BERGMAN TYPE PROJECTION IN THE BESOV SPACES

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A Bergman type operator is constructed on the unit disc, that continuously projects a Besov space onto its harmonic subspace.

§1. INTRODUCTION

Let $h(\mathbb{D})$ $(H(\mathbb{D}))$ denote the set of all harmonic (respectively holomorphic) functions in the unit disc $\mathbb{D}$. For a measurable in $\mathbb{D}$ function $f(z) = f(re^{i\theta})$ we denote

$$M_p(f;r) = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |f(re^{i\theta})|^p\, d\theta\right)^{1/p}, & 0 < p < \infty, \\ \operatorname*{ess\,sup}_{-\pi<\theta\le\pi} |f(re^{i\theta})|, & p = \infty, \end{cases}$$

where $0 \le r < 1$. The set of those harmonic (holomorphic) functions $f(z)$, for which

$$\|f\|_{h^p} = \sup_{0\le r<1} M_p(f;r) < +\infty,$$

is the usual Hardy space h^p (respectively H^p).

The quasi-normed space $L(p,q,\alpha)$ $(0 < p, q \le \infty,\ \alpha \in \mathbb{R})$ with mixed norm is the set of the measurable in $\mathbb{D}$ functions $f(z)$, for which the quasi-norm defined by

$$\|f\|_{p,q,\alpha} = \begin{cases} \left(\displaystyle\int_0^1 (1-r)^{\alpha q-1} M_p^q(f;r)dr\right)^{1/q}, & 0 < q < \infty, \\ \operatorname*{ess\,sup}_{0\le r<1} (1-r)^{\alpha} M_p(f;r), & q = \infty, \end{cases}$$

is finite. The harmonic and holomorphic spaces with mixed norms are defined to be the subspaces of $L(p,q,\alpha)$ consisting of harmonic or holomorphic functions:

$$h(p,q,\alpha)=h(\mathbb{D})\cap L(p,q,\alpha),\quad H(p,q,\alpha)=H(\mathbb{D})\cap L(p,q,\alpha).$$

Note that $h(p,\infty,0)=h^p$ and $H(p,\infty,0)=H^p$. The first results on mixed norm spaces are contained in the classical works by Hardy and Littlewood [1], [2]. Observe that for $p=q<\infty$ the spaces $h(p,q,\alpha)$ and $H(p,q,\alpha)$ coincide with the well-known weighted Bergman spaces (see [3] – [6]). Later, the methods and results of [1], [2] were essentially improved and developed by Flett [7]. Analytic projections in Bergman and mixed norm spaces were investigated in [8], [9].

The present paper considers Besov spaces that are closely related to the mixed norm spaces and study Bergman type projections in these spaces.

§2. NOTATION AND PRELIMINARIES

Throughout the paper, the letters $C(\alpha,\beta,\dots),C_\alpha$ etc. will denote different positive constants depending on the parameters as indicated. For $A,B>0$ the notation $A\approx B$ stands for the two-sided estimate $c_1A\le B\le c_2A$ with some positive constants c_1 and c_2. The symbol dm_2 denotes the Lebesgue measure on the disc normalized so that $m_2(\mathbb{D})=1$. If T is a bounded operator mapping the space X to Y, i.e. $\|Tf\|_Y\le C\|f\|_X$, $f\in X$, then we write $T:X\longmapsto Y$.

For a function $f(z)=f(re^{i\theta})$ defined on $\mathbb{D}$, by $D^\alpha\equiv D_r^\alpha$ we denote the Riemann–Liouville integro-differential operator given by

$$D^{-\alpha}f(z)=\frac{r^\alpha}{\Gamma(\alpha)}\int_0^1(1-\eta)^{\alpha-1}f(\eta z)\,d\eta,$$

$$D^0f(z)=f(z),\quad D^\alpha f(z)=\left(\frac{\partial}{\partial r}\right)^k D^{-(k-\alpha)}f(z),$$

where $\alpha>0$ and k is an integer such that $k-1<\alpha\le k$. We also set

$$\mathcal{D}^{-\alpha}f(rw)=r^{-\alpha}D^{-\alpha}f(rw),\quad \mathcal{D}^\alpha f(rw)=D^\alpha\{r^\alpha f(rw)\}.$$

