GENERALIZED LITTLEWOOD PROBLEM

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A new family of Littlewood–Paley type g-functions is defined and the related L^p-inequalities are proved for n-harmonic and holomorphic functions on the unit polydisc of $\mathbb{C}^n$. The paper generalizes and improves the results of author's recent work, that gave a positive answer to Littlewood's question on extension of L^p-inequalities to the case of several complex variables.

§1. INTRODUCTION

We write $U^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| < 1, 1 \le j \le n\}$ for the unit polydisc in $\mathbb{C}^n$ and $T^n = \{w = (w_1, \dots, w_n) \in \mathbb{C}^n : |w_j| = 1,\ 1 \le j \le n\}$ for the distinguished boundary of U^n (i.e. an n-dimensional torus). In U^n we consider n-harmonic functions, i.e. functions harmonic in each variable z_j separately.

The g-function of Littlewood and Paley [1] is defined by

$$g(f)(\vartheta) = \left(\int_0^1 (1-r)|f'(re^{i\vartheta})|^2 dr \right)^{1/2}, \quad \vartheta \in (-\pi, \pi), \tag{1}$$

where $f(z)$ is a holomorphic function in the unit disc U^1. One of the first results on this function is the equivalence of norms $\|g(f)\|_{L^p}$ and $\|f\|_{L^p}$ on the unit circle for $p > 1$ (see [1] and [2], Chapter XIV). A similar result in the upper half-space of $\mathbb{R}^{n+1}$ is established by Stein [3] (Chapter IV).

Different authors, in particular Flett [4], extended the concept of a g-function in the unit disc using fractional derivatives and gave applications in some theorems on multipliers. Littlewood [5]

(p. 43, Problem 28) posed the conjecture about validity of L^p-estimates for the g-function in case of two complex variables and spoke about eschewing the "flat" complex methods.

In author's work [6], some Littlewood–Paley type g-functions have been defined and the related L^p-estimates for n-harmonic functions in the polydisc established by the use of Riemann–Liouville fractional derivatives D^α. This gave a positive answer to the mentioned Littlewood problem [5]. In [6] some L^p-estimates containing D^α derivatives were proved only for small or integer values of α. This diminished the applicability of the L^p-estimates of [6].

This paper constructs a new family of Littlewood–Paley type g-functions, using Hadamard's $\mathcal{F}^\alpha$ and the Riemann–Liouville $\mathcal{D}^\alpha$ fractional derivatives. The corresponding L^p-estimates for arbitrary values of the multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ are proved in the polydisc for n-harmonic functions with $p > 1$ and for holomorphic functions with $p > 0$. Some new applications of the L^p-estimates are given.

§2. NOTATION AND THE STATEMENTS OF MAIN THEOREMS

We set

$$I^n = [0,1)^n, \quad \zeta \in \mathbb{C}^n, \quad r \in I^n, \quad dr = dr_1 \cdots dr_n, \quad r\zeta = (r_1\zeta_1, \dots, r_n\zeta_n)$$

and by $\mathbb{Z}_+^n$ we denote the set of all multiindices $m = (m_1, \dots, m_n)$ possessing nonnegative integer coordinates $m_j \in \mathbb{Z}_+$. Besides, assuming that $q \in \mathbb{R}$, $\alpha = (\alpha_1, \dots, \alpha_n)$ we set

$$(1-r)^\alpha = \prod_{j=1}^n (1-r_j)^{\alpha_j}, \quad r^\alpha = \prod_{j=1}^n r_j^{\alpha_j}, \quad \Gamma(\alpha) = \prod_{j=1}^n \Gamma(\alpha_j),$$

$$\left(\frac{\partial}{\partial r}\right)^m = \left(\frac{\partial}{\partial r_1}\right)^{m_1} \cdots \left(\frac{\partial}{\partial r_n}\right)^{m_n}, \qquad \alpha q + 1 = (\alpha_1 q + 1, \dots, \alpha_n q + 1).$$

For a function $f(z) = f(rw)$ $(r \in I^n,\ w \in T^n)$ defined in U^n, by $\mathcal{F}^\alpha \equiv \mathcal{F}_r^\alpha$ we denote Hadamard's operator of fractional integro-differentiation by the variable $r \in I^n$:

$$\mathcal{F}^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_{I^n} \prod_{j=1}^n \left(\log \frac{1}{\eta_j}\right)^{\alpha_j - 1} f(\eta z)\, d\eta,$$

$$\mathcal{F}^m f(z) = \left(\frac{\partial}{\partial r} \cdot r\right)^m f(z), \qquad \mathcal{F}^\alpha f(z) = \mathcal{F}^{-(m-\alpha)} \mathcal{F}^m f(z),$$

where $\alpha_j > 0$, $m \in \mathbb{Z}_+^n$, $m_j - 1 < \alpha_j \leq m_j$ and $1 \leq j \leq n$. Note that the properties and some equivalent definitions of the one dimensional operator $\mathcal{F}^\alpha$ are given, for instance, in [7] and [4].

