MATHEMATICAL NOTES, VOL. 75, NO. 4, 2004, PP. 453–461.
Translated from Matematicheskie Zametki, vol. 75, no. 4, 2004, pp. 483–492.

Inequalities of Littlewood–Paley Type for n-Harmonic Functions on the Polydisk

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Received February 25, 2002

Abstract—For n-harmonic functions on the unit polydisk in the space $\mathbb{C}^n$ we define g-functions of Littlewood–Paley type and establish L^p-inequalities related to them. In the present paper, the main theorems deal with the extension of results of Littlewood, Paley, and Flett to the polydisk and their generalization to fractional derivatives of arbitrary order. This gives an answer to a question posed by Littlewood.

KEY WORDS: *Littlewood–Paley inequalities, Littlewood–Paley g function, n-harmonic functions in the polydisk, fractional derivative of arbitrary order, interpolation of operators.*

1. INTRODUCTION

Suppose that

$$U^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| < 1,\ 1 \le j \le n\}$$

is the unit polydisk in the space $\mathbb{C}^n$ and

$$T^n = \{w = (w_1, \dots, w_n) \in \mathbb{C}^n : |w_j| = 1,\ 1 \le j \le n\}$$

is the n-dimensional torus (the frame of the polydisk). We consider *n-harmonic* functions on the polydisk U^n, i.e., functions harmonic in each variable z_j. In their studies of the theory of Fourier series carried out in the 1930s, Littlewood and Paley [1] introduced the so-called *g-function*, which is now named after them:

$$g(f)(\theta) = \left(\int_0^1 (1-r)|f'(re^{i\theta})|^2\,dr\right)^{1/2}, \qquad \theta \in (-\pi, \pi), \tag{1}$$

where $f(z)$ is a holomorphic function on the unit disk U^1. One of the main results due to Littlewood and Paley in connection with the g-function is the equivalence of the norms $\|g(f)\|_{L^p}$ and $\|f\|_{L^p}$ on the unit circle for $p > 1$ (see [1; 2, Chap. 14]). A similar result in the upper half-space of the space $\mathbb{R}^{n+1}$ was established by Stein [3, Chap. 4]. A number of researchers, notably, Flett [4], considerably extended the definition of the g-function on the unit disk by using the fractional derivative and applied it to theorems on multipliers. Littlewood [5, Problem 28, p. 43] conjectured the validity of L^p-inequalities for the g-function in the case of two complex variables and expressed the desire to avoid using "plane" complex methods.

In the present paper, fractional derivatives D^α (α is an arbitrary positive multi-index) are used to define the g-functions of Littlewood–Paley type and to establish L^p-inequalities, related to them, for n-harmonic functions on the polydisk. Although the operator interpolation method applied in the proof of Theorem 1 is well known and was used by Zygmund [2]; nevertheless, in contrast to [1, 2, 4], our proofs are not based on complex methods and, therefore, are suitable for the extension to other domains in the spaces $\mathbb{R}^n$ and $\mathbb{C}^n$. On the other hand, the extension of real-variable methods [3] (such, as Green's formula and Hilbert space methods) to fractional derivatives of arbitrary order and the parameter values $1 < p < \infty$ $0 < q \le \infty$ is highly problematic.

2. NOTATION AND STATEMENT OF THE MAIN THEOREMS

Suppose that

$$I^n = [0,1)^n, \qquad \zeta \in \mathbb{C}^n, \qquad r \in I^n, \qquad dr = dr_1 \cdots dr_n, \qquad r\zeta = (r_1\zeta_1, \dots, r_n\zeta_n).$$

By $\mathbb{Z}_+^n$ we denote the set of all multi-indices $m = (m_1, \dots, m_n)$ with nonnegative integer coordinates $m_j \in \mathbb{Z}_+$. Also assuming that $q \in \mathbb{R}$ $\alpha = (\alpha_1, \dots, \alpha_n)$, we set

$$(1-r)^\alpha = \prod_{j=1}^n (1-r_j)^{\alpha_j}, \qquad r^\alpha = \prod_{j=1}^n r_j^{\alpha_j}, \qquad \Gamma(\alpha) = \prod_{j=1}^n \Gamma(\alpha_j),$$

$$\left(\frac{\partial}{\partial r}\right)^m = \left(\frac{\partial}{\partial r_1}\right)^{m_1} \cdots \left(\frac{\partial}{\partial r_n}\right)^{m_n}, \qquad \alpha q + 1 = (\alpha_1 q + 1, \dots, \alpha_n q + 1).$$

In what follows, the product $\prod$ is extended from $j = 1$ to $j = n$ unless otherwise stated.

