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Lacunary Series and Sharp Estimates in Weighted Spaces of Holomorphic Functions*

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Abstract—The well-known Paley inequalities for lacunary series are applied in investigation of weighted spaces $H(p,\alpha)$ and $H\big(p,\log(\alpha)\big)$ of functions holomorphic in the unit disc of the complex plane. These are spaces which are similar to the Bloch and Hardy spaces and naturally arise as the images of some fractional operators.

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1. We denote by $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ the unit disc, by $\mathbb{T}=\partial\mathbb{D}$ the unit circle of the complex plane, by $H(\mathbb{D})$ the set of holomorphic functions in $\mathbb{D}$. The integral mean of a measurable function $f(z)=f(re^{i\theta})$ we denote as

$$M_p(f;r)=\big\|f(r\cdot)\big\|_{L^p(\mathbb{T};dm)},\quad 0\le r<1,\quad 0<p\le\infty,$$

where dm stands for the normed Lebesgue measure on $\mathbb{T}$. Note that the set of holomorphic functions $f(z)$ that satisfy $\|f\|_{H^p}=\sup\limits_{0<r<1}M_p(f;r)<\infty$, is the ordinary Hardy space H^p.

By $H(p,\alpha)$ $(0<p\le\infty,\ \alpha>0)$ we denote the space of all functions $f(z)$ holomorphic in $\mathbb{D}$, which have finite quasinorms

$$\|f\|_{p,\alpha}=\sup_{0<r<1}(1-r)^{\alpha}M_p(f;r)$$

and note that the norm $|f(0)|+\|\nabla f\|_{\infty,1}$ coincides with that of Bloch spaces $\mathcal{B}$, see [1], [2] for the basic theory.

By $H_0(p,\alpha)$ $(0<p\le\infty,\alpha>0)$ we denote the space of those functions $f(z)$ holomorphic in the disc $\mathbb{D}$, for which

$$(1-r)^{\alpha}M_p(f;r)=o(1)\quad\text{as}\quad r\to 1-.$$

Note that if $(1-r)M_\infty(\nabla f;r)=o(1)$, then f is said to belong to the little Bloch space $\mathcal{B}_0$. In addition, we define $H\big(p,\log(\alpha)\big)$ $\big(0<p\le\infty,\alpha>0\big)$ to be the class of those functions $f(z)$ holomorphic in $\mathbb{D}$, for which

$$\|f\|_{p,\log(\alpha)}=\sup_{0<r<1}\left(\log\frac{e}{1-r}\right)^{-\alpha}M_p(f;r)<+\infty.$$

2. The positive constants we are going to denote $C(\alpha,\beta,\dots)$, C_α etc, displaying the dependence on parameters. Besides, dm_2 will stand for the Lebesgue measure over $\mathbb{D}$ and for any $A,B>0$ the notation $A\approx B$ will mean the existence of the two-sided estimate $c_1A\le B\le c_2A$ with some non-essential positive constants c_1 and c_2, not depending on the variable.

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For any function representable by the series $f(z) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$, Hadamard's fractional integrodifferentiation operator $\mathcal{F}^\alpha$ of the order $\alpha \in \mathbb{R}$ is defined:

$$\mathcal{F}^\alpha f(z) = \sum_{n=0}^{\infty}(1+n)^\alpha a_n r^n e^{in\theta}.$$

An evident inversion formula $\mathcal{F}^\alpha \mathcal{F}^{-\alpha} f(z) = f(z)$, $\alpha \in \mathbb{R}$ is valid.

A sequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ is called *lacunary or Hadamard*, if there exists a number $\lambda > 1$ such that $n_{k+1}/n_k \geq \lambda$ for all $k = 1, 2, \ldots$ In addition, the corresponding power series is called lacunary, and the following well-known theorem is true (see [3], Chapter 5, Theorem 8.20).

Theorem 1. (Paley [3]) *Let $\{n_k\}_{k=1}^{\infty}$ be a lacunary sequence and f be a holomorphic function representable in $\mathbb{D}$ by the convergent lacunary series $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$.*

Then for any p, $0 < p < \infty$, the function f is of the Hardy space H^p if and only if $\{a_k\} \in \ell^2$, and the corresponding norms are equivalent: $\|f\|_{H^p} \approx \|\{a_k\}\|_{\ell^2}$.