If a function $u(z)$ is n-harmonic (holomorphic), then the function $\mathcal{F}^\alpha u(z)$ ($\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{R}$) is of the same type; besides, for any $\alpha = (\alpha_1, \ldots, \alpha_n)$,

$$\mathcal{F}^\alpha f = \mathcal{F}_{r_1}^{\alpha_1} \mathcal{F}_{r_2}^{\alpha_2} \ldots \mathcal{F}_{r_n}^{\alpha_n} f, \tag{2}$$

where $\mathcal{F}_{r_j}^{\alpha_j}$ stands for the same operator applied by the variable r_j.

Henceforth, by $C(\alpha, \beta, \ldots), c_\alpha$ etc. we mean positive constants depending on the parameters α, β etc. For any p ($1 \leq p \leq \infty$) by p' we denote the conjugate index, i.e. $p' = p/(p-1)$, and all inequalities $A \leq B$ in the statements of theorems mean that if B is finite, then A is finite too and $A \leq B$.

For a function $f(z)$ defined in U^n and for any values of the parameters $\alpha_j > 0$ ($1 \leq j \leq n$) and $0 < q \leq \infty$, define the Littlewood–Paley g-function as follows (compare with [4], [6]):

$$\mathcal{G}_{q,\alpha}(f)(w) = \left(\int_{I^n} (1-r)^{\alpha q - 1} |\mathcal{F}^\alpha f(rw)|^q dr \right)^{1/q}, \quad 0 < q < \infty,$$

$$\mathcal{G}_{\infty,\alpha}(f)(w) = \operatorname*{ess\,sup}_{r \in I^n} \ (1-r)^\alpha |\mathcal{F}^\alpha f(rw)|.$$

For $q = 2$ and $\alpha = (1, 1, \ldots, 1)$ the function $\mathcal{G}_{q,\alpha}(f)$ corresponds to the classical g-function (1). The following theorems are the main result of the present article.

Theorem 1. Let $\alpha_j > 0$ ($1 \leq j \leq n$), $1 < p < \infty$ and $2 \leq q < \infty$ be arbitrary and $u(z)$ be the Poisson integral of some $f \in L^p(T^n)$. Then

$$\|\mathcal{G}_{q,\alpha}(u)\|_{L^p} \leq C(p, q, \alpha, n) \|f\|_{L^p}. \tag{3}$$

If $u(z)$ is holomorphic in U^n, then for any $p > 0$

$$\|\mathcal{G}_{q,\alpha}(u)\|_{L^p} \leq C(p, q, \alpha, n) \|u\|_{H^p}, \tag{4}$$

where H^p is Hardy's holomorphic class in the polydisc.

Theorem 2. Let $\alpha_j > 0$ $(1 \le j \le n)$, $1 < p < \infty$ and $0 < q \le 2$ be arbitrary and $u(z)$ be an n-harmonic function in U^n, such that $\mathcal{G}_{q,\alpha}(u) \in L^p(T^n)$. Then $u(z)$ is the Poisson integral of its boundary function $f \in L^p(T^n)$, and

$$\|f\|_{L^p} \le C(p,q,\alpha,n)\|\mathcal{G}_{q,\alpha}(u)\|_{L^p}.$$

We give also some analogs of Theorems 1 and 2, using the Riemann–Liouville fractional integro-differentiation:

$$D^{-\alpha} f(z) = \frac{r^\alpha}{\Gamma(\alpha)} \int_{I^n} (1-\eta)^{\alpha-1} f(\eta z) d\eta, \quad D^\alpha f(z) = D^{-(m-\alpha)} \left(\frac{\partial}{\partial r}\right)^m f(z),$$

$$\mathcal{D}^{-\alpha} f(z) = r^{-\alpha} D^{-\alpha} f(z), \quad \mathcal{D}^\alpha f(z) = D^\alpha\{r^\alpha f(z)\}, \quad z = rw \in U^n,$$

where $\alpha_j > 0$, $m \in \mathbb{Z}_+^n$, $m_j - 1 < \alpha_j \le m_j$, $1 \le j \le n$. Eventually, the corresponding Littlewood–Paley type g-function will take the form

$$g_{q,\alpha}(f)(w) = \left(\int_{I^n} (1-r)^{\alpha q-1} |\mathcal{D}^\alpha f(rw)|^q dr\right)^{1/q}, \quad 0 < q < \infty,$$

$$g_{\infty,\alpha}(f)(w) = \operatorname*{ess\,sup}_{r \in I^n} \ (1-r)^\alpha |\mathcal{D}^\alpha f(rw)|.$$