It is easy to see, that for harmonic f the functions $\mathcal{D}^\alpha f$ and $\mathcal{D}^{-\alpha}f$ also are harmonic, and the following inversion formulas are valid:

$$\mathcal{D}^\alpha\mathcal{D}^{-\alpha}f(z)=\mathcal{D}^{-\alpha}\mathcal{D}^\alpha f(z)=f(z). \tag{1}$$

We will consider Poisson type kernels P_α and conjugate kernels Q_α ([10, Ch. IX]):

$$P_\alpha(z) = \Gamma(\alpha+1)\left[\operatorname{Re}\frac{2}{(1-z)^{\alpha+1}} - 1\right], \quad z \in \mathbb{D}, \quad \alpha \geq 0,$$

$$Q_\alpha(z) = \Gamma(\alpha+1)\operatorname{Im}\frac{2}{(1-z)^{\alpha+1}}, \quad z \in \mathbb{D}, \quad \alpha \geq 0.$$

Notice that $P_0(z) = P(z)$ and $Q_0(z) = Q(z)$ are the usual Poisson and conjugate Poisson kernels. The kernels

$$P_\alpha(z,\zeta) = P_\alpha(z\bar{\zeta}), \quad Q_\alpha(z,\zeta) = Q_\alpha(z\bar{\zeta}), \quad z,\zeta \in \mathbb{D}$$

are harmonic with respect to both z and ζ. Clearly,

$$P_\alpha(z,\zeta) = P_\alpha(\zeta,z) = P_\alpha(\bar{z},\bar{\zeta}).$$

Also for $\alpha \geq 0$

$$P_0(z,\zeta) = \mathcal{D}^{-\alpha}P_\alpha(z,\zeta), \quad Q_0(z,\zeta) = \mathcal{D}^{-\alpha}Q_\alpha(z,\zeta),$$

$$P_\alpha(z,\zeta) = \mathcal{D}^{\alpha}P_0(z,\zeta), \quad Q_\alpha(z,\zeta) = \mathcal{D}^{\alpha}Q_0(z,\zeta).$$

Lemma 1. For any $\alpha \geq 0$ and p satisfying $\frac{1}{1+\alpha} < p \leq \infty$ the following estimates hold:

$$|P_\alpha(z,\zeta)| \leq C(\alpha)\frac{1}{|1-\bar{\zeta}z|^{\alpha+1}}, \quad z,\zeta \in \mathbb{D},$$

$$|Q_\alpha(z,\zeta)| \leq C(\alpha)\frac{1}{|1-\bar{\zeta}z|^{\alpha+1}}, \quad z,\zeta \in \mathbb{D},$$

$$M_p(P_\alpha;r) \leq C(\alpha,p)\frac{1}{(1-r)^{\alpha+1-1/p}}, \quad 0 \leq r < 1,$$

$$M_p(Q_\alpha;r) \leq C(\alpha,p)\frac{1}{(1-r)^{\alpha+1-1/p}}, \quad 0 \leq r < 1.$$

The proof is straightforward, and so we omit it.

The next lemma contains versions of the Hardy's well-known inequality [11].

Lemma 2. If $g(t) \geq 0$ ($0 \leq t < 1$), $1 \leq q < \infty$, $\beta < -1 < \alpha$, then

$$\int_0^1 (1-r)^\alpha \left(\int_0^r g(t)\,dt\right)^q dr \leq C(\alpha,q)\int_0^1 (1-r)^{\alpha+q} g^q(r)\,dr, \tag{2}$$

$$\int_0^1 x^\beta \left(\int_0^x g(t)\,dt\right)^q dx \leq C(\beta,q)\int_0^1 x^{\beta+q} g^q(x)\,dx. \tag{3}$$

The next lemma presents some continuous embeddings of Hardy–Littlewood and Flett type.

