x_k). \tag{3.3}$$

It is easy to verify that for $j = 2N+1, \ldots, 4N+1$

$$\frac{2}{2N+1} \sum_{k=-N}^{N} \varphi_j(x_k) \overline{\varphi_n}(x_k) = \begin{cases} 0\,, & n = 0 \\ (-1)^n \delta_{4N+2-n,j}\,, & 1 \le n \le 2N \end{cases} \tag{3.4}$$

This, together with (2.2), completes the proof due to (3.3).

We can rewrite Lemma 3.1 for coefficients f_n^s.

Remark 3.1. *Assume that $f \in C^2[-1,1]$ and $f'' \in BV[-1,1]$. Then, the following identity holds*

$$\check{f}_n^s = f_n^s + \sum_{j \neq 0} f_{n+(2N+1)j}^s, \; n = 1, \dots, N. \tag{3.5}$$

The next theorem describes the convergence in the L_2-norm.

Theorem 3.1. *Let f be odd function on $[-1,1]$. Assume that $f \in C^{2q+1}[-1,1]$ and $f^{(2q+1)} \in BV[-1,1]$, $q \geq 0$. Then, the following estimate holds*

$$\lim_{N \to \infty} N^{2q+\frac{3}{2}} ||R_{N,q}||_{L_2} = |B_{2q+1}(f)| \frac{\sqrt{a(q)}}{\pi^{2q+2}}, \tag{3.6}$$

where

$$a(q) = \frac{1}{4q+3} + \int_0^1 \left(\sum_{s \neq 0} \frac{(-1)^s}{(2s+x)^{2q+2}} \right)^2 dx. \tag{3.7}$$

Proof. We can rewrite $R_{N,q}(f,x)$ for odd f as follows

$$R_{N,q}(f,x) = \sum_{n=1}^{N} (F_n^s - \check{F}_n^s) \sin \pi(n - \tfrac{1}{2})x + \sum_{n=N+1}^{\infty} F_n^s \sin \pi(n - \tfrac{1}{2})x. \tag{3.8}$$

Due to the orthonormality of the basis functions of $\mathcal{H}$, we get

$$||R_{N,q}||_{L_2}^2 = \sum_{n=1}^{N} \left(F_n^s - \check{F}_n^s\right)^2 + \sum_{n=N+1}^{\infty} \left(F_n^s\right)^2. \tag{3.9}$$

Taking into account that function F obeys the first q odd derivative conditions (1.9), we derive the following asymptotic expansion of its modified Fourier coefficients by means of integration by parts

$$F_n^s = B_{2q+1}(f) \frac{(-1)^{n+1}}{(\pi(n - \frac{1}{2}))^{2q+2}} + o(n^{-2q-2}). \tag{3.10}$$

Then, application of Remark 3.1 leads to the following estimate for $n = 1, \dots, N$

$$\check{F}_n^s - F_n^s = B_{2q+1}(f) \frac{(-1)^{n+1}}{(\pi N)^{2q+2}} \sum_{j \neq 0} \frac{(-1)^j}{(2j + \frac{n}{N})^{2q+2}} + o(N^{-2q-2}). \tag{3.11}$$

Estimates (3.10) and (3.11), together with (3.9), complete the proof.

When $q = 0$, Theorem 3.1 shows convergence rate $O(N^{-\frac{3}{2}})$ in the L_2-norm. The classical interpolation has convergence rate $O(N^{-\frac{1}{2}})$ in the L_2-norm for odd functions on $[-1,1]$ (see [17]). Hence, the improvement is by factor N^{-1}.

4. POINTWISE CONVERGENCE

In this section, we investigate the pointwise convergence of the modified interpolations on $|x| < 1$ and the endpoints $x = \pm 1$. We need some auxiliary estimates for the proof of the main results. We will frequently use the properties of the following numbers

$$\omega_{p,m} = \sum_{s=0}^{p} \binom{p}{s} (-1)^s s^m, \tag{4.1}$$

which are connected with the Stirling numbers of the second kind ([18]). In [15] it was verified that

$$\omega_{p,m} = 0, \quad 0 \leq m < p, \quad \omega_{p,p} = (-1)^p p!, \quad \omega_{p,p+1} = (-1)^p \frac{p(p+1)!}{2}. \tag{4.2}$$

Let $c = \{c_n\}$ be a sequence of complex numbers and by $\Delta_n^p(c)$ denote the following finite differences

$$\Delta_n^p(c) = \sum_{s=0}^{2p} \binom{2p}{s} c_{n+p-s}, \quad p \geq 0. \tag{4.3}$$

Let

$$Q_n(m) = \frac{(-1)^{n+1}}{(\pi(n-\frac{1}{2}))^{2m+2}}, \quad Q(m) = \{Q_n(m)\}_{n=1}^{\infty}. \tag{4.4}$$

From (1.13) and asymptotic expansion (3.10), it follows that $Q_n(m)$ are the modified Fourier coefficients of correction polynomial $Q_m(x)$. Then, denote $\check{Q}(m) = \{\check{Q}_n^s(m)\}_{n=1}^{N}$, where $\check{Q}_n^s(m)$ are the discrete modified coefficients of $Q_m(x)$.