Theorem 3. Let $\alpha_j > 0$ $(1 \le j \le n)$, $1 < p < \infty$ and $2 \le q < \infty$ be arbitrary and $u(z)$ be the Poisson integral of a function $f \in L^p(T^n)$. Then

$$\|g_{q,\alpha}(u)\|_{L^p} \le C(p,q,\alpha,n)\|f\|_{L^p}.$$

If $u(z)$ is holomorphic in U^n, then for any $p > 0$

$$\|g_{q,\alpha}(u)\|_{L^p} \le C(p,q,\alpha,n)\|u\|_{H^p}.$$

Theorem 4. Let $\alpha_j > 0$ $(1 \le j \le n)$, $1 < p < \infty$ and $0 < q \le 2$ be arbitrary and $u(z)$ be an n-harmonic function in U^n, such that $g_{q,\alpha}(u) \in L^p(T^n)$. Then $u(z)$ is the Poisson integral of its boundary function $f \in L^p(T^n)$, and

$$\|f\|_{L^p} \le C(p,q,\alpha,n)\|g_{q,\alpha}(u)\|_{L^p}.$$

Note that for functions holomorphic in the unit ball of $\mathbb{C}^n$, the analogs of the above Theorems $1-4$ are given in [8]. In spite of similarity of Theorems 1-2 and 3-4, there are essential differences in their proofs (for Hadamard's operator a semigroup formula $\mathcal{F}^{\alpha+\beta}=\mathcal{F}^{\alpha}\mathcal{F}^{\beta}$ holds, that has no analog for the Riemann–Liouville operator $\mathcal{D}^{\alpha}$).

As applications of Theorems $1-4$, we give the following embedding theorems, where

$$M_p(f;r)=\left\|f(r\,\cdot)\right\|_{L^p(T^n;dm_n)},\qquad r=(r_1,\ldots,r_n)\in I^n,$$

and dm_n is the Lebesgue measure in T^n, while $\mathcal{J}^{\alpha}$ stands for $\mathcal{F}^{\alpha}$ as well as for $\mathcal{D}^{\alpha}$.

Theorem 5. Let $\alpha_j>0$ $(1\le j\le n)$, $1<p\le q\le\infty$, $2\le q\le\infty$ *and* $p\le p_0\le\infty$ *be arbitrary. Then*

$$\left(\int_{I^n}(1-r)^{\alpha q-1}M_p^q(\mathcal{J}^{\alpha}u;r)\,dr\right)^{1/q}\le C\|u\|_{h^p}, \tag{5}$$

$$\left(\int_{I^n}(1-r)^{(\alpha+1/p-1/p_0)q-1}M_{p_0}^q(\mathcal{J}^{\alpha}u;r)dr\right)^{1/q}\le C\|u\|_{h^p}. \tag{6}$$

If $u\in H^p$ *is a holomorphic function, then the inequalities (5) and (6) are true for all* $p>0$.

Theorem 6. Let $\alpha_j>0$ $(1\le j\le n)$, $1<p<\infty$, $0<q\le 2$ *and* $q\le p$ *be arbitrary. Then*

$$\|\mathcal{J}^{-\alpha}u\|_{h^p}\le C\left(\int_{I^n}(1-r)^{\alpha q-1}M_p^q(u;r)\,dr\right)^{1/q}.$$

§3. AUXILIARY STATEMENTS

Recall that the Poisson kernel for the polydisc U^n is given by the formula

$$P(z,\zeta)=\prod_{j=1}^n P(z_j,\zeta_j)=\prod_{j=1}^n\frac{1-|z_j|^2}{|\zeta_j-z_j|^2},\qquad z\in U^n,\quad \zeta\in T^n.$$

Lemma 1. If $\alpha_j\ge 0\,(1\le j\le n)$, *then*

$$|\mathcal{F}^{\alpha}P(z,\zeta)|\le C(\alpha,n)\prod_{j=1}^n\frac{1}{|\zeta_j-z_j|^{\alpha_j+1}},\qquad z\in U^n,\quad \zeta\in T^n.$$

Proof: by direct estimation.