3. While the above Theorem 1 characterizes lacunary series in the Hardy space H^p, the next theorem, which we prove, relates to a similar characterization in the weighted spaces $H(p, \alpha)$.

Theorem 2. *Let $\alpha > 0$ be arbitrary, let $\{n_k\}_{k=1}^{\infty}$ be a lacunary sequence and let a function f be representable by the convergent lacunary series $f(z) = \sum_{k=1}^{\infty} a_k n_k^\alpha z^{n_k}$. Then the following conditions are equivalent:*

(i) $f(z) \in H(\infty, \alpha)$,

(ii) $f(z) \in H(p, \alpha)$ *for some* p, $0 < p < \infty$,

(iii) $f(z) \in H(p, \alpha)$ *for all* p, $0 < p < \infty$,

(iv) $\{a_k\}_{k=1}^{\infty} \in \ell^\infty$,

and the corresponding norms are equivalent.

Proof: The implication $(i) \Rightarrow (ii)$ is obvious by the embedding $H(\infty, \alpha) \subset H(p, \alpha)$. Also, the implication $(ii) \Rightarrow (iii)$ is obvious by Theorem 1, according to which $M_p(f; r) \approx M_q(f; r)$ for any q, $0 < q < \infty$.

To prove the implication $(iii) \Rightarrow (iv)$, assume that $f(z) \in H(p, \alpha)$ for any $p, 0 < p < \infty$, and particularly $(1-r)^\alpha M_1(f; r) \leq \|f\|_{1,\alpha}$. Then by the Cauchy integral formula

$$|a_k|\, n_k^\alpha = \left| \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)\, d\zeta}{\zeta^{1+n_k}} \right| \leq \frac{1}{r^{n_k}} M_1(f; r) \leq \frac{\|f\|_{1,\alpha}}{(1-r)^\alpha\, r^{n_k}}$$

for any $0 < r < 1$ and $k = 1, 2, \ldots$ Hence, taking $r = 1 - 1/n_k$ we get

$$|a_k| \leq \|f\|_{1,\alpha} \left(1 - \frac{1}{n_k}\right)^{-n_k} \leq 4\|f\|_{1,\alpha},$$

i.e. $\{a_k\} \in \ell^\infty$. To prove the implication $(iv) \Rightarrow (i)$, assume that

$$|a_k| \leq M \leq \left\|\{a_k\}\right\|_{\ell^\infty} \quad \text{for all} \quad k = 1, 2, \ldots$$

Then we apply Hadamard's operator to the function $f(z)$

$$\mathcal{F}^{1-\alpha} f(z) = \sum_{k=1}^{\infty}(1+n_k)^{1-\alpha} a_k n_k^\alpha\, z^{n_k},$$

and estimate

$$\left|\mathcal{F}^{1-\alpha} f(z)\right| \le C_\alpha M \sum_{k=1}^{\infty} n_k\, r^{n_k}.$$

One can see that

$$n_{k+1} \le \frac{\lambda}{\lambda-1}(n_{k+1}-n_k), \quad k=1,2,\dots,$$

hence

$$n_{k+1} r^{n_{k+1}} \le C(\lambda)\Big[r^{1+n_k} + r^{2+n_k} + \dots + r^{n_{k+1}}\Big].$$

Therefore,

$$\left|\mathcal{F}^{1-\alpha} f(z)\right| \le C(\alpha,\lambda)\, M \sum_{k=1}^{\infty} r^k \le C(\alpha,\lambda)\, \big\|\{a_k\}\big\|_{\ell^\infty} \frac{r}{1-r}.$$

By the inclusion $\mathcal{F}^{1-\alpha} f \in H(\infty,1)$ and invertibility of Hadamard's operator

$$f(z) = \mathcal{F}^{\alpha-1}\mathcal{F}^{1-\alpha} f(z) \in H(\infty, 1+(\alpha-1)) = H(\infty,\alpha),$$

where the corresponding norms are equivalent. This completes the proof.