Lemma 4.1. *For any $p \geq 0$ and $m \geq 0$ the following estimate holds*

$$\Delta_n^p(Q(m)) = \frac{(-1)^{n+p+1}(2m+2p+1)!}{(\pi(n-\frac{1}{2}))^{2m+2}(n-\frac{1}{2})^{2p}(2m+1)!} + O(n^{-2m-2p-3}), \quad n \to \infty. \tag{4.5}$$

Proof. From definition of $\Delta_n^p(Q(m))$, we have

$$\begin{aligned}\Delta_n^p(Q(m)) &= \sum_{s=0}^{2p} \binom{2p}{s} Q_{n+p-s}(m) = \frac{(-1)^{n+p+1}}{(\pi(n-\frac{1}{2}))^{2m+2}} \sum_{s=0}^{2p} \frac{\binom{2p}{s}(-1)^k}{\left(1+\frac{p-k}{n-\frac{1}{2}}\right)^{2m+2}} \\ &= \frac{(-1)^{n+p+1}}{(\pi(n-\frac{1}{2}))^{2m+2}} \sum_{s=0}^{\infty} \binom{s+2m+1}{2m+1} \frac{(-1)^s}{(n-\frac{1}{2})^s} \sum_{j=0}^{s} \binom{s}{j} (-1)^j p^{s-j} \omega_{2p,j},\end{aligned} \tag{4.6}$$

where $\omega_{2p,j}$ are defined by (4.1). This completes the proof in view of (4.2).

Lemma 4.2. *For any $p \geq 0$ and $m \geq 0$ the following estimates hold*

$$\Delta_n^p(\check{Q}(m) - Q(m)) = \frac{(-1)^{n+p+1}(2m+2p+1)!}{(\pi N)^{2m+2} N^{2p}(2m+1)!} \sum_{j \neq 0} \frac{(-1)^j}{(2j+\frac{n}{N})^{2m+2p+2}} + O(N^{-2m-2p-3}), \tag{4.7}$$

$n = 1, \ldots, N$, $N \to \infty$.

Proof. According to Remark 3.1, we can write

$$\Delta_n^p(\check{Q}(m) - Q(m)) = \sum_{j \neq 0} \Delta_{n+(2N+1)j}^p (Q(m))$$

$$= \frac{(-1)^{n+p+1}}{(\pi N)^{2m+2}} \sum_{k=0}^{2p} \binom{2p}{k} (-1)^k \sum_{j \neq 0} \frac{(-1)^j}{(2j+\frac{n}{N})^{2m+2}} \frac{1}{\left(1+\frac{j+p-k-\frac{1}{2}}{N(2j+\frac{n}{N})}\right)^{2m+2}}$$

$$= \frac{(-1)^{n+p}}{(\pi N)^{2m+2}} \sum_{t=0}^{\infty} \frac{(-1)^t}{N^t} \binom{2m+1+t}{2m+1} \sum_{s=0}^{t} \binom{t}{s} (-1)^s \omega_{2p,s} \sum_{j \neq 0} \frac{(-1)^j (p+j-\frac{1}{2})^{t-s}}{(2j+\frac{n}{N})^{2m+2}},$$

where $\omega_{2p,s}$ are defined by (4.1). This completes the proof in view of (4.2).

Lemma 4.3. *For any $m \geq 0$ the following estimate holds as $N \to \infty$*

$$\Delta_N^p(\check{Q}(m)) = \frac{(-1)^{N+p}(2m+2p+2)!}{(\pi N)^{2m+2} N^{2p+1}(2m+1)!} \sum_{j=-\infty}^{\infty} \frac{(-1)^j (j-\frac{1}{2})}{(2j+1)^{2m+2p+3}} + O(N^{-2m-2p-4}). \tag{4.8}$$