Lemma 2. Let $f(z)$ be an n-harmonic function in U^n and $0 < p, q \leq \infty$, $\alpha_j > 0$ $(1 \leq j \leq n)$ be arbitrary. Then

$$|\mathcal{F}^\alpha f(z)| \leq C(p,q,\alpha,n)\|\mathcal{G}_{q,\alpha}(f)\|_{L^p} \prod_{j=1}^n \frac{1}{(1-|z_j|)^{\alpha_j+1/p}}, \quad z \in U^n.$$

Proof: is standard, based on Hölder's inequality and the n-subharmonic behaviour of $|\mathcal{F}^\alpha f|^p$ $(p > 0)$.

Lemma 3. Let $f(z)$ be n-harmonic in U^n and $1 \leq q < \infty$, α_j, $\beta_j > 0$ $(1 \leq j \leq n)$ be arbitrary. Then

$$\mathcal{G}_{q,\beta}(f)(w) \leq C(\alpha,\beta,q,n)\,\mathcal{G}_{q,\beta+\alpha}(f)(w), \quad w \in T^n.$$

Proof: by applying n times the one-dimensional version of the same inequality (see [4]).

For any fixed δ $(0 < \delta < 1)$ and $\zeta = e^{i\vartheta} \in T^1$ we consider the standard domain $\Gamma_\delta(\zeta) \equiv \Gamma_\delta(\vartheta)$ in the unit disc U^1, bounded by two tangents to the circle $|z| = \delta$, containing the point $\zeta = e^{i\vartheta}$, and the lergest arc of $|z| = \delta$. For any fixed δ_j, $0 < \delta_j < 1$ $\big(1 \leq j \leq n\big)$ and $\zeta = (\zeta_1, \ldots, \zeta_n) \in T^n$ we define $\Gamma_\delta(\zeta) = \Gamma_{\delta_1}(\zeta_1) \times \cdots \times \Gamma_{\delta_n}(\zeta_n)$.

Lemma 4. Let $\alpha_j > 0$, $\delta_j > 0$, $1 \leq j \leq n$ be arbitrary and $f(z)$ be an n-harmonic function in U^n. Then the non-tangential maximal function of $f(z)$, that is

$$f^*_\delta(\zeta) = \sup\big\{|f(z)|;\ z \in \Gamma_\delta(\zeta)\big\}, \quad \zeta \in T^n$$

admits the estimate

$$|\mathcal{F}^\alpha f(rw)| \leq C(\alpha,\delta,n)\frac{f^*_\delta(w)}{(1-r)^\alpha}, \quad z = rw \in U^n.$$

Proof: Denote by $B = B_z$ the polydisc centered at z, with the radius $\big(\delta_1(1-r_1)/2, \ldots, \delta_n(1-r_n)/2\big)$. Then for any point $z = rw \in U^n$ we have $B \subset \Gamma_\delta(w)$. We get the desired inequality differentiating the Poisson representation of f in B by means of $\mathcal{F}^\alpha$ and using the well-known estimates of the Poisson kernel.

§4. PROOFS OF MAIN THEOREMS

Assuming that $u(z)$ is n-harmonic in U^n, we introduce the following version of Lusin's area integral:

$$S_\delta(u)(\zeta) = \left(\int_{\Gamma_\delta(\zeta)} |\mathcal{F}^1 u(z)|^2 dm_{2n}(z)\right)^{1/2}, \quad \zeta \in T^n,$$

where m_{2n} is Lebesgue's mesure in U^n. The following lemma was proved by Marcinkiewicz and Zygmund [2] (Chapter XIV, Th. 3.1) in the particular case of the unit disc, for $k = 1$ and the classical g-function (1).

Lemma 5. Let $u(z)$ be an n-harmonic function in U^n, $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ $(k_j \geq 1)$ and $0 < \delta_j < 1$ $(1 \leq j \leq n)$. Then

$$\mathcal{G}_{2,k}(u)(\zeta) \leq C(n, k, \delta)\, S_\delta(u)(\zeta), \quad \zeta \in T^n.$$

Proof: First, we give a proof for the case $n = 1$, i.e. for the unit disc U^1. To this end, observe that if $z = re^{i\vartheta} \in U^1$ is fixed, then from the Poisson formula in the disc

$$B = \big\{\xi \in \mathbb{C} : |\xi - z| < \delta(1 - r)/2\big\} \subset \Gamma_\delta(e^{i\vartheta}) \equiv \Gamma_\delta(\vartheta)$$

it follows that

$$|\mathcal{F}^k u(re^{i\vartheta})|^2 \leq \frac{C(k)}{|B|(1 - r)^{2(k-1)}} \iint_B |\mathcal{F}^1 u(\rho e^{it})|^2 \rho \, d\rho \, dt,$$

where $|B|$ is the area of B. Clearly

$$C'_\delta(1 - r) < 1 - \rho < C''_\delta(1 - r), \quad \rho e^{it} \in B.$$

Consequently

$$(1 - r)^{2k-1} |\mathcal{F}^k u(re^{i\vartheta})|^2 \leq C(k, \delta) \iint_B |\mathcal{F}^1 u(\rho e^{it})|^2 \, \frac{\rho \, d\rho \, dt}{1 - \rho}. \tag{7}$$