For a function $f(z) = f(rw)$, $r \in I^n$, $w \in T^n$, defined on U^n, consider the operator $D^\alpha \equiv D_r^\alpha$ of fractional Riemann–Liouville integro-differentiation with respect to the variable $r \in I^n$:

$$D^{-\alpha} f(z) = \frac{r^\alpha}{\Gamma(\alpha)} \int_{I^n} (1-\eta)^{\alpha-1} f(\eta z)\, d\eta,$$

$$D^\alpha f(z) = \left(\frac{\partial}{\partial r}\right)^m D^{-(m-\alpha)} f(z),$$

where $\alpha_j > 0$, $m \in \mathbb{Z}_+^n$, $m_j - 1 < \alpha_j \le m_j$, $1 \le j \le n$. We shall sometimes write $D^{(\alpha_1, \dots, \alpha_n)}$ instead of D^α. Some of the coordinates α_j of the multi-index α can be equal to zero; for example, by the operator $D^{(0, -\alpha_2, \dots, -\alpha_n)}$ we mean

$$D^{(0,-\alpha_2,\dots,-\alpha_n)} f(z) = \frac{\prod_{j=2}^n r_j^{\alpha_j}}{\prod_{j=2}^n \Gamma(\alpha_j)} \int_{I^{n-1}} \prod_{j=2}^n (1-\eta_j)^{\alpha_j - 1} f(\eta z)\, d\eta_2 \cdots d\eta_n.$$

The definition of the operator D^α implies that for any $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j \ge 0$, we have

$$D^{\pm\alpha} f = D_{r_1}^{\pm\alpha_1} D_{r_2}^{\pm\alpha_2} \cdots D_{r_n}^{\pm\alpha_n} f, \tag{2}$$

where $D_{r_j}^{\alpha_j}$ denotes the same operator only acting in the variable r_j. In particular, we have

$$D^1 f \equiv D^{(1,1,\dots,1)} f = D_{r_1}^1 D_{r_2}^1 \cdots D_{r_n}^1 f.$$

The symbols $C(\alpha, \beta, \dots)$, c_α, etc., will everywhere denote positive constants that may be different in different places and depend only on the parameters $\alpha, \beta, \dots$. For any p, $1 \le p \le \infty$, the number p' means the conjugate index, i.e., $p' = p/(p-1)$ (we assume $1/(+\infty) = 0$ and $1/0 = +\infty$). All inequalities $A \le B$ in the statements mean the following: if B is finite, then A is finite as well, with $A \le B$.

For a function $f(z)$ defined on U^n and the parameters $\alpha_j > 0$, $1 \le j \le n$, $0 < q \le \infty$, we define a *g-function of Littlewood–Paley type* (cf. [4]):

$$g_{q,\alpha}(f)(w) \equiv g_q(D^\alpha f)(w) \overset{\text{def}}{=} \begin{cases} \left(\displaystyle\int_{I^n} (1-r)^{\alpha q - 1} |D^\alpha f(rw)|^q \, dr\right)^{1/q}, & 0 < q < \infty, \\ \operatorname*{ess\,sup}_{r \in I^n} (1-r)^\alpha |D^\alpha f(rw)|, & q = \infty. \end{cases}$$

We can easily see that for $q = 2$ and $\alpha = (1, 1, \dots, 1)$ the function $g_{q,\alpha}(f)$ corresponds to the classical g-function (1).

The main results of this paper are the following two theorems.