Lemma 3. For $\alpha, \alpha_0, \beta \in \mathbb{R}$, $0 < p, q \le \infty$ *the following continuous embeddings hold:*

(i) $h(p,q,\alpha) \subset h(p,q,\beta), \qquad \beta > \alpha,$

(ii) $h(p,q,\alpha) \subset h(p_0,q,\alpha), \qquad 0 < p_0 < p \le \infty,$

(iii) $h(p,q,\alpha) \subset h(p,q_0,\alpha), \qquad 0 < q < q_0 \le \infty,$

(iv) $h(p,q,\alpha) \subset h(p_0,q,\alpha_0), \qquad \alpha_0 \ge \alpha + 1/p - 1/p_0,\ 0 < p \le p_0 \le \infty,$

(v) $h(p,q,\alpha) \subset h(p_0,q_0,\beta), \qquad \beta > \alpha + 1/p,\ 0 < p_0, q_0 \le \infty,$

(vi) $h(p,q,\alpha) \subset h(p,q_0,\beta), \qquad \beta > \alpha,\ 0 < q_0 \le \infty.$

Besides, for $\alpha > 0$ *the condition* $\alpha_0 \ge \alpha + 1/p - 1/p_0$ *is necessary for embedding (iv).*

Proof: The first two embeddings (i) and (ii) are trivial; the embeddings (iii) and (iv) are proved in [7], the embeddings (v) and (vi) can be found in [12]. Now we prove the necessity of the condition $\alpha_0 \ge \alpha + 1/p - 1/p_0$ for the embedding (iv) when $\alpha > 0$. Let

$$\|u\|_{p_0,q,\alpha_0} \le C\|u\|_{p,q,\alpha}, \quad u \in h(p,q,\alpha),$$

where the constant C does not depend on the function u. Assume that the reverse inequality $\alpha_0 + 1/p_0 < \alpha + 1/p$ holds. For any $a \in \mathbb{D}$ and index $\gamma > \max\{\alpha_0 + 1/p_0, \alpha + 1/p\}$ we define the function $f_{\gamma,a}(z) = 1/(1 - \bar{a}z)^\gamma$. A simple calculation shows that

$$M_p(f_{\gamma,a}; r) \approx \frac{1}{(1 - |a|r)^{\gamma - 1/p}},$$

$$\|f_{\gamma,a}\|_{p,q,\alpha} \approx \left(\int_0^1 \frac{(1-r)^{\alpha q - 1}}{(1 - |a|r)^{\gamma q - q/p}}\, dr \right)^{1/q} \approx \frac{1}{(1 - |a|)^{\gamma - 1/p - \alpha}}.$$

Similarly

$$\|f_{\gamma,a}\|_{p_0,q,\alpha_0} \approx \frac{1}{(1 - |a|)^{\gamma - 1/p_0 - \alpha_0}}.$$

This implies

$$\frac{\|f_{\gamma,a}\|_{p_0,q,\alpha_0}}{\|f_{\gamma,a}\|_{p,q,\alpha}} \approx (1 - |a|)^{(\alpha_0 + 1/p_0) - (\alpha + 1/p)}.$$

It is easy to see, that the right-hand side becomes infinitely large as $|a| \to 1$, and so we get a contradiction with $h(p,q,\alpha) \subset h(p_0,q,\alpha_0)$. Lemma 3 is proved.

§3. BERGMAN TYPE PROJECTIONS

Area integral representations in the weighted Bergman spaces $H(p,p,\alpha)$ on the disc are well-known (see [3]–[6]). Below we give a short elementary proof for integral representations of harmonic functions from $h(p,q,\alpha)$ with mixed norm.

Theorem 1. Let $\alpha > 0$ *and* $u \in h(p,q,\alpha)$. *If either* $0 < p,q \le \infty$, $\beta > \max\{\alpha + 1/p - 1,\, \alpha\}$, *or* $1 \le p \le \infty$, $0 < q \le 1$, $\beta \ge \alpha$, *then*

$$u(z) = \frac{1}{\Gamma(\beta)} \iint\limits_{\mathbb{D}} (1 - |\zeta|^2)^{\beta-1}\, P_\beta(z,\zeta)\, u(\zeta)\, dm_2(\zeta), \qquad z \in \mathbb{D}, \tag{4}$$

$$v(z) = \frac{1}{\Gamma(\beta)} \iint\limits_{\mathbb{D}} (1 - |\zeta|^2)^{\beta-1}\, Q_\beta(z,\zeta)\, u(\zeta)\, dm_2(\zeta), \qquad z \in \mathbb{D}, \tag{5}$$

where $v(z)$ *is the harmonic conjugate of* $u(z)$ *normalized so that* $v(0) = 0$.