Now if $0 \leq r \leq \delta/(2 + \delta)$, then we extend the integration domain in (7) to become

$$E \equiv E(r, \delta) = \{\xi \in \mathbb{C} : |\xi| < r_2 \equiv r + \delta(1 - r)/2\}$$

and observe that since $E(r, \delta) \subset \Gamma_\delta(\vartheta)$,

$$(1 - r)^{2k-1} |\mathcal{F}^k u(re^{i\vartheta})|^2 \leq C(k, \delta) \iint_{\Gamma_\delta(\vartheta)} \mathcal{X}_{(0, r_2)}(\rho) |\mathcal{F}^1 u(\rho e^{it})|^2 \, \frac{\rho \, d\rho \, dt}{1 - \rho},$$

where $\mathcal{X}_{(0, r_2)}$ stands for the characteristic function of the interval $(0, r_2)$. Integrating by r and estimating the inner integral, we get

$$\int_0^1 (1 - r)^{2k-1} |\mathcal{F}^k u(re^{i\vartheta})|^2 dr \leq C(k, \delta) \iint_{\Gamma_\delta(\vartheta)} \left(\int_0^1 \mathcal{X}_{(0, r_2)}(\rho) \, dr \right) |\mathcal{F}^1 u(\rho e^{it})|^2 \, \frac{\rho d\rho dt}{1 - \rho} \leq$$

$$\leq C(k,\delta)\iint_{\Gamma_\delta(\vartheta)} |\mathcal{F}^1 u(\rho e^{it})|^2 \rho\, d\rho\, dt.$$

If $\delta/(2+\delta) < r < 1$, then we extend the integration domain in (7) to a ring sector, whose sides are tangent to the disc B, i.e.

$$(1-r)^{2k-1}|\mathcal{F}^k u(re^{i\vartheta})|^2 \leq C(k,\delta)\int_{r_1}^{r_2}\int_{\vartheta_1}^{\vartheta_2} |\mathcal{F}^1 u(\rho e^{it})|^2 \,\frac{\rho\, d\rho\, dt}{1-\rho}, \tag{8}$$

where

$$r_{1,2} = r \mp \frac{\delta(1-r)}{2}, \qquad \vartheta_{1,2} = \vartheta \mp \arcsin\frac{\delta(1-r)}{2r}.$$

Then, integrating (8) by r we get

$$\int_0^1 (1-r)^{2k-1}|\mathcal{F}^k u(re^{i\vartheta})|^2\, dr \leq$$

$$\leq C(k,\delta)\int_0^1\int_{-\pi}^{\pi}\left(\int_0^1 \mathcal{X}_{(r_1,r_2)}(\rho)\mathcal{X}_{(\vartheta_1,\vartheta_2)}(t)\, dr\right)|\mathcal{F}^1 u(\rho e^{it})|^2\,\frac{\rho\, d\rho\, dt}{1-\rho}. \tag{9}$$

To get a suitable estimate of the inner integral, we observe that

$$\mathcal{X}_{(r_1,r_2)}(\rho)\,\mathcal{X}_{(\vartheta_1,\vartheta_2)}(t) = \begin{cases} 1 & \text{if} \quad r_1<\rho<r_2,\ \vartheta_1<t<\vartheta_2,\\ 0 & \text{otherwise.}\end{cases}$$

The requirement $r_1 < \rho < r_2$ is equivalent to $\rho_1 < r < \rho_2$, where $\rho_{1,2} = (\rho \mp \delta/2)/(1\mp\delta/2)$. Besides, $\vartheta_1 < t < \vartheta_2$ is equivalent to

$$r < r_0 \equiv \frac{1}{1+(2/\delta)\sin|t-\vartheta|}.$$

Consequently

$$\int_0^1 \mathcal{X}_{(r_1,r_2)}(\rho)\,\mathcal{X}_{(\vartheta_1,\vartheta_2)}(t)\, dr = \int_{\rho_1}^{\rho_2}\mathcal{X}_{(0,r_0)}(r)\, dr \leq \begin{cases} \rho_2-\rho_1 & \text{if} \quad \rho_1<r_0<1,\\ 0 & \text{if} \quad 0<r_0\leq\rho_1.\end{cases}$$