Proof. From Remark 3.1, we have

$$\Delta_N^p(\check{Q}(m)) = \sum_{j=-\infty}^{\infty} \Delta_{N+(2N+1)j}^p(Q(m))$$

$$= \frac{(-1)^{N+p+1}}{(\pi N)^{2m+2}} \sum_{t=0}^{\infty} \frac{(-1)^t}{N^t} \binom{2m+1+t}{2m+1} \sum_{s=0}^{t} \binom{t}{s} (-1)^s \omega_{2p,s} \sum_{j=-\infty}^{\infty} \frac{(-1)^j (p+j-\frac{1}{2})^{t-s}}{(2j+1)^{2m+2}},$$

where $\omega_{2p,s}$ are defined by (4.1). Taking into account that

$$\sum_{j=-\infty}^{\infty} \frac{(-1)^j}{(2j+1)^{2m+2}} = 0, \quad m = 0, 1, \ldots \tag{4.9}$$

and identities (4.2), we complete the proof.

The next Theorem demonstrates the pointwise convergence of the modified interpolation away from the endpoints.

Theorem 4.1. *Let f be an odd function on $[-1,1]$. Assume that $f \in C^{2q+3}[-1,1]$ and $f^{(2q+3)} \in BV[-1,1]$, $q \geq 0$. Then, the following estimate holds for $|x| < 1$*

$$R_{N,q}(f,x) = B_{2q+1}(f) \frac{(-1)^N}{N^{2q+3}} \frac{\pi |E_{2q+2}|}{2^{2q+5}(2q+1)!} \frac{\sin \pi (N+\frac{1}{2})x}{\cos^2 \frac{\pi x}{2}} + o(N^{-2q-3}), \quad N \to \infty,$$

where E_k is the k-th Euler number.

Proof. We put $f_{-n}^s = -f_{n+1}^s$, $\check{f}^s{}_{-n} = -\check{f}_{n+1}^s$, to rewrite interpolation error (3.8) in a more convenient form

$$R_{N,q}(f,x) = \frac{1}{2i} \sum_{n=-N+1}^{N} \left(F_n^s - \check{F}_n^s\right) e^{i\pi(n-\frac{1}{2})x} + \frac{1}{2i} \sum_{n=N+1}^{\infty} F_n^s e^{i\pi(n-\frac{1}{2})x} + \frac{1}{2i} \sum_{n=-\infty}^{-N} F_n^s e^{i\pi(n-\frac{1}{2})x}.$$

We proceed by application of the Abel transformation and derive

$$\begin{aligned} R_{N,q}(f,x) &= \frac{1}{2(1+\cos \pi x)} \left(\check{F}_{N+1}^s \sin \pi (N-\tfrac{1}{2})x - \check{F}_N^s \sin \pi (N+\tfrac{1}{2})x\right) \\ &+ \frac{1}{4(1+\cos \pi x)^2} \left(\Delta_{N+1}^1(\check{F}^s) \sin \pi (N-\tfrac{1}{2}) - \Delta_N^1(\check{F}^s) \sin \pi (N+\tfrac{1}{2})\right) \\ &+ \frac{e^{-i\frac{\pi x}{2}}}{8(1+\cos \pi x)^2} \left(\sum_{n=1}^{N} \Delta_n^2(F^s - \check{F}^s) e^{i\pi n x} + \sum_{n=N+1}^{\infty} \Delta_n^2(F^s) e^{i\pi n x}\right) \\ &+ \frac{e^{i\frac{\pi x}{2}}}{8(1+\cos \pi x)^2} \left(\sum_{n=-N}^{-1} \Delta_n^2(F^s - \check{F}^s) e^{i\pi n x} + \sum_{n=-\infty}^{-N-1} \Delta_n^2(F^s) e^{i\pi n x}\right). \end{aligned} \tag{4.10}$$

Taking into account the following asymptotic expansion of the modified coefficients

$$F_n^s = \sum_{m=q}^{q+1} B_{2m+1}(f) Q_n(m) + o(n^{-2q-4}), \quad n \to \infty, \tag{4.11}$$

we get

$$\Delta_n^p(F^s) = \sum_{m=q}^{q+1} B_{2m+1}(f) \Delta_n^p(Q(m)) + o(n^{-2q-4}), \quad n \to \infty.$$

Now, according to Lemma 4.1, we have $\Delta_n^2(F^s) = o(n^{-2q-4})$, and the infinite sums on the right-hand side of (4.10) are $o(N^{-2q-3})$. Again from (4.11), we write

$$\Delta_n^2(\check{F}^s - F^s) = \sum_{m=q}^{q+1} B_{2m+1}(f)\Delta_n^2(\check{Q}(m) - Q(m)) + o(N^{-2q-4}), \tag{4.12}$$

and from Lemma 4.2, we get

$$\Delta_n^2(F^s - \check{F}^s) = o(N^{-2q-4}), \quad n = \pm 1, \pm 2, \ldots, \pm N. \tag{4.13}$$

Hence, the finite sums on the right-hand side of (4.10) are $o(N^{-2q-3})$.

