Equivalent Conditions for Bergman Space and Littlewood-Paley Type Inequalities

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Abstract: In this paper we show that the following integrals

$$\int_B |f(z)|^p(1-|z|)^\alpha dV(z), \quad \int_B |f(z)|^{p-q}|\nabla f(z)|^q(1-|z|)^{\alpha+q}dV(z),$$

$$\text{and} \quad \int_B |f(z)|^{p-q}|\mathcal{R}f(z)|^q(1-|z|)^{\alpha+q}dV(z),$$

where $p>0$, $q\in[0,p]$, $\alpha\in(-1,\infty)$, and where f is a holomorphic function on the unit ball B in $\mathbb{C}^n$ are comparable. This result confirms a conjecture proposed by the second author at several meetings, for example, at the International two-day meeting on complex, harmonic, and functional analysis and applications, Thessaloniki, December 12 and 13, 2003. Also we generalize the well-known inequality of Littlewood-Paley in the unit ball.

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1 Introduction

Let $z=(z_1,\dots,z_n)$ and $w=(w_1,\dots,w_n)$ be points in the complex vector space $\mathbb{C}^n$. By $\langle z,w\rangle \equiv z\,\overline{w}=\sum_{k=1}^n z_k\bar{w}_k$ we denote the inner product of z and w, and $|z|=\sqrt{\langle z,z\rangle}$.

Let B denote the unit ball of $\mathbb{C}^n$, $B(a,r) = \{ z \in \mathbb{C}^n \mid |z-a| < r \}$ the open ball centered at a of radius r, dV the normalized Lebesgue measure on $\mathbb{C}^n$ and $d\sigma$ the normalized surface measure on the boundary S of B.

By $H(B)$ we denote the class of all functions holomorphic in B. For $f \in H(B)$ we usually write

$$M_p(f,r) = \left(\int_S |f(r\zeta)|^p d\sigma(\zeta) \right)^{1/p}, \quad p \in (0,\infty) \quad \text{for} \quad 0 \le r < 1$$

for the integral means of f and

$$M_\infty(f,r) = \sup_{\zeta \in S} |f(r\zeta)| \quad \text{for} \quad 0 \le r < 1.$$

For $p \in (0,\infty)$ and $\alpha \in (-1,\infty)$, the weighted Bergman space $\mathcal{A}^p_\alpha(B)$ is the space of all holomorphic functions f on B such that

$$||f||_{p,\alpha} = \left(\int_B |f(z)|^p (1-|z|)^\alpha dV(z) \right)^{1/p} < \infty.$$

Weighted Bergman spaces of analytic functions of one variable have been studied, for example, in [7, 8, 17, 20, 23, 27], while weighted Bergman spaces of analytic functions of several variables have been studied, for example, in [3, 5, 10, 13, 15, 16, 21, 25, 26] (see, also the references therein).

In papers [20, 21] we have investigated relationships among various type of integrals on the Bergman space on the unit disk, unit ball and unit polydisc. In [22] we posed several open problems and conjectures concerning this topic. Among other conjectures, we posed the following:

Conjecture 1. *Let $p > 0$, $q \in [0,p]$, $\alpha \in (-1,\infty)$, and $f \in H(B)$. Show that*

$$\int_B |f(z)|^p (1-|z|)^\alpha dV(z) \asymp |f(0)|^p + \int_B |f(z)|^{p-q} |\nabla f(z)|^q (1-|z|)^{\alpha+q} dV(z). \tag{1}$$

The above means that there are finite positive constants C and C' independent of f such that the left and right hand sides $L(f)$ and $R(f)$ satisfy

$$CR(f) \le L(f) \le C'R(f)$$

for all analytic f.

Remark 1. Note that for $q = 0$ the relationship (1) is obvious. On the other hand, we know that

$$|f(0)|^p + \int_B |\nabla f(z)|^p (1-|z|)^{\alpha+p} dV(z) \asymp \int_B |f(z)|^p (1-|z|)^\alpha dV(z) \tag{2}$$

see, for example, [16, 21, 25], and hence (1) holds also when $p = q$.

The paper is organized as follows. In Section 2 we give several auxiliary results which we use in the proof of the main results. In Section 3 we confirm Conjecture 1, that is, we prove the following result:

Theorem 1. *Let $p > 0$, $q \in [0, p]$, $\alpha \in (-1, \infty)$, and $f \in H(B)$. Then*

$$\int_B |f(z)|^p (1 - |z|)^\alpha dV(z) \asymp |f(0)|^p + \int_B |f(z)|^{p-q} |\nabla f(z)|^q (1 - |z|)^{\alpha+q} dV(z).$$

Some generalizations of the Littlewood-Paley inequality on the unit ball are given in Sections 4 and 5.