Proof: First consider the case where $p = q = 1$, $\beta = \alpha$ and $u(z)$ is an arbitrary function in $h(1,1,\alpha)$. In view of inversion formula (1)

$$u(z) = \mathcal{D}_r^{-\alpha} \mathcal{D}_r^{\alpha} u(z) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \mathcal{D}_r^{\alpha} u(\tau z)\, d\tau.$$

By changing the variable $\tau = \rho^2$ and applying Poisson formula, we get

$$u(z) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\rho^2)^{\alpha-1} \mathcal{D}_r^{\alpha} u(\rho^2 z)\, 2\rho\, d\rho =$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\rho^2)^{\alpha-1} \mathcal{D}_r^{\alpha} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} P_0(z, \rho e^{i\theta}) u(\rho e^{i\theta})\, d\theta \right\} 2\rho\, d\rho =$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^1 \int_{-\pi}^{\pi} (1-\rho^2)^{\alpha-1}\, \mathcal{D}_r^{\alpha} P_0(z, \rho e^{i\theta})\, u(\rho e^{i\theta})\, \rho\, d\rho\, \frac{d\theta}{\pi} =$$

$$= \frac{1}{\Gamma(\alpha)} \iint\limits_{\mathbb{D}} (1 - |\zeta|^2)^{\alpha-1}\, P_\alpha(z,\zeta)\, u(\zeta)\, dm_2(\zeta),$$

where the integral converges by Lemma 1. For other admissible p, q, β the proof follows from the embedding $h(p,q,\alpha) \subset h(1,1,\beta)$ in Lemma 3.

If $v(z)$ is the harmonic conjugate of $u(z)$ normalized by $v(0) = 0$, then we apply the conjugate Poisson formula instead of the Poisson formula. The rest of the proof repeats the above arguments. Theorem 1 is proved.

The representations (4) and (5) generate the following Bergman type linear integral operators:

$$T_\beta(f)(z) = \frac{1}{\Gamma(\beta)} \iint\limits_{\mathbb{D}} (1-|\zeta|^2)^{\beta-1}\, P_\beta(z,\zeta)\, f(\zeta)\, dm_2(\zeta),$$

$$\widetilde{T}_\beta(f)(z) = \frac{1}{\Gamma(\beta)} \iint\limits_{\mathbb{D}} (1-|\zeta|^2)^{\beta-1}\, Q_\beta(z,\zeta)\, f(\zeta)\, dm_2(\zeta),$$

The well-known Forelli–Rudin type theorems (see [5], [8], [9]) realize an analytic projection of the space $L(p,q,\alpha)$ into its holomorphic subspace. A similar question is of interest for other function spaces, in particular, for Besov spaces.

Definition. We say that a function $f(z)$ defined on the disc $\mathbb{D}$ *belongs to the Besov space* $\Lambda^{p,q}_\alpha$ $(0 < p, q \le \infty,\ \alpha \ge 0)$, if

$$\mathcal{D}^{\widetilde{\alpha}} f(z) \in L(p,q,\widetilde{\alpha}-\alpha),$$

where $\widetilde{\alpha}$ is the smallest integer greater than α, and $\mathcal{D}^\alpha$ is the Riemann–Liouville integro-differential operator. The Besov space $\Lambda^{p,q}_\alpha$ is equipped with the norm (quasinorm)

$$\|f\|_{\Lambda^{p,q}_\alpha} = \|\mathcal{D}^{\widetilde{\alpha}} f\|_{p,q,\widetilde{\alpha}-\alpha}.$$