Theorem 1. *Suppose that* $\alpha_j > 0$, $1 \le j \le n$, $1 < p < \infty$, $2 \le q < \infty$, *and* $u(z)$ *is the Poisson integral of a function* $f \in L^p(T^n)$. *Then the function*

$$g_q(r^\alpha D^\alpha u)(w) \overset{\text{def}}{=} \left(\int_{I^n} (1-r)^{\alpha q-1} |r^\alpha D^\alpha u(rw)|^q \, dr\right)^{1/q}, \qquad w \in T^n, \tag{3}$$

satisfies the inequality

$$\|g_q(r^\alpha D^\alpha u)\|_{L^p} \le C(p,q,\alpha,n)\|f\|_{L^p}. \tag{3$'$}$$

If, furthermore, the α_j *are integers or* $0 < \alpha_j < 1$, $1 \le j \le n$, *then the stronger inequality holds*:

$$\|g_q(D^\alpha u)\|_{L^p} \equiv \|g_{q,\alpha}(u)\|_{L^p} \le C(p,q,\alpha,n)\|f\|_{L^p}. \tag{3$''$}$$

For arbitrary $\alpha_j > 0$, *inequality* $(3'')$ *is valid under the additional assumption*

$$D^{(k_1,\dots,k_n)} u(rw) = 0 \tag{4}$$

for all $r = (r_1,\dots,r_n)$, $\prod r_j = 0$, $k_j \in \mathbb{Z}_+$, *and* $0 \le k_j \le [\alpha_j] - 1$.

Theorem 2. *Suppose that* $\alpha_j > 0$, $1 \le j \le n$, $1 < p < \infty$, $0 < q \le 2$,*and* $u(z)$ *is an* n*-harmonic function on* U^n *satisfying condition* (4) *and* $g_{q,\alpha}(u) \in L^p(T^n)$. *Then* $u(z)$ *is the Poisson integral of its boundary function* $f \in L^p(T^n)$, *with*

$$\|f\|_{L^p} \le C(p,q,\alpha,n)\|g_{q,\alpha}(u)\|_{L^p}.$$

In the case $0 < \alpha_j < 1$, $1 \le j \le n$, *assumption* (4) *becomes redundant.*

Remark. For $n = 2$, $q = 2$, and $\alpha = (1,1)$ Theorems 1 and 2 yield a positive answer to Littlewood's question [5, pp. 39, 43].

3. AUXILIARY ASSERTIONS

The Poisson kernel in the polydisk U^n is given by

$$P(z,\zeta) = \prod P(z_j,\zeta_j) = \prod \frac{1-|z_j|^2}{|\zeta_j - z_j|^2}, \qquad z \in U^n, \quad \zeta \in T^n.$$

Lemma 1. *If* $\alpha_j \ge 0$, $1 \le j \le n$, *then*

$$|r^\alpha D^\alpha P(z,\zeta)| \le C(\alpha,n) \prod \frac{1}{|\zeta_j - z_j|^{\alpha_j+1}}, \qquad z \in U^n, \quad \zeta \in T^n.$$

But if, furthermore, the α_j *are integers, then the factor* r^α *on the left-hand side can be dropped.*

Proof. The proof of this estimate can be obtained directly. □

Lemma 2. *Suppose that* $f(z)$ *is an* n*-harmonic function on* U^n *and* $0 < p, q \le \infty$, $\alpha_j > 0$, $1 \le j \le n$. *Then*

$$|r^\alpha D^\alpha f(z)| \le C(p,q,\alpha,n)\|g_{q,\alpha}(f)\|_{L^p} \prod \frac{1}{(1-|z_j|)^{\alpha_j+1/p}}, \qquad z \in U^n.$$

Proof. The proof is omitted, since it is fairly standard and can be reduced to Hölder's inequality and the n-subharmonic behavior of the function $|r^\alpha D^\alpha f|^p$, $p>0$. $\square$

The following lemma, in particular, shows that the function $g_{q,\alpha}(f)$ is "essentially" increasing in α.