2 Auxiliary results

In order to prove the main results we need several auxiliary results which are incorporated in the following lemmas. Throughout the following we will use C to denote a positive constant which may vary from line to line.

Lemma 1. *Suppose $0 \le p < \infty$ and $f \in H(B)$. Then*

$$\big||f(\rho\zeta)|^p - |f(r\zeta)|^p\big| \le (\rho - r) \sup_{r<s<\rho} p|f(s\zeta)|^{p-1} |\nabla f(s\zeta)| \tag{3}$$

almost everywhere, where $r < \rho$ and $\zeta \in \partial B$.

Proof. For $f \equiv 0$ the result is obvious. If $f \not\equiv 0$, at points z where f is not zero we have

$$\Big|\frac{d}{ds}(|f(z)|^p)\Big| = p|f(z)|^{p-1} \big|\langle \nabla |f|(s\zeta), \zeta\rangle\big| \le p|f(z)|^{p-1} |\nabla f(z)|, \tag{4}$$

where $z = s\zeta$. Integrating (4) in s from r to ρ we obtain (3). □

Lemma 2. *Suppose $0 < q \le p < \infty$ and $\alpha > -1$. Then, there is a constant $C = C(p, q, \alpha, n)$ such that*

$$M_\infty^p(f, 1/2) \le C \left(|f(0)|^p + \int_B |f(z)|^{p-q} |\nabla f(z)|^q (1 - |z|)^{q+\alpha} dV(z) \right),$$

for all $f \in H(B)$.

Proof. By Lemma 1, we have

$$\big||f|^{p/q}(z) - |f|^{p/q}(0)\big| \le \frac{p}{q} |z| \sup_{|w|<1/2} |f(w)|^{\frac{p}{q}-1} |\nabla f(w)|,$$

for every $|z| < 1/2$. Hence

$$|f(z)|^p \le C \left(|f(0)|^p + \sup_{|w|<1/2} |f(w)|^{p-q} |\nabla f(w)|^p \right), \tag{5}$$

for some positive constant C independent of f.

We have

$$|\nabla f(w)|^q \asymp \sum_{k=1}^{n} \left|\frac{\partial f}{\partial z_k}(w)\right|^q. \tag{6}$$

From (6) and since the functions $|f(w)|^{p-q}|\frac{\partial f}{\partial z_k}(w)|^q$, $k \in \{1,\dots,n\}$ are subharmonic, we have that there is a positive constant C independent of f such that

$$\begin{aligned}
|f(z)|^{p-q}|\nabla f(z)|^q &\leq C\sum_{k=1}^{n}|f(z)|^{p-q}\left|\frac{\partial f}{\partial z_k}(z)\right|^q \\
&\leq C\sum_{k=1}^{n}\int_{|w|<3/4}|f(w)|^{p-q}\left|\frac{\partial f}{\partial z_k}(w)\right|^q dV(w) \\
&\leq C\int_{|w|<3/4}|f(w)|^{p-q}|\nabla f(w)|^q dV(w), \qquad (7)
\end{aligned}$$

for every $|z| < 1/2$.

From (5) and (7), it follows that

$$\begin{aligned}
|f(z)|^p &\leq C\left(|f(0)|^p + \int_{|w|<3/4}|f(w)|^{p-q}|\nabla f(w)|^q dV(w)\right) \\
&\leq C\left(|f(0)|^p + \int_{|w|<3/4}|f(w)|^{p-q}|\nabla f(w)|^q(1-|w|)^\alpha dV(w)\right) \\
&\leq C\left(|f(0)|^p + \int_{B}|f(w)|^{p-q}|\nabla f(w)|^q(1-|w|)^\alpha dV(w)\right),
\end{aligned}$$

for every $|z| < 1/2$, as desired. □

Lemma 3. *Let $0 < p < \infty$, $q \in [0,p]$ and $0 \leq r < 1$. Then there is a constant C independent of f and r such that*

$$\int_S \sup_{0\leq\tau<1}|f(\tau r\zeta)|^{p-q}|\nabla f(\tau r\zeta)|^q d\sigma(\zeta) \leq C\int_S |f(r\zeta)|^{p-q}|\nabla f(r\zeta)|^q d\sigma(\zeta)$$

for all $f \in H(B)$.