Let $h\Lambda^{p,q}_\alpha$ be the subspace of $\Lambda^{p,q}_\alpha$ consisting of harmonic functions. For a function $f \in h\Lambda^{p,q}_\alpha$ the index $\widetilde{\alpha}$ can be replaced by any other index $\gamma > \alpha$ with an equivalent norm

$$\|f\|_{h\Lambda^{p,q}_\alpha} \approx \|\mathcal{D}^{\gamma} f\|_{p,q,\gamma-\alpha}.$$

Usually the space $h\Lambda^{p,q}_\alpha$ for $1 \le p, q \le \infty$ is defined in terms of boundary values $f(e^{i\theta})$ of a function $f(z)$ with an equivalent norm (see [11])

$$\|f\|_{L^p} + \left(\int_{-\pi}^{\pi} \frac{\|\Delta_t^k f(e^{i\theta})\|^q_{L^p(d\theta)}}{|t|^{1+\alpha q}}\, dt \right)^{1/q} < +\infty, \quad 1 \le q < \infty,$$

where

$$\Delta_t^1 f(e^{i\theta}) = f(e^{i(\theta+t)}) - f(e^{i\theta}), \quad \Delta_t^k f(e^{i\theta}) = \Delta_t^1 \Delta_t^{k-1} f(e^{i\theta}), \quad k \in \mathbb{Z}, \quad k > \alpha.$$

In the case $q = \infty$ the L_q-norm becomes into sup-norm.

The Lipschitz spaces are invariant under the Bergman projection, see [13] – [15]. Below we prove similar results for Besov spaces and the following Bergman type projection operators:

$$\Phi_{\widetilde{\alpha}}(f)(z) = \frac{1}{\Gamma(\widetilde{\alpha})} \iint\limits_{\mathbb{D}} (1-|\zeta|^2)^{\widetilde{\alpha}-1}\, P(z,\zeta)\, \mathcal{D}^{\widetilde{\alpha}} f(\zeta)\, dm_2(\zeta),$$

$$\widetilde{\Phi}_{\widetilde{\alpha}}(f)(z) = \frac{1}{\Gamma(\widetilde{\alpha})} \iint\limits_{\mathbb{D}} (1-|\zeta|^2)^{\widetilde{\alpha}-1}\, Q(z,\zeta)\, \mathcal{D}^{\widetilde{\alpha}} f(\zeta)\, dm_2(\zeta).$$

We need two additional lemmas.

Lemma 4. For $1 \le p \le \infty$, $0 < q \le \infty$ the embeddings

$$h\Lambda_{\alpha}^{p,q} \subset h(1,1,\beta), \quad \alpha \ge 0, \quad \beta > 0,$$

$$h\Lambda_{\alpha}^{p,q} \subset h\Lambda_{0}^{1,1}, \quad \alpha > 0$$

are continuous.

Proof: Let $u \in h\Lambda_{\alpha}^{p,q}$. We have $\mathcal{D}^{\gamma} u \in h(p,q,\gamma-\alpha)$ for any $\gamma > \alpha$. According to the embeddings (ii), (vi) of Lemma 3, we get $\mathcal{D}^{\gamma} u \in h(1,1,\beta+\gamma)$. Using the properties of fractional integro-differential operators in the spaces $h(p,q,\alpha)$ (see [7]), we obtain $u \in h(1,1,\beta)$, yielding the first embedding. To prove the second embedding, observe that $h\Lambda_{\alpha}^{p,q} \subset h\Lambda_{\beta}^{p,q}$ for any $0 < \beta < \alpha$, meaning that $\mathcal{D}^{\delta} u \in h(p,q,\delta-\beta)$ for any $\delta > \beta$. Again according to the embeddings (ii), (vi) and the properties of fractional integro-differential operator in the spaces $h(p,q,\alpha)$, we get $\mathcal{D}^{\delta} u \in h(1,1,\delta)$ for any $\delta > 0$, i.e. $u \in h\Lambda_{0}^{1,1}$. Lemma 4 is proved.

Corollary. The operator T_{β} is the identity map on $h\Lambda_{\alpha}^{p,q}$ for any $1 \le p \le \infty$, $0 < q \le \infty$, $\alpha \ge 0$, $\beta > 0$.