Lemma 4.3 shows that

$$\Delta_N^1(\check{F}^s) = o(N^{-2q-3}), \quad \Delta_{N+1}^1(\check{F}^s) = o(N^{-2q-3}). \tag{4.14}$$

All these lead to the following estimate

$$R_{N,q}(f,x) = \frac{1}{4\cos^2\frac{\pi x}{2}}\left(\check{F}_{N+1}^s \sin\pi(N-\tfrac{1}{2})x - \check{F}_N^s \sin\pi(N+\tfrac{1}{2})x\right) + o(N^{-2q-3}). \tag{4.15}$$

According to Lemma 4.3, we get

$$\check{F}_N^s = B_{2q+1}(f)\frac{(-1)^N(2q+2)}{(\pi N)^{2q+2}N}\sum_{j=-\infty}^{\infty}\frac{(-1)^j(j-\frac{1}{2})}{(2j+1)^{2m+3}} + o(N^{-2q-3}). \tag{4.16}$$

From the other side

$$\check{F}_{N+1}^s = \frac{2}{2N+1}\sum_{k=-N}^{N} f(x_k)\sin\pi k = 0. \tag{4.17}$$

Hence,

$$R_{N,q}(f,x) = B_{2q+1}(f)\frac{(-1)^{N+1}(q+1)}{2\pi^{2q+2}N^{2q+3}}\frac{\sin\pi(N+\frac{1}{2})x}{\cos^2\frac{\pi x}{2}}\sum_{j=-\infty}^{+\infty}\frac{(-1)^j(j-\frac{1}{2})}{(2j+1)^{2q+3}} + o(N^{-2q-3}),$$

which completes the proof.

When $q = 0$, Theorem 4.1 implies the convergence rate $O(N^{-3})$ as $N \to \infty$ for an odd function. The classical interpolation (see [17]) has convergence rate $O(N^{-1})$ for the grid $x_k = 2k/2N+1$ and convergence rate $O(N^{-2})$ for the optimal grid $x_k = (2k \pm 1)/2N + 1$. Hence, improvement is by factor $O(N)$ as $N \to \infty$.

Next theorem explores the convergence of the modified interpolations at the endpoints $x = \pm 1$.

Theorem 4.2. *Let f be an odd function on $[-1,1]$. Assume that $f \in C^{2q+2}[-1,1]$ and $f^{(2q+2)} \in BV[-1,1]$, $q \geq 0$. Then, the following estimate holds*

$$R_{N,q}(f,\pm 1) = \pm B_{2q+1}(f)\frac{(-1)^{N+1}}{N^{2q+1}}\frac{|E_{2q}|}{2^{2q+1}\pi(2q+1)!} + o(N^{-2q-1}), \quad N \to \infty,$$

where E_k is the k-th Euler number.

Proof. We use (3.8) and get

$$R_{N,q}(f,\pm 1) = \sum_{n=1}^{N}(F_n^s - \check{F}_n^s)(-1)^{n+1} + \sum_{n=N+1}^{\infty} F_n^s(-1)^{n+1}. \tag{4.18}$$

Taking into account the following asymptotic expansion of the modified Fourier coefficients

$$F_n^s = B_{2q+1}(f)Q_n(q) + o(n^{-2q-2}), \quad n \to \infty, \tag{4.19}$$

and applying Remark 3.1, for $n = 1, \dots, N$ and $N \to \infty$ we get

$$\check{F}_n^s - F_n^s = \frac{B_{2q+1}(f)(-1)^{n+1}}{(\pi N)^{2q+2}} \sum_{j\neq 0}(-1)^j \frac{1}{(2j+\frac{n}{N})^{2q+2}} + o(N^{-2q-2}). \tag{4.20}$$

Equation (4.18), together with (4.19) and (4.20), implies

$$R_{N,q}(f,\pm 1) = \pm B_{2q+1}(f)\frac{(-1)^N}{\pi^{2q+2}N^{2q+1}}\left(\frac{1}{2q+1} - \int_0^1 \sum_{j\neq 0}\frac{(-1)^j}{(2j+x)^{2q+2}}dx\right) + o(N^{-2q-1}),$$

which completes the proof.

When $q = 0$, Theorem 4.2 shows convergence rate $O(1/N)$. In this case, as $f(1) \neq f(-1)$, the classical interpolation doesn't converge at the endpoints. Hence, the modified interpolations have better convergence rate at the endpoints and the improvement is by factor $O(N)$.

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