Proof. By [18, p.165] there is a positive constant C independent of nonnegative subharmonic function u on the unit ball $B \subset \mathbb{R}^m$ such that

$$\int_S \sup_{0\leq\tau<1} u(\tau r\zeta)\,d\sigma(\zeta) \leq C\int_S u(r\zeta)\,d\sigma(\zeta)$$

for every $r \in (0,1)$. From this, (6), using the fact that the functions $|f(w)|^{p-q}|\frac{\partial f}{\partial z_k}(w)|^q$, $k \in \{1,\dots,n\}$ are subharmonic, and choosing $m = 2n$ we can easily obtain the result. □

We also need the following technical lemma.

Lemma 4. ([14]) *Suppose that* $g(r)$ *is a nonnegative continuous function on the interval* $[0,1)$, $b>0$ *and* $\alpha>-1$. *Then there is a constant* $C=C(\alpha,b)$ *such that*

$$\int_0^1 g^b(r)(1-r)^\alpha dr \le C\left(\max_{r\in[0,1/2]} g^b(r)+\int_0^1 \left|g\left(\frac{1+r}{2}\right)-g(r)\right|^b (1-r)^\alpha dr\right).$$

3 Proof of Theorem 1

In this section we prove the main result of this paper.

Proof of Theorem 1. The existence of a positive constant C such that

$$\int_B |f(z)|^{p-q}|\nabla f(z)|^q(1-|z|)^{\alpha+q}dV(z)\le C\int_B |f(z)|^p(1-|z|)^\alpha dV(z)$$

follows from Theorem 2 in [20], with $\omega(z)=(1-|z|)^\alpha$. Assume first that $q\le 1$. By Lemma 4 (the case $b=1$), Lemma 2, Lemma 3 and polar coordinates, we obtain

$$\begin{aligned}
||f||_{p,\alpha}^p &= 2n\int_0^1 M_p^p(f,r)(1-r)^\alpha r^{2n-1}dr\\
&\le C\left(M_p^p(f,1/2)+\int_0^1 \left|M_p^p(f,(1+r)/2)-M_p^p(f,r)\right|(1-r)^\alpha dr\right)\\
&\le C\left(M_p^p(f,1/2)+\int_0^1 \left|M_q^q(|f|^{p/q},(1+r)/2)-M_q^q(|f|^{p/q},r)\right|(1-r)^\alpha dr\right)\\
&\le C\left(M_\infty^p(f,1/2)+\int_0^1\int_S \left||f|^{p/q}((1+r)\zeta/2)-|f|^{p/q}(r\zeta)\right|^q d\sigma_N(\zeta)(1-r)^\alpha dr\right)\\
&\le C\left(M_\infty^p(f,1/2)+\int_0^1\int_S \left|\frac{p}{q}\sup_{r<\rho<\frac{1+r}{2}}|f(\rho\zeta)|^{\frac{p}{q}-1}|\nabla f(\rho\zeta)|\right|^q d\sigma_N(\zeta)(1-r)^{\alpha+q}dr\right)\\
&\le C\left(M_\infty^p(f,1/2)+\left(\frac{p}{q}\right)^q\int_0^1\int_S \sup_{0<\rho<\frac{1+r}{2}}|f(\rho\zeta)|^{p-q}|\nabla f(\rho\zeta)|^q d\sigma_N(\zeta)(1-r)^{\alpha+q}dr\right)\\
&\le C\left(M_\infty^p(f,1/2)+\left(\frac{p}{q}\right)^q\int_0^1\int_S \left|f\left(\frac{1+r}{2}\zeta\right)\right|^{p-q}\left|\nabla f\left(\frac{1+r}{2}\zeta\right)\right|^q d\sigma_N(\zeta)(1-r)^{\alpha+q}dr\right)\\
&= C\left(M_\infty^p(f,1/2)+2^{\alpha+q+1}\left(\frac{p}{q}\right)^q\int_{1/2}^1\int_S |f(r\zeta)|^{p-q}|\nabla f(r\zeta)|^q d\sigma_N(\zeta)(1-r)^{\alpha+q}dr\right)\\
&\le C\left(M_\infty^p(f,1/2)+2^{\alpha+q+2n}\left(\frac{p}{q}\right)^q\int_0^1\int_S |f(r\zeta)|^{p-q}|\nabla f(r\zeta)|^q d\sigma_N(\zeta)(1-r)^{\alpha+q}r^{2n-1}dr\right)\\
&\le C\left(|f(0)|^p+\int_B |f(z)|^{p-q}|\nabla f(z)|^q(1-|z|)^{\alpha+q}dV(z)\right),
\end{aligned}$$

finishing the proof in this case.