Lemma 5. For $1 \le p \le \infty$, $0 < q \le \infty$, $\alpha > 0$ and $\delta > 0$, any $u(z) \in h\Lambda_{\alpha}^{p,q}$ is representable in the form

$$u(z) = \frac{1}{\Gamma(\delta)} \iint\limits_{\mathbb{D}} (1-|\zeta|^2)^{\delta-1}\, P(z,\zeta)\, \mathcal{D}^{\delta} u(\zeta)\, dm_2(\zeta), \quad z \in \mathbb{D}. \tag{6}$$

Proof: According to the second embedding of Lemma 4, $\mathcal{D}^{\delta} u(z) \in h(1,1,\delta)$ for any $\delta > 0$. By Theorem 1 $\mathcal{D}^{\delta} u(z) = T_{\delta}(\mathcal{D}^{\delta} u)(z)$. The inversion formula (1) and integration using the operator $\mathcal{D}^{-\delta}$ complete the proof.

Theorem 2. For $1 \le p,q \le \infty$, $\alpha > 0$, $m = \widetilde{\alpha}$ *we have*

$$\Phi_m : \Lambda^{p,q}_\alpha \longmapsto h\Lambda^{p,q}_\alpha, \quad \widetilde{\Phi}_m : \Lambda^{p,q}_\alpha \longmapsto h\Lambda^{p,q}_\alpha, \tag{7}$$

and the operator Φ_m *continuously projects the Besov space* $\Lambda^{p,q}_\alpha$ *onto the whole subspace* $h\Lambda^{p,q}_\alpha$.

Proof: Given a function $f(z)$ (not harmonic in general) from $\Lambda^{p,q}_\alpha$, we need to prove the inequality

$$\|\Phi_m(f)\|_{h\Lambda^{p,q}_\alpha} \le C\|f\|_{\Lambda^{p,q}_\alpha}, \tag{8}$$

or

$$\|\mathcal{D}^\gamma \Phi_m(f)\|_{p,q,\gamma-\alpha} \le C\|\mathcal{D}^m f\|_{p,q,m-\alpha}, \tag{9}$$

where $m \in \mathbb{Z}_+$, $\alpha < m \le \alpha+1$, $\gamma > \alpha$. Differentiating $\Phi_m(f)(z)$ by means of the operator $\mathcal{D}^\gamma$ and using Lemma 1, we obtain

$$|\mathcal{D}^\gamma \Phi_m(f)(z)| \le C \iint\limits_{\mathbb{D}} (1-|\zeta|^2)^{m-1} |\mathcal{D}^\gamma P(z,\zeta)|\ |\mathcal{D}^m f(\zeta)|\, dm_2(\zeta) \le$$

$$\le C \iint\limits_{\mathbb{D}} \frac{(1-|\zeta|^2)^{m-1}}{|1-\bar{\zeta} r|^{\gamma+1}} |\mathcal{D}^m f(\zeta e^{i\theta})|\, dm_2(\zeta), \quad z = re^{i\theta}.$$

By Minkowski's inequality

$$M_p(\mathcal{D}^\gamma \Phi_m(f); r) \le C \iint\limits_{\mathbb{D}} \frac{(1-|\zeta|^2)^{m-1}}{|1-\bar{\zeta} r|^{\gamma+1}} M_p(\mathcal{D}^m f; \rho)\, d\rho \le C \int_0^1 \frac{(1-\rho)^{m-1}}{(1-\rho r)^\gamma} M_p(\mathcal{D}^m f; \rho)\, d\rho.$$

If $1 \le q < \infty$, then

$$M_p(\mathcal{D}^\gamma \Phi_m(f); r) \le C \left(\int_0^r + \int_r^1 \right) \frac{(1-\rho)^{m-1}}{(1-\rho r)^\gamma} M_p(\mathcal{D}^m f; \rho)\, d\rho.$$