Lemma 3. *Suppose that $f(z)$ is an n-harmonic function on U^n, $1\le q<\infty$, $\alpha_j,\beta_j>0$, $1\le j\le n$, and $D^{-\alpha}D^{\alpha+\beta}f=D^\beta f$. Then*

$$g_{q,\beta}(f)(w)\le C(\alpha,\beta,q,n)\,g_{q,\beta+\alpha}(f)(w),\qquad w\in T^n.$$

Proof. The proof is obtained by the n-fold repetition of the one-dimensional version of this inequality (see [4]). Note that if $0<\alpha_j+\beta_j<1$, then the identity $D^{-\alpha}D^{\alpha+\beta}f=D^\beta f$ necessarily holds. $\square$

Further, for a fixed δ, $0<\delta<1$, and $\zeta=e^{i\theta}\in T^1$ we consider the standard domain $\Gamma_\delta(\zeta)\equiv\Gamma_\delta(\theta)$ in the unit disk U^1 bounded by two tangents to the circle $|z|=\delta$ issuing from the point $\zeta=e^{i\theta}$ and by the largest arc of the circle $|z|=\delta$. For fixed δ_j, $0<\delta_j<1$, $1\le j\le n$, and $\zeta=(\zeta_1,\dots,\zeta_n)\in T^n$, we define $\Gamma_\delta(\zeta)$ as $\Gamma_\delta(\zeta)=\Gamma_{\delta_1}(\zeta_1)\times\dots\times\Gamma_{\delta_n}(\zeta_n)$.

Lemma 4. *Suppose that $\alpha_j>0$, $\delta_j>0$, $1\le j\le n$, and $f(z)$ is an n-harmonic function on U^n. Then its nontangential maximal function*

$$f^*_\delta(\zeta)=\sup\bigl\{|f(z)|:z\in\Gamma_\delta(\zeta)\bigr\}$$

satisfies the estimate

$$|r^\alpha D^\alpha f(rw)|\le C(\alpha,\delta,n)\,\frac{f^*_\delta(w)}{(1-r)^\alpha},\qquad z=rw\in U^n. \tag{5}$$

But if, furthermore, the α_j are integers, then the factor r^α on the left-hand side of (5) *can be dropped.*

Proof. Choose a point $z=rw\in U^n$. Let $B=B_z$ be the polydisk centered at this point with radius $(\delta_1(1-r_1)/2,\dots,\delta_n(1-r_n)/2)$, and note that $B\subset\Gamma_\delta(w)$. We write the Poisson representation of the function f on B:

$$f(z)=\int_{T_B}P_B(z,\zeta)f(\zeta)\,dm_n(\zeta),$$

where P_B is the Poisson kernel in the polydisk B, T_B is the frame of the polydisk B, and m_n is the surface Lebesgue measure on T_B. Differentiating by means of the operator D^α and using the estimates of the Poisson kernel, we obtain the required inequality (5). $\square$

4. THE PROOF OF THE MAIN THEOREMS

For an n-harmonic function $u(z)$ on U^n, using the domain $\Gamma_\delta(\zeta)$, we define the following version of the Luzin area integral:

$$S_\delta(u)(\zeta)=\biggl(\int_{\Gamma_\delta(\zeta)}|D^1u(z)|^2\,dm_{2n}(z)\biggr)^{1/2},\qquad \zeta\in T^n,$$

where m_{2n} is the Lebesgue measure on the polydisk U^n. We need the following lemma proved by Marcinkiewicz and Zygmund [2, p. 315 of the Russian translation] in the unit disk for $k=1$ and the classical g-function (1).

Lemma 5. *Suppose that $u(z)$ is an n-harmonic function on U^n, $k=(k_1,\dots,k_n)\in\mathbb{Z}^n_+$, $k_j\ge 1$, $0<\delta_j<1$, $1\le j\le n$. Then*

$$g_{2,k}(u)(\zeta)\le C(n,k,\delta)\,S_\delta(u)(\zeta),\qquad \zeta\in T^n. \tag{6}$$

Proof. First, we prove the lemma for $n=1$, i.e., for the unit disk U^1. Choose a point $z=re^{i\theta}\in U^1$ and consider the disk

$$B=\left\{\xi\in\mathbb{C}:|\xi-z|<\frac{\delta(1-r)}{2}\right\}\subset\Gamma_\delta(e^{i\theta})\equiv\Gamma_\delta(\theta).$$