Inserting $G_\delta = \{\xi = \rho e^{it}\in U^1 : \rho_1 < r_0\}$ and $\rho_2-\rho_1 = C_\delta(1-\rho)$ in (9) yields

$$\int_0^1 (1-r)^{2k-1}|\mathcal{F}^k u(re^{i\vartheta})|^2 dr \leq C(k,\delta)\iint_{G_\delta}|\mathcal{F}^1 u(\rho e^{it})|^2\rho\, d\rho\, dt. \tag{10}$$

It remains to prove the inclusion $G_\delta\subset\Gamma_\delta(\vartheta)$. To this end, suppose $\xi=\rho e^{it}\in G_\delta$ and $\delta\leq\rho<1$. Then the set $\Gamma_\delta(\vartheta)\backslash\{|\xi|<\delta\}$ is described by the following three conditions:

$$\delta\leq\rho<1, \quad |t| = |\arg\xi| < \arccos\delta, \quad |\arg(1-\xi)| < \arcsin\delta. \tag{11}$$

At the same time, the set G_δ is defined by the condition $\rho_1 < r_0$ or

$$\sin|t| < \frac{\delta}{2}\,\frac{1-\rho}{\rho-\delta/2}.$$

The last two conditions of (11) follow from the obvious estimates

$$\sin|t| < \frac{\delta}{2}\,\frac{1-\rho}{\rho-\delta/2} \le 1-\delta < \sqrt{1-\delta^2},$$

$$\sin|t| < \frac{\delta}{2}\,\frac{1-\rho}{\rho-\delta/2} \le \frac{\delta}{\rho}\sqrt{1-2\rho\cos t+\rho^2},$$

and the inclusion $G_\delta \subset \Gamma_\delta(\vartheta)$ is proved. Thus, (10) implies the desired inequality for $n=1$:

$$\int_0^1 (1-r)^{2k-1}|\mathcal{F}^k u(re^{i\vartheta})|^2 dr \le C(k,\delta) \iint_{\Gamma_\delta(\vartheta)} |\mathcal{F}^1 u(\xi)|^2 dm_2(\xi). \tag{12}$$

For $n>1$, the proof follows by applying (12) n times and using the expansion (2).

Proof of Theorem 1. First, suppose that $\alpha_j \ge 1$. By Lemma 1,

$$|\mathcal{F}^\alpha u(z)| \le \int_{T^n} |\mathcal{F}^\alpha P(z,\zeta)|\,|f(\zeta)| dm_n(\zeta) \le \frac{C(\alpha,n)}{(1-|z|)^\alpha}\int_{T^n} P(z,\zeta)\,|f(\zeta)|\,dm_n(\zeta),$$

hence a pointwise estimate by the maximal function u^*_δ follows:

$$\mathcal{G}_{\infty,\alpha}(u)(w) \le C(\alpha,n)\sup_{r\in I^n}\int_{T^n} P(rw,\zeta)\,|f(\zeta)|\,dm_n(\zeta) \le C(\alpha,n)\,u^*_\delta(w), \quad w\in T^n,$$

where $0<\delta_j<1$ $(1\le j\le n)$ are some numbers. Assuming that $u(z)$ is of Hardy's holomorphic class H^p or of the n-harmonic class h^p $(1<p<\infty)$ and using Zygmund's maximal function [9]

$$\|u^*_\delta\|_{L^p} \le C\|u\|_{H^p} \tag{13}$$

we get

$$\|\mathcal{G}_{\infty,\alpha}(u)\|_{L^p} \le C\|u\|_{H^p}. \tag{14}$$

On the other hand, Lemma 5 and (13) together with the Gundy–Stein theorem [10] on equivalence of the L^p-norms of the functions $S_\delta(u)$ and u^*_δ yield

$$\|\mathcal{G}_{2,\alpha}(u)\|_{L^p} \le C\|S_\delta(u)\|_{L^p} \le C\|u^*_\delta\|_{L^p} \le C\|u\|_{H^p}. \tag{15}$$

By a version of the Riesz–Thorin interpolation theorem for the quasi-normed spaces (see [11], p. 316 and [12]) the inequalities (14) and (15) imply (3) and (4) for all q $(2 \le q \le \infty)$ and $\alpha_j \ge 1$.