By the triangle inequality

$$\|\mathcal{D}^\gamma \Phi_m(f)\|_{p,q,\gamma-\alpha} = \left\|(1-r)^{\gamma-\alpha} M_p(\mathcal{D}^\gamma \Phi_m(f); r)\right\|_{L^q(dr/(1-r))} \le C(I_1 + I_2),$$

where

$$I_1 = \left\| (1-r)^{\gamma-\alpha} \int_0^r (1-\rho)^{m-\gamma-1} M_p(\mathcal{D}^m f; \rho) d\rho \right\|_{L^q(dr/(1-r))},$$

$$I_2 = \left\| (1-r)^{-\alpha} \int_r^1 (1-\rho)^{m-1} M_p(\mathcal{D}^m f; \rho) d\rho \right\|_{L^q(dr/(1-r))}.$$

We estimate I_1 and I_2 separately, using Hardy's inequalities (2) and (3),

$$I_1^q \le C\int_0^1 (1-r)^{(\gamma-\alpha)q-1}\left(\frac{1-r}{(1-r)^{m-\gamma-1}}M_p(\mathcal{D}^m f;r)\right)^q dr = C\|\mathcal{D}^m f\|^q_{p,q,m-\alpha},$$

$$I_2^q \le C\int_0^1 (1-r)^{-\alpha q-1}\big[(1-r)^{m+1}M_p(\mathcal{D}^m f;r)\big]^q dr = C\|\mathcal{D}^m f\|^q_{p,q,-\alpha},$$

yielding the inequalities (8) and (9). If $q=\infty$, then

$$M_p(\mathcal{D}^\gamma\Phi_m(f);r) \le \frac{\|\mathcal{D}^m f\|_{p,\infty,m-\alpha}}{(1-\rho)^{m-\alpha}}.$$

Therefore

$$M_p(\mathcal{D}^\gamma\Phi_m(f);r) \le C\|\mathcal{D}^m f\|_{p,\infty,m-\alpha}\int_0^1 \frac{(1-\rho)^{\alpha-1}}{(1-\rho r)^\gamma}\,d\rho \le C\|\mathcal{D}^m f\|_{p,\infty,m-\alpha}\frac{1}{(1-r)^{\gamma-\alpha}}.$$

Thus,

$$\|\mathcal{D}^\gamma\Phi_m(f)\|_{p,\infty,\gamma-\alpha} \le C\|\mathcal{D}^m f\|_{p,\infty,m-\alpha},$$

and the result follows. To complete the proof it remains to observe that the surjectivity of the first mapping (7) follows immediately from Lemma 5.

Theorem 3. For $1\le p,q\le\infty$, $\alpha\ge 0$, $\beta>0$ *the operators*

$$T_\beta : L(p,q,-\alpha) \longmapsto h\Lambda^{p,q}_\alpha, \quad \widetilde{T}_\beta : L(p,q,-\alpha) \longmapsto h\Lambda^{p,q}_\alpha$$

are bounded.

Proof: It suffices to prove the theorem for the operator T_β. Given a function $\varphi(z)\in L(p,q,-\alpha)$, $1\le p,q\le\infty$, $\alpha\ge 0$, we need to prove that for any $\beta>0$

$$\|T_\beta(\varphi)\|_{h\Lambda^{p,q}_\alpha} \le C\|\varphi\|_{p,q,-\alpha}. \tag{10}$$

Setting $f(z)=T_\beta(\varphi)(z)$, for any $\gamma>\alpha$ the inequality (10) can be written in the form

$$\|\mathcal{D}^\gamma f\|_{p,q,\gamma-\alpha} \le C\|\varphi\|_{p,q,-\alpha}. \tag{11}$$

To prove (11), we differentiate the equality $f(z)=T_\beta(\varphi)(z)$ applying the operator $\mathcal{D}^\gamma$:

$$\mathcal{D}^\gamma f(z) = \frac{1}{\Gamma(\beta)}\iint\limits_{\mathbb{D}} (1-|\zeta|^2)^{\beta-1}\mathcal{D}^\gamma P_\beta(z,\zeta)\varphi(\zeta)\,dm_2(\zeta).$$