From the integral Poisson formula in the disk B, it is easy to obtain the inequality

$$|D^k u(re^{i\theta})|^2\le\frac{C(k)}{|B|(1-r)^{2(k-1)}}\iint_B|D^1u(\rho e^{it})|^2\rho\,d\rho\,dt,$$

where $|B|$ is the area of the disk B. For $\rho e^{it}\in B$, we have $C'_\delta(1-r)<1-\rho<C''_\delta(1-r)$ and, therefore,

$$(1-r)^{2k-1}|D^ku(re^{i\theta})|^2\le C(k,\delta)\iint_B|D^1u(\rho e^{it})|^2\frac{\rho\,d\rho\,dt}{1-\rho}. \tag{7}$$

Now consider two cases. First, let $0\le r\le\delta/(2+\delta)$. Then we expand the domain of integration in (7) up to the disk

$$E\equiv E(r,\delta)=\left\{\xi\in\mathbb{C}:|\xi|<r_2\equiv r+\frac{\delta(1-r)}{2}\right\}.$$

We can easily see that $E(r,\delta)\subset\Gamma_\delta(\theta)$ and, therefore,

$$(1-r)^{2k-1}|D^ku(re^{i\theta})|^2\le C(k,\delta)\iint_{\Gamma_\delta(\theta)}\mathcal{X}_{(0,r_2)}(\rho)|D^1u(\rho e^{it})|^2\frac{\rho\,d\rho\,dt}{1-\rho},$$

where $\mathcal{X}$ denotes the characteristic function of the interval indicated in the subscript. Integrating over r and estimating of the internal integral, we obtain

$$\begin{aligned}\int_0^1&(1-r)^{2k-1}|D^ku(re^{i\theta})|^2\,dr\\&\le C(k,\delta)\iint_{\Gamma_\delta(\theta)}\left(\int_0^1\mathcal{X}_{(0,r_2)}(\rho)\,dr\right)|D^1u(\rho e^{it})|^2\frac{\rho\,d\rho\,dt}{1-\rho}\\&\le C(k,\delta)\iint_{\Gamma_\delta(\theta)}|D^1u(\rho e^{it})|^2\rho\,d\rho\,dt.\end{aligned}$$

Now, let $\delta/(2+\delta)<r<1$. Then we expand the domain of integration in (7) up to the annular sector whose sides touch the disk B:

$$(1-r)^{2k-1}|D^ku(re^{i\theta})|^2\le C(k,\delta)\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2}|D^1u(\rho e^{it})|^2\frac{\rho\,d\rho\,dt}{1-\rho}, \tag{8}$$

where we denote

$$r_{1,2}=r\mp\frac{\delta(1-r)}{2},\qquad\theta_{1,2}=\theta\mp\arcsin\frac{\delta(1-r)}{2r}.$$

Integration of inequality (8) over r leads to the estimate

$$\int_0^1 (1-r)^{2k-1}|D^k u(re^{i\theta})|^2 dr \le C(k,\delta)\int_0^1\int_{-\pi}^{\pi}\left(\int_0^1 \mathcal{X}_{(r_1,r_2)}(\rho)\mathcal{X}_{(\theta_1,\theta_2)}(t)\,dr\right)|D^1u(\rho e^{it})|^2\frac{\rho\,d\rho\,dt}{1-\rho}. \tag{9}$$

It remains to estimate the internal integral. We have

$$\mathcal{X}_{(r_1,r_2)}(\rho)\,\mathcal{X}_{(\theta_1,\theta_2)}(t)=\begin{cases}1 & \text{if } r_1<\rho<r_2,\ \theta_1<t<\theta_2,\\ 0 & \text{otherwise.}\end{cases}$$

The condition $r_1<\rho<r_2$ is equivalent to the condition $\rho_1<r<\rho_2$, where $\rho_{1,2}=(\rho\mp\delta/2)/(1\mp\delta/2)$. Moreover, $\theta_1<t<\theta_2$ is equivalent to the condition

$$r<r_0\equiv\frac{1}{1+2/(\delta\sin|t-\theta|)}.$$

Therefore,

$$\int_0^1 \mathcal{X}_{(r_1,r_2)}(\rho)\,\mathcal{X}_{(\theta_1,\theta_2)}(t)\,dr=\int_{\rho_1}^{\rho_2}\mathcal{X}_{(0,r_0)}(r)\,dr\le\begin{cases}\rho_2-\rho_1 & \text{if } \rho_1<r_0<1,\\ 0 & \text{if } 0<r_0\le\rho_1.\end{cases}$$