In the general case $m_j - 1 < \alpha_j \le m_j$, $m_j \in \mathbb{Z}_+$ $(1 \le j \le n)$ we represent the derivative $\mathcal{F}^\alpha$ in the form $\mathcal{F}^\alpha u = \mathcal{F}^{-(m-\alpha)} \mathcal{F}^m u$. By virtue of Lemma 3, this permits to reduce the estimation to the above considered case of integer $\alpha_j \ge 1$. Integrating $|\mathcal{F}^\alpha u| \le \mathcal{F}^{-(m-\alpha)} |\mathcal{F}^m u|$ in the degree q over $(0,1)$, by the mesure $(1-r)^{\alpha q - 1} dr$, and then integrating the degree p of the result over the circle T^1, we come to (3) – (4).

Proof of Theorem 2. In view of Lemma 3, it suffices to give a proof only for $0 < \alpha_j < 1$. First suppose $1 < q \le 2$; for any fixed $r \in I^n$ consider the linear functional, generated over $L^{p'}(T^n)$ by $u(z)$:

$$F_u(v) = \int_{T^n} u(rw)\, v(w) dm_n(w), \quad v(w) \in L^{p'}(T^n).$$

Let $v(rw)$ be the Poisson integral of the function $v(w)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a small positive multiindex, such that $0 < \alpha_j + \gamma_j < 1$. Then using the identities $\mathcal{F}_r^\alpha u(\eta r w) = \mathcal{F}_\eta^\alpha u(\eta r w)$ and $\mathcal{F}_r^{-\alpha-\gamma} \mathcal{F}_r^{\alpha+\gamma} u = u$, we get

$$F_u(v) = \frac{1}{\Gamma(\alpha+\gamma)} \int_{I^n} \prod_{j=1}^n \left(\log \frac{1}{\eta_j} \right)^{\alpha_j + \gamma_j - 1} \left[\int_{T^n} v(w)\, \mathcal{F}_\eta^\alpha\, \mathcal{F}_r^\gamma u(\eta r w) dm_n(w) \right] d\eta =$$

$$= C \int_{I^n} \prod_{j=1}^n \left(\log \frac{1}{\eta_j} \right)^{\alpha_j + \gamma_j - 1} \left[\int_{T^n} \mathcal{F}_\eta^\alpha u(\sqrt{\eta}\, \zeta)\, \mathcal{F}_r^\gamma v(\sqrt{\eta}\, r\zeta)\, dm_n(\zeta) \right] d\eta =$$

$$= C \int_{T^n} \left[\int_{I^n} \prod_{j=1}^n \left(\log \frac{1}{\eta_j} \right)^{\alpha_j + \gamma_j - 1} \mathcal{F}_\eta^\alpha u(\sqrt{\eta}\, \zeta)\, \mathcal{F}_r^\gamma v(\sqrt{\eta}\, r\zeta)\, d\eta \right] dm_n(\zeta).$$

Further, denoting

$$h_{q',\gamma}(r\zeta) = \left(\int_{I^n} (1-\eta)^{\gamma q' - 1} \left| \mathcal{F}_r^\gamma v(\sqrt{\eta}\, r\zeta) \right|^{q'} d\eta \right)^{1/q'},$$

by a repeated application of Hölder's inequality and Theorem 1, we obtain

$$|F_u(v)| \le C \int_{T^n} \mathcal{G}_{q,\alpha}(u)(\zeta)\, h_{q',\gamma}(r\zeta)\, dm_n(\zeta) \le$$

$$\le C \|\mathcal{G}_{q,\alpha}(u)\|_{L^p}\, \|h_{q',\gamma}(r\zeta)\|_{L^{p'}(T^n)} \le C(p,q,\alpha,\gamma,n) \|\mathcal{G}_{q,\alpha}(u)\|_{L^p}\, \|v\|_{L^{p'}(T^n)}.$$

By the duality $(L^{p'})^* = L^p$,

$$\|u(rw)\|_{L^p(T^n)} = \sup\{ |F_u(v)|;\, \|v\|_{L^{p'}} = 1 \} \le C\, \|\mathcal{G}_{q,\alpha}(u)\|_{L^p}.$$

Now suppose $0 < q < 1$ (for $q = 1$ our assertion is obvious). Then by Lemma 4

$$|u(rw)| \le \frac{1}{\prod_{j=1}^{n}\Gamma(\alpha_j)} \int_{I^n} (1-\eta)^{\alpha-1} |\mathcal{F}^{\alpha} u(\eta r w)|\, d\eta \le$$

$$\le C(\alpha, \delta, n) \big(u_{\delta}^{*}(w)\big)^{1-q} \int_{I^n} (1-\eta)^{\alpha q-1} \big|\mathcal{F}^{\alpha} u(\eta r w)\big|^{q} d\eta,$$

where $u_{\delta}^{*}(w)$ is the nontangential maximal function. Further, by Hölder's inequality,

$$\|u(rw)\|_{L^p(T^n)}^{p} \le C \|u_{\delta}^{*}\|_{L^p}^{p(1-q)} \, \|\mathcal{G}_{q,\alpha}(u)\|_{L^p}^{pq}.$$

Consequently, by (13)

$$\|f\|_{L^p} = \|u(rw)\|_{L^p(T^n)} \le C \|u_{\delta}^{*}\|_{L^p}^{1-q} \, \|\mathcal{G}_{q,\alpha}(u)\|_{L^p}^{q} \le C \|f\|_{L^p}^{1-q} \, \|\mathcal{G}_{q,\alpha}(u)\|_{L^p}^{q}.$$

This completes the proof of Theorem 2.