Let $p=q=1$. Then

$$\|\mathcal{D}^\gamma f\|_{1,1,\gamma-\alpha} \le C \iint\limits_{\mathbb{D}} (1-|z|^2)^{\gamma-\alpha-1} |\mathcal{D}^\gamma f(z)|\, dm_2(z) \le$$

$$\le C \iint\limits_{\mathbb{D}} (1-|z|^2)^{\gamma-\alpha-1} \left[\iint\limits_{\mathbb{D}} (1-|\zeta|^2)^{\beta-1} |\mathcal{D}^\gamma \mathcal{D}^\beta P(z,\zeta)|\ |\varphi(\zeta)|\, dm_2(\zeta) \right] dm_2(z) \le$$

$$\le C \iint\limits_{\mathbb{D}} (1-|\zeta|^2)^{\beta-1} |\varphi(\zeta)| \left[\iint\limits_{\mathbb{D}} (1-|z|^2)^{\gamma-\alpha-1} |\mathcal{D}^\gamma \mathcal{D}^\beta P(z,\zeta)|\ dm_2(z) \right] dm_2(\zeta).$$

Using Lemma 1, we obtain

$$\|\mathcal{D}^\gamma f\|_{1,1,\gamma-\alpha} \le C \iint\limits_{\mathbb{D}} (1-|\zeta|^2)^{\beta-1} |\varphi(\zeta)| \left[\iint\limits_{\mathbb{D}} \frac{(1-|z|^2)^{\gamma-\alpha-1}}{|1-\bar\zeta z|^{\gamma+\beta+1}}\, dm_2(z) \right] dm_2(\zeta) \le$$

$$\le C(\alpha,\beta,\gamma) \iint\limits_{\mathbb{D}} (1-|\zeta|^2)^{-\alpha-1} |\varphi(\zeta)|\, dm_2(\zeta).$$

Therefore,

$$\|\mathcal{D}^\gamma f\|_{1,1,\gamma-\alpha} \le C(\alpha,\beta,\gamma)\, \|\varphi\|_{1,1,-\alpha}. \tag{12}$$

Let now $p=q=\infty$. Then

$$|\mathcal{D}^\gamma f(z)| \le \frac{1}{\Gamma(\beta)} \iint\limits_{\mathbb{D}} (1-|\zeta|^2)^{\alpha+\beta-1} |\mathcal{D}^\gamma P_\beta(z,\zeta)|\, (1-|\zeta|^2)^{-\alpha} |\varphi(\zeta)|\, dm_2(\zeta) \le$$

$$\le C(\alpha,\beta)\, \|\varphi\|_{\infty,\infty,-\alpha} \iint\limits_{\mathbb{D}} (1-|\zeta|^2)^{\alpha+\beta-1} |\mathcal{D}^\gamma P_\beta(z,\zeta)|\, dm_2(\zeta) \le$$

$$\le C(\alpha,\beta,\gamma)\, \|\varphi\|_{\infty,\infty,-\alpha} \iint\limits_{\mathbb{D}} \frac{(1-|\zeta|^2)^{\alpha+\beta-1}}{|1-\bar\zeta z|^{\gamma+\beta+1}}\, dm_2(\zeta) \le C(\alpha,\beta,\gamma)\, \|\varphi\|_{\infty,\infty,-\alpha}\, \frac{1}{(1-|z|)^{\gamma-\alpha}}.$$

Therefore,

$$\|\mathcal{D}^\gamma f\|_{\infty,\infty,\gamma-\alpha} \le C(\alpha,\beta,\gamma)\, \|\varphi\|_{\infty,\infty,-\alpha}. \tag{13}$$

By a version of Riesz-Thorin interpolation theorem [16] the inequalities (12) and (13) imply (11) for all $1 \le p,q \le \infty$. This completes the proof of Theorem 3.

Note that for $p=q=\infty$, $\alpha=0$ Theorem 3 asserts, in particular, the boundedness of the operator T_β from $L^\infty(\mathbb{D})$ into the Bloch space $\mathcal{B}h = h\Lambda_0^{\infty,\infty}$ of harmonic functions. This fact is well known for holomorphic functions, see [5], [17].

R E F E R E N C E S

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