We define the set $G_\delta=\{\xi=\rho e^{it}\in U^1:\rho_1<r_0\}$ and calculate $\rho_2-\rho_1=C_\delta(1-\rho)$. After the substitution in (9), we obtain

$$\int_0^1(1-r)^{2k-1}|D^ku(re^{i\theta})|^2\,dr\le C(k,\delta)\iint_{G_\delta}|D^1u(\rho e^{it})|^2\rho\,d\rho\,dt. \tag{10}$$

Further, it remains to establish the embedding $G_\delta\subset\Gamma_\delta(\theta)$. Suppose that $\xi=\rho e^{it}\in G_\delta$ and $\delta\le\rho<1$. The set $\Gamma_\delta(\theta)\setminus\{|\xi|<\delta\}$ is characterized by three conditions:

$$\delta\le\rho<1,\qquad |t|=|\arg\xi|<\arccos\delta,\qquad |\arg(1-\xi)|<\arcsin\delta. \tag{11}$$

At the same time, the set G_δ is defined by the condition $\rho_1<r_0$ or

$$\sin|t|<\frac{\delta}{2}\,\frac{1-\rho}{\rho-\delta/2}.$$

The following two estimates are valid:

$$\sin|t|<\frac{\delta}{2}\,\frac{1-\rho}{\rho-\delta/2}\le 1-\delta<\sqrt{1-\delta^2},$$
$$\sin|t|<\frac{\delta}{2}\,\frac{1-\rho}{\rho-\delta/2}\le\frac{\delta}{\rho}\sqrt{1-2\rho\cos t+\rho^2};$$

they yield two last conditions in (11), which proves the embedding $G_\delta\subset\Gamma_\delta(\theta)$. Thus, from (10) we obtain the required inequality

$$\int_0^1(1-r)^{2k-1}|D^ku(re^{i\theta})|^2\,dr\le C(k,\delta)\iint_{\Gamma_\delta(\theta)}|D^1u(\xi)|^2\,dm_2(\xi). \tag{12}$$

The case $n=1$ is proved. The general case $n>1$ can be obtained by the n-fold application of (12) with the use of the expansion (2). Lemma 5 is proved. □

Proof of Theorem 1. First, we prove the theorem for integer values of $\alpha_j \geq 1$. The differentiation of the function u using the operator D^α and the estimate using Lemma 1 yields

$$|D^\alpha u(z)| \leq \int_{T^n} |D^\alpha P(z,\zeta)|\,|f(\zeta)|\,dm_n(\zeta) \leq \frac{C(\alpha,n)}{(1-|z|)^\alpha}\int_{T^n} P(z,\zeta)\,|f(\zeta)|\,dm_n(\zeta).$$

Hence we obtain the following pointwise estimate in terms of the nontangential maximal function u_δ^*:

$$g_{\infty,\alpha}(u)(w) \leq C(\alpha,n)\sup_{r\in I^n}\int_{T^n} P(rw,\zeta)\,|f(\zeta)|\,dm_n(\zeta) \leq C(\alpha,n)\,u_\delta^*(w), \qquad w\in T^n,$$

where the $0<\delta_j<1$, $1\leq j\leq n$, are fixed. Using the Zygmund maximal Theorem [6]

$$\|u_\delta^*\|_{L^p} \leq C\|f\|_{L^p}, \tag{13}$$

we find that

$$\|g_{\infty,\alpha}(u)\|_{L^p} \leq C\|f\|_{L^p}. \tag{14}$$

On the other hand, successively applying Lemma 5, the Gundy–Stein Theorem [7] on the equivalence of the L^p-norms of the functions $S_\delta(u)$ and u_δ^*, and then again inequality (13), we obtain the estimate

$$\|g_{2,\alpha}(u)\|_{L^p} \leq C\|S_\delta(u)\|_{L^p} \leq C\|u_\delta^*\|_{L^p} \leq C\|f\|_{L^p}. \tag{15}$$

By one version of the Riesz–Torin interpolation theorem for spaces with mixed norm (see [8, p. 316]) inequalities (14) and (15) imply (3″) for all q, $2\leq q\leq\infty$, and integers $\alpha_j\geq 1$.