Corollary. Let $\alpha_j > 0$ $(1 \le j \le n)$, $1 < p < \infty$ and $0 < q \le 2$ be any numbers and $u(z)$ be an n-harmonic function in U^n. Then

$$\big\|\mathcal{F}^{-\alpha}|u|\big\|_{h^p} \le C(p, q, \alpha, n) \Big\| \big\|(1-r)^{\alpha} u\big\|_{L^q(dr/(1-r))} \Big\|_{L^p(T^n)}. \tag{16}$$

The proof of this assertion is similar to Proof of Theorem 2 with

$$\Phi_u(v) = \int_{T^n} \mathcal{F}^{-\alpha} |u(rw)|\, v(w) dm_n(w), \quad v(w) \in L^{p'}(T^n),$$

replacing the functional F_u.

Now, we briefly outline the proofs of Theorems 3 – 6.

Proof of Theorem 3: Let an n-harmonic (or holomorphic) function $u(z)$ belong to h^p (or H^p). Given the numbers $\alpha_j > 0$, $m \in \mathbb{Z}_{+}^{n}$ $(m_j - 1 < \alpha_j \le m_j,\ 1 \le j \le n)$, for each $j \in [1, n]$

$$\mathcal{D}_{r_j}^{\alpha_j} u = D_{r_j}^{\alpha_j}\{r_j^{\alpha_j} u\} = D_{r_j}^{-(m_j-\alpha_j)} \left(\frac{\partial}{\partial r_j}\right)^{m_j} \{r_j^{\alpha_j} u\} =$$

$$= r_j^{m_j-\alpha_j} \mathcal{D}_{r_j}^{-(m_j-\alpha_j)} \left\{ r_j^{\alpha_j} \frac{\partial^{m_j} u}{\partial r_j^{m_j}} + \text{L.O.T. (low order terms)} \right\}.$$

The highest term of this sum can be written in the form

$$r_j^{m_j} \mathcal{D}_{r_j}^{-(m_j-\alpha_j)} D_{r_j}^{m_j} u = r_j^{\alpha_j} D_{r_j}^{\alpha_j} u.$$

Hence the estimate established in [6]:

$$\Big\| \|(1-r)^\alpha r^\alpha D^\alpha u\|_{L^q(dr/(1-r))} \Big\|_{L^p(T^n)} \le C\|u\|_{h^p}.$$

Proof of Theorem 4: It suffices to recall the definition of the fractional integrals and to use the inequality (16):

$$\|\mathcal{D}^{-\alpha} u\|_{h^p} \le C\|\mathcal{F}^{-\alpha}|u|\|_{h^p} \le C\Big\| \|(1-r)^\alpha u\|_{L^q(dr/(1-r))} \Big\|_{L^p(T^n)}.$$

Proof of Theorem 5: According to Minkowski's integral inequality and Theorems 1 and 3,

$$\Big\|(1-r)^\alpha M_p(\mathcal{J}^\alpha u; r)\Big\|_{L^q(dr/(1-r))} \le \Big\| \|(1-r)^\alpha \mathcal{J}^\alpha u\|_{L^q(dr/(1-r))} \Big\|_{L^p(T^n)} \le C\|u\|_{h^p}.$$

Now the inequality (6) follows from (5) and by the inclusion (see [13])

$$\Big\|(1-r)^{\alpha+1/p-1/p_0} M_{p_0}(f; r)\Big\|_{L^q(dr/(1-r))} \le C\Big\|(1-r)^\alpha M_p(f; r)\Big\|_{L^q(dr/(1-r))}.$$

Proof of Theorem 6: By Minkowski's inequality and Theorems 2 and 4,

$$\|\mathcal{J}^{-\alpha} u\|_{h^p} \le C\Big\| \|(1-r)^\alpha u\|_{L^q(dr/(1-r))} \Big\|_{L^p(T^n)} \le C\Big\|(1-r)^\alpha M_p(u; r)\Big\|_{L^q(dr/(1-r))}.$$

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