Note that replacing $f(z)$ by $f'(z)$ and taking $\alpha=1$ in Theorem 2 we come to the characterization of the Bloch space $\mathcal{B}$ given in [1], [2], [4]. Also, an "little oh" version of Theorem 2, giving a characterization of the little Bloch space $\mathcal{B}_0$, is true. The proof of this version is similar to that of Theorem 2, and the statement is obtained by replacing $H(p,\alpha)$ by $H_0(p,\alpha)$ and the condition $\{a_k\} \in \ell^\infty$ by $\lim\limits_{k\to\infty} a_k = 0$ in Theorem 2.

4. The next theorem gives some sharp estimates for Hadamard's integrodifferential operator in the spaces $H(p,\alpha)$ and $H\big(p,\log(\alpha)\big)$.

Theorem 3. *If $u(z) \in H(\mathbb{D})$ and $\alpha > 0$, then*

$$\|\mathcal{F}^{-\alpha} u\|_{p,\log(1/p)} \le C\|u\|_{p,\alpha}, \qquad 0 < p \le 2, \tag{1}$$

$$\|\mathcal{F}^{-\alpha} u\|_{p,\log(1/2)} \le C\|u\|_{p,\alpha}, \qquad 2 \le p < \infty, \tag{2}$$

$$\|\mathcal{F}^{-\alpha} u\|_{\infty,1/p} \le C\|u\|_{p,\alpha}, \qquad 0 < p < \infty, \tag{3}$$

$$\|\mathcal{F}^{-\alpha} u\|_{\infty,\log(1)} \le C\|u\|_{\infty,\alpha}, \tag{4}$$

$$\|\mathcal{F}^{-\alpha} u\|_{p,\log(1/2)} \le C\|u\|_{\infty,\alpha}, \qquad 0 < p < \infty. \tag{5}$$

The estimates (1)-(5) are the best possible in the sense that for each estimate $\|\mathcal{F}^{-\alpha} u\|_Y \le C\|u\|_X$ there exists a function $f_0(z)$ for which $\|\mathcal{F}^{-\alpha} f_0\|_Y \approx \|f_0\|_X$.

Proof: For $u(z) \in H(\mathbb{D})$ we use Flett's inequalities [5], [6]

$$\|\mathcal{F}^{-\alpha} u\|_{H^p} \le C\left(\int_{\mathbb{D}} (1-|z|)^{\alpha p-1} |u(z)|^p dm_2(z)\right)^{1/p}, \quad \alpha>0, \quad 0<p\le 2, \tag{6}$$

$$\|\mathcal{F}^{-\alpha} u\|_{H^p} \le C\left(\int_0^1 (1-r)^{2\alpha-1} M_p^2(u;r)dr\right)^{1/2}, \quad \alpha>0, \quad 2\le p<\infty. \tag{7}$$

Applying (6) to the dilated function $u_\rho(z) = u(\rho z)$ we get

$$M_p(\mathcal{F}^{-\alpha} u;\rho) \le C\left(\int_{\mathbb{D}} (1-|z|)^{\alpha p-1} |u(\rho z)|^p dm_2(z)\right)^{1/p}, \quad \rho \in (0,1).$$

Hence

$$M_p^p(\mathcal{F}^{-\alpha}u;\rho)\le C\int_0^1(1-r)^{\alpha p-1}M_p^p(u;\rho r)dr$$

$$\le C\|u\|_{p,\alpha}^p\int_0^1\frac{(1-r)^{\alpha p-1}}{(1-\rho r)^{\alpha p}}dr\le C\|u\|_{p,\alpha}^p\log\frac{1}{1-\rho}.$$

This inequality is the best possible in view of the example $f_1(z)=(1-z)^{-\alpha-1/p}$. In fact one can easily verify that

$$(1-r)^\alpha M_p(f_1;r)\approx 1\quad\text{and}\quad M_p(\mathcal{F}^{-\alpha}f_1;r)\approx\left(\log\frac{e}{1-r}\right)^{1/p}.$$