In the general case $m_j-1<\alpha_j\leq m_j$, $m_j\in\mathbb{Z}_+$, $1\leq j\leq n$, we prove the theorem, first, for $n=1$. The derivative D^α can be expressed as (see, for example, [9, p. 52])

$$D^\alpha u = D^{-(m-\alpha)}D^m u + \sum_{k=1}^m D^{m-k}u(0)\,\frac{r^{m-\alpha-k}}{\Gamma(1+m-\alpha-k)}, \tag{16}$$

$$|r^\alpha D^\alpha u| \leq D^{-(m-\alpha)}|D^m u| + C_\alpha\sum_{k=1}^m |D^{m-k}u(0)|; \tag{17}$$

in view of Lemma 3, this allows us to reduce the estimate to the case of integer values of α examined above. The qth power integration of inequality (17) with measure $(1-r)^{\alpha q-1}dr$ on $(0,1)$, and then the pth power integration on the circle T^1 yields (3′). For $0<\alpha<1$, we obtain the stronger inequality (3″), since there is no divergence of the integral at zero and we can estimate (16) without multiplication by r^α.

For $n=2$, the expression (16) takes the form

$$\begin{aligned} D^{(\alpha_1,\alpha_2)}u(z_1,z_2) &= D^{(-(m_1-\alpha_1),-(m_2-\alpha_2))}D^{(m_1,m_2)}u(z_1,z_2)\\ &\quad+\sum_{k_1=1}^{m_1} D_{r_1}^{m_1-k_1}D_{r_2}^{\alpha_2}u(0,z_2)\,\frac{r_1^{m_1-\alpha_1-k_1}}{\Gamma(1+m_1-\alpha_1-k_1)}\\ &\quad+\sum_{k_2=1}^{m_2} D_{r_1}^{\alpha_1}D_{r_2}^{m_2-k_2}u(z_1,0)\,\frac{r_2^{m_2-\alpha_2-k_2}}{\Gamma(1+m_2-\alpha_2-k_2)}\\ &\quad-\sum_{k_1=1}^{m_1}\sum_{k_2=1}^{m_2} D^{(m_1-k_1,m_2-k_2)}u(0,0)\,\frac{r^{m-\alpha-k}}{\Gamma(1+m-\alpha-k)}. \end{aligned}$$

Using similar arguments and the estimate in the one-dimensional case, we obtain (3′) and (3″). Obviously, the procedure can be extended to all $n\geq 1$. □

Proof of Theorem 2. In view of Lemma 3 and conditions (4), it suffices to prove the theorem only for $0 < \alpha_j < 1$. First, let $1 < q \le 2$. For an arbitrary fixed $r \in I^n$, consider the linear functional on $L^{p'}(T^n)$ generated by the function $u(z)$:

$$F_u(v) = \int_{T^n} u(rw)v(w)\,dm_n(w), \qquad v(w) \in L^{p'}(T^n).$$

Assuming that $v(rw)$ is the Poisson integral of the function $v(w)$ and $\gamma = (\gamma_1, \dots, \gamma_n)$ is a small positive multi-index such that $0 < \alpha_j + \gamma_j < 1$ and using the identity $r^\alpha D_r^\alpha = \eta^\alpha D_\eta^\alpha$, we see that

$$\begin{aligned} F_u(v) &= \int_{T^n} D_r^{-\alpha-\gamma}\, D_r^{\alpha+\gamma} u(rw)v(w)\,dm_n(w) \\ &= \frac{r^\gamma}{\prod \Gamma(\alpha_j+\gamma_j)} \int_{I^n} (1-\eta)^{\alpha+\gamma-1}\eta^\alpha \left[\int_{T^n} v(w)\, D_\eta^\alpha\, D_r^\gamma u(\eta r w)\,dm_n(w)\right] d\eta. \end{aligned} \tag{18}$$