Now assume that $q > 1$. Then by Lemma 4 with $b = q$ and Minkowski's inequality, we have

$$\begin{aligned}
||f||_{p,\alpha}^p &= 2n\int_0^1 (M_p^{p/q}(f,r))^q(1-r)^\alpha r^{2n-1}dr \\
&\le C\left(M_p^p(f,1/2) + \int_0^1 \left|M_p^{p/q}(f,(1+r)/2) - M_p^{p/q}(f,r)\right|^q (1-r)^\alpha dr\right) \\
&\le C\left(M_p^p(f,1/2) + \int_0^1 \left|M_q(|f|^{p/q},(1+r)/2) - M_q(|f|^{p/q},r)\right|^q (1-r)^\alpha dr\right) \\
&\le C\left(M_\infty^p(f,1/2) + \int_0^1\int_S \left||f|^{p/q}(\frac{1+r}{2}\zeta) - |f|^{p/q}(r\zeta)\right|^q d\sigma_N(\zeta)(1-r)^\alpha dr\right) \\
&\le C\left(M_\infty^p(f,1/2) + \int_0^1\int_S \left|\frac{p}{q}\sup_{r<\rho<\frac{1+r}{2}} |f(\rho\zeta)|^{\frac{p}{q}-1}|\nabla f(\rho\zeta)|\right|^q d\sigma_N(\zeta)(1-r)^{\alpha+q}dr\right) \\
&\le C\left(M_\infty^p(f,1/2) + \left(\frac{p}{q}\right)^q\int_0^1\int_S \sup_{0<\rho<\frac{1+r}{2}} |f(\rho\zeta)|^{p-q}|\nabla f(\rho\zeta)|^q d\sigma_N(\zeta)(1-r)^{\alpha+q}dr\right).
\end{aligned}$$

The rest of the proof is the same as in the first case and will be omitted. □.

4 Fractional derivative

For holomorphic functions in the ball consider fractional integrodifferentiation of order $\alpha \in \mathbb{R}$. If $f \in H(B)$ has a series expansion

$$f(z) = \sum_{k\in Z_+^n} a_k z^k, \qquad z \in B,$$

then define

$$\mathcal{D}^\alpha f(z) = \sum_{k\in Z_+^n} (1+|k|)^\alpha a_k z^k, \qquad z \in B.$$

Theorem 2. *Suppose $0 < q \le p < \infty$, $\alpha > 0$, $f(z) \in H^p(B)$, and a holomorphic function $g(z)$ belongs to the mixed norm space $H(p,q,\alpha)$ in B, that is*

$$\|g\|_{H(p,q,\alpha)}^q = \int_0^1 M_p^q(g,r)(1-r)^{\alpha q-1}dr < +\infty.$$

Then

$$\int_B |f(z)|^{p-q}|g(z)|^q(1-|z|)^{\alpha q-1}dV(z) \le C\|f\|_{H^p}^{p-q}\|g\|_{H(p,q,\alpha)}^q.$$

In particular, if $\mathcal{D}^\alpha f \in H(p,q,\alpha)$, then

$$\int_B |f(z)|^{p-q} |\mathcal{D}^\alpha f(z)|^q (1-|z|)^{\alpha q-1} dV(z) \leq C \|f\|_{H^p}^{p-q} \|\mathcal{D}^\alpha f\|_{H(p,q,\alpha)}^q.$$

Proof. Assuming that $\|f\|_{H^p} \neq 0$, we can apply Jensen's inequality to the integral

$$\begin{aligned}
&\int_S |f(r\zeta)|^{p-q} |g(r\zeta)|^q d\sigma(\zeta) \\
= \;& M_p^p(f,r) \left[\frac{1}{M_p^p(f,r)} \int_S \left| \frac{g(r\zeta)}{f(r\zeta)} \right|^q |f(r\zeta)|^p d\sigma(\zeta) \right]^{\frac{p}{q}\frac{q}{p}} \\
\leq \;& M_p^p(f,r) \left[\frac{1}{M_p^p(f,r)} \int_S \left| \frac{g(r\zeta)}{f(r\zeta)} \right|^p |f(r\zeta)|^p d\sigma(\zeta) \right]^{q/p} \\
= \;& M_p^{p-q}(f,r) \left[\int_S |g(r\zeta)|^p d\sigma(\zeta) \right]^{q/p} = M_p^{p-q}(f,r) M_p^q(g,r). \qquad (8)
\end{aligned}$$