Thus, the inequality (1) is proved. Further, by (7) and Fatou's lemma

$$M_p^2(\mathcal{F}^{-\alpha}u;\rho)\le C\int_0^1(1-r)^{2\alpha-1}M_p^2(u;\rho r)dr$$

$$\le C\|u\|_{p,\alpha}^2\int_0^1\frac{(1-r)^{2\alpha-1}}{(1-\rho r)^{2\alpha}}dr\le C\|u\|_{p,\alpha}^2\ \log\frac{e}{1-\rho},$$

for any $\rho\in(0,1)$, and this estimate is the best in view of the example $f_2(z)=\sum_{k=0}^\infty 2^{\alpha k}\,z^{2^k}$. Indeed, by Theorem 1

$$M_p(f_2;r)\approx\left(\sum_{k=0}^\infty 2^{2\alpha k}\,r^{2^{k+1}}\right)^{1/2}\approx\frac{r}{(1-r)^\alpha},\qquad r\in(0,1).$$

On the other hand,

$$\mathcal{F}^{-\alpha}f_2(z)=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^\infty 2^{\alpha k}\left(\int_0^1(1-\eta)^{\alpha-1}\eta^{2^k}d\eta\right)\,z^{2^k},$$

and $M_p^2(\mathcal{F}^{-\alpha}f_2;r)\approx\log\frac{e}{1-r}$. This completes the proof of the inequality (2).

Now let $u(z)\in H(p,\alpha)$ be arbitrary. Then by the continuous embedding $H(p,\alpha)\subset H(\infty,\alpha+1/p)$ (see [6] or [7]) and the relation $\mathcal{F}^{-\alpha}\big(H(\infty,\alpha+1/p)\big)=H(\infty,1/p)$ (see [6]) we get the estimate

$$\|\mathcal{F}^{-a}u\|_{\infty,1/p}\le C\|u\|_{\infty,\alpha+1/p}\le C\|u\|_{p,\alpha}.$$

It is sharp due to the example $f_1(z)=(1-z)^{-\alpha-1/p}$, and the proof of (3) is complete.

Assume now that $u(z)\in H(\infty,\alpha)$. Then

$$M_\infty(\mathcal{F}^{-\alpha}u;r)\le\frac{1}{\Gamma(\alpha)}\int_0^1(1-\eta)^{\alpha-1}M_\infty(u;\eta r)d\eta$$

$$\le\frac{1}{\Gamma(\alpha)}\|u\|_{\infty,\alpha}\int_0^1\frac{(1-\eta)^{\alpha-1}}{(1-\eta r)^\alpha}\,d\eta\le C_\alpha\|u\|_{\infty,\alpha}\log\frac{e}{1-r}.$$

The sharpness of this inequality is proved by the example $f_3(z)=(1-z)^{-\alpha}$, so the inequality (4) is proved.

It remains to see that by (2) and the monotone increasing property of M_p in p

$$\|\mathcal{F}^{-a}u\|_{p,\log(1/2)}\le\|\mathcal{F}^{-a}u\|_{\max\{2,p\},\log(1/2)}\le C\|u\|_{\max\{2,p\},\alpha}\le C\|u\|_{\infty,\alpha}.$$

This is a sharp estimate due to the example $f_2(z)=\sum_{k=0}^\infty 2^{\alpha k}z^{2^k}$, since

$$M_\infty(f_2;r)\le\sum_{k=0}^\infty 2^{\alpha k}\,r^{2^k}\approx\frac{r}{(1-r)^\alpha},\quad r\in(0,1),$$

while $M_p^2(\mathcal{F}^{-\alpha} f_2; r) \approx \log \dfrac{e}{1-r}$. Thus, (5) is established and the proof is complete.

Remarks. For the particular case $\alpha = 1$, most of the statements Theorem 3 can be found in [1], [4], [8]–[10]. The proof of the inequality (5) for $\alpha = 1$ found in [9] (p. 364) and [10] (p. 374) are considerably more complicated than those of Theorem 3. In [4] (p. 467) the inequality (3) is proved for $\alpha = 1$, $1/2 \le p < \infty$.

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