Let us transform the internal integral:

$$\begin{aligned} &\int_{T^n} v(w)\, D_\eta^\alpha\, D_r^\gamma u(\eta r w)\,dm_n(w) \\ &\quad= \int_{T^n} v(w)\, D_r^\gamma \left[\int_{T^n} P(\sqrt{\eta}\, rw, \zeta)\, D_\eta^\alpha u(\sqrt{\eta}\,\zeta)\,dm_n(\zeta)\right] dm_n(w) \\ &\quad= \int_{T^n} D_\eta^\alpha u(\sqrt{\eta}\,\zeta) \left[\int_{T^n} D_r^\gamma P(\sqrt{\eta}\, rw, \zeta)\, v(w)\,dm_n(w)\right] dm_n(\zeta) \\ &\quad= \int_{T^n} D_\eta^\alpha u(\sqrt{\eta}\,\zeta)\, D_r^\gamma v(\sqrt{\eta}\, r\zeta)\,dm_n(\zeta). \end{aligned}$$

Substituting the result into (18) and changing the order of integration, we obtain

$$F_u(v) = C(\alpha,\gamma,n)\, r^\gamma \int_{T^n} \left[\int_{I^n} \eta^\alpha (1-\eta)^{\alpha+\gamma-1} D_\eta^\alpha u(\sqrt{\eta}\,\zeta)\, D_r^\gamma v(\sqrt{\eta}\, r\zeta)\,d\eta\right] dm_n(\zeta).$$

Denoting

$$h_{q',\gamma}(r\zeta) = \left(\int_{I^n} (1-\eta)^{\gamma q'-1} \big|D_r^\gamma v(\sqrt{\eta}\, r\zeta)\big|^{q'}\,d\eta\right)^{1/q'}$$

and applying Hölder's inequality twice and then Theorem 1, we obtain

$$\begin{aligned} |F_u(v)| &\le C \int_{T^n} g_{q,\alpha}(u)(\zeta)\, h_{q',\gamma}(r\zeta)\,dm_n(\zeta) \\ &\le C \|g_{q,\alpha}(u)\|_{L^p}\, \|h_{q',\gamma}(r\zeta)\|_{L^{p'}(T^n)} \\ &\le C(p,q,\alpha,\gamma,n) \|g_{q,\alpha}(u)\|_{L^p}\, \|v\|_{L^{p'}(T^n)}. \end{aligned}$$

By the duality of $(L^{p'})^* = L^p$, we have

$$\|u(rw)\|_{L^p(T^n)} = \sup\big\{|F_u(v)| : \|v\|_{L^{p'}} = 1\big\} \le C \|g_{q,\alpha}(u)\|_{L^p}.$$

Now, let $0 < q < 1$ (for $q = 1$, the theorem is obvious). By Lemma 4, we have

$$\begin{aligned} |u(rw)| &\le \frac{1}{\prod \Gamma(\alpha_j)} \int_{I^n} (1-\eta)^{\alpha-1} |D^\alpha u(\eta r w)|\,d\eta \\ &\le C(\alpha,\delta,n) \big(u_\delta^*(w)\big)^{1-q} \int_{I^n} \frac{(1-\eta)^{\alpha-1}}{(1-\eta r)^{\alpha(1-q)}} |D^\alpha u(\eta r w)|^q\,d\eta \\ &\le C(\alpha,\delta,n) \big(u_\delta^*(w)\big)^{1-q} \int_{I^n} (1-\eta)^{\alpha q-1} |D^\alpha u(\eta r w)|^q\,d\eta, \end{aligned}$$

where $u^*_\delta(w)$ is the nontangential maximal function. Further, applying Hölder's inequality with the indices $1/q$ and $1/(1-q)$, we obtain

$$\|u(rw)\|^p_{L^p(T^n)} \le C\|u^*_\delta\|^{p(1-q)}_{L^p}\|g_{q,\alpha}(u)\|^{pq}_{L^p}.$$

Therefore, inequality (13) yields

$$\|f\|_{L^p} = \|u(rw)\|_{L^p(T^n)} \le C\|u^*_\delta\|^{1-q}_{L^p}\|g_{q,\alpha}(u)\|^q_{L^p} \le C\|f\|^{1-q}_{L^p}\|g_{q,\alpha}(u)\|^q_{L^p},$$

which concludes the proof of Theorem 2. □

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