Multiplying (8) by $(1-r)^{\alpha q-1} r^{2n-1} dr$, then integrating from 0 to 1 it follows that

$$\begin{aligned}
&\int_B |f(z)|^{p-q} |g(z)|^q (1-|z|)^{\alpha q-1} dV(z) \\
\leq \;& C \int_0^1 M_p^{p-q}(f,r) M_p^q(g,r) (1-r)^{\alpha q-1} dr \\
\leq \;& C \|f\|_{H^p}^{p-q} \int_0^1 M_p^q(g,r)(1-r)^{\alpha q-1} dr,
\end{aligned}$$

and the proof is complete. □

Now we introduce some more notation to formulate several auxiliary lemmas. In what follows, for a fixed $\delta > 1$ let $\Gamma_\delta(\zeta) = \{z \in B : |1-\overline{\zeta}z| \leq \delta(1-|z|)\}$ be the admissible approach region whose vertex is at $\zeta \in S$. Let also $I_{\zeta,t} = \{\eta \in S : |1-\overline{\zeta}\eta| < t\}$ and $\widehat{I}_{\zeta,t} = \{z \in B : |1-\overline{\zeta}z| < t\}$.

Following [6, 12], consider the functions

$$\begin{aligned}
A_p(f)(\zeta) &= \left(\int_{\Gamma_\delta(\zeta)} \frac{|f(z)|^p}{(1-|z|)^{n+1}} dV(z) \right)^{1/p}, \qquad p < \infty, \\
A_\infty(f)(\zeta) &= \sup\{|f(z)| : z \in \Gamma_\delta(\zeta)\}, \\
C_p(f)(\zeta) &= \sup_t \left(\frac{1}{|I_{\zeta,t}|} \int_{\widehat{I}_{\zeta,t}} \frac{|f(z)|^p}{1-|z|} dV(z) \right)^{1/p}, \quad p < \infty, \quad \zeta \in S.
\end{aligned}$$

Lemma A. ([6, 12]) For any functions $f(z)$ and $g(z)$ measurable in B

$$\int_B \frac{|f(z)||g(z)|}{1-|z|} dV(z) \le C \int_S A_p(f)(\zeta)\, C_{p'}(g)(\zeta)\, d\sigma(\zeta), \quad 1 < p \le \infty,$$

where $p' = p/(p-1)$ is the conjugate index.

Lemma B. ([6, 12]) For $0 < q < \infty, \alpha > 0, \beta > 0$ and a function $f(z)$ measurable in B

$$\left\| C_q\big(|f(z)|(1-|z|)^\alpha\big) \right\|_{L^\infty}^q \asymp \sup_{w\in B} (1-|w|)^\beta \int_B \frac{|f(z)|^q (1-|z|)^{\alpha q - 1}}{|1-\overline{w}z|^{\beta+n}} dV(z).$$

Theorem 3. *Let $0 < q < 2$, $q < p$, $\gamma > 0$, $0 < \alpha < \gamma q/n$. Then for any $\lambda > (p-q)/\alpha$,*

$$\int_B |f(z)|^{p-q} |\mathcal{D}^\gamma f(z)|^q (1-|z|)^{\gamma q - 1} dV(z) \le C \|f\|_{H^\lambda}^{p-q} \left\| \mathcal{D}^{\alpha n/q} f \right\|_{H^q}^q. \qquad (9)$$

Proof. Denote by L the integral on the left-hand side of (9). Choosing any α, $0 < \alpha < \gamma q/n$ and estimating L by Lemma A gives

$$\begin{aligned} L &= \int_B |\mathcal{D}^\gamma f(z)|^q (1-|z|)^{\gamma q - \alpha n} \cdot |f(z)|^{p-q} (1-|z|)^{\alpha n} \frac{dV(z)}{1-|z|} \\ &\le C \int_S A_{2/q}\Big(|\mathcal{D}^\gamma f(z)|^q (1-|z|)^{\gamma q - \alpha n} \Big)(\zeta) \cdot C_{(2/q)'}\Big(|f(z)|^{p-q} (1-|z|)^{\alpha n} \Big)(\zeta)\, d\sigma(\zeta) \\ &\le C \left\| C_{(2/q)'}\Big(|f(z)|^{p-q} (1-|z|)^{\alpha n} \Big) \right\|_{L^\infty} \int_S A_{2/q}\Big(|\mathcal{D}^\gamma f(z)|^q (1-|z|)^{\gamma q - \alpha n} \Big)(\zeta)\, d\sigma(\zeta). \end{aligned}$$

We estimate here L^∞-norm and the last integral separately. By Lemma B, choosing $\beta > 0$ large enough, the L^∞-norm can be estimated as follows

$$\begin{aligned} & \left\| C_{2/(2-q)}\Big(|f(z)|^{p-q} (1-|z|)^{\alpha n} \Big) \right\|_{L^\infty}^{2/(2-q)} \\ \le\ & C \sup_{w\in B} (1-|w|)^\beta \int_B |f(z)|^{2(p-q)/(2-q)} \frac{(1-|z|)^{2\alpha n/(2-q) - 1}}{|1-\overline{w}z|^{\beta+n}} dV(z) \\ \le\ & C \|f\|_{H^\lambda}^{2(p-q)/(2-q)} \sup_{w\in B} (1-|w|)^\beta \int_B \frac{(1-|z|)^{2\alpha n/(2-q) - (2n/\lambda)(p-q)/(2-q) - 1}}{|1-\overline{w}z|^{\beta+n}} dV(z) \\ \le\ & C \|f\|_{H^\lambda}^{2(p-q)/(2-q)} \sup_{w\in B} (1-|w|)^{2n/(2-q)\cdot(\alpha - (p-q)/\lambda)} \\ \le\ & C \|f\|_{H^\lambda}^{2(p-q)/(2-q)}. \end{aligned}$$

where $|f(z)| \le C\|f\|_{H^\lambda}(1-|z|)^{-n/\lambda}$, $z \in B$, and another well-known inequality ([15]) in the unit ball are used. Hence for any $\lambda > (p-q)/\alpha$

$$\left\| C_{2/(2-q)}\Big(|f(z)|^{p-q} (1-|z|)^{\alpha} \Big) \right\|_{L^\infty} \le C \|f\|_{H^\lambda}^{p-q}. \qquad (10)$$

On the other hand,

$$\begin{aligned} J &\equiv \int_S A_{2/q}\Big(|\mathcal{D}^\gamma f(z)|^q(1-|z|)^{\gamma q-\alpha n}\Big)(\zeta)\,d\sigma(\zeta) \\ &= \int_S \left[\int_{\Gamma_\delta(\zeta)} |\mathcal{D}^\gamma f(z)|^2(1-|z|)^{2(\gamma-\alpha n/q)-n-1}\,dV(z)\right]^{q/2} d\sigma(\zeta). \end{aligned}$$

According to a result on fractional differentiation ([11, pp. 179, 186])

$$J \le C\big\|\mathcal{D}^{\alpha n/q} f\big\|^q_{H^q}. \tag{11}$$

This completes the proof of Theorem 3. □

Remark 2. Note that taking $p=2$ and $\gamma=1$ in (9) and formally passing to the limit as $q\to 2-$ and $\alpha\to 0+$ we get the classical Littlewood-Paley inequality for the unit ball.

5 Radial derivative

In this section consider radial derivative

$$\mathcal{R}f(z)=\sum_{k=1}^n z_k\frac{\partial f(z)}{\partial z_k}.$$

If $f\in H(B)$ has a series expansion

$$f(z)=\sum_{k\in Z_+^n} a_k z^k, \qquad z\in B,$$

then

$$\mathcal{R}f(z)=\sum_{k\in Z_+^n} |k| a_k z^k, \qquad z\in B.$$

Theorem 4. *Suppose* $0<q\le p<\infty$, $f\in H(B)$ *and*

$$\mathcal{L}_{p,q}(f)=\int_B |f(z)|^{p-q}|\mathcal{R}f(z)|^q(1-|z|)^{q-1}dV(z).$$

Then

$$|f(0)|^p+\mathcal{L}_{p,q}(f)\le C\|f\|^p_{H^p}, \qquad q\ge 2. \tag{12}$$

Conversely, if $f(0)=0$, *then*

$$\|f\|^p_{H^p}\le C\mathcal{L}_{p,q}(f), \qquad q\le 2. \tag{13}$$

Proof. First we prove that for $q = 2$

$$\|f\|_{H^p}^p \asymp |f(0)|^p + \mathcal{L}_{p,2}(f) \tag{14}$$

(compare with [19]). Indeed, we can apply a one variable analogue of (14) (see, e.g., [24]) to the slice function $f_\zeta(\lambda) = f(\lambda\zeta), \lambda \in U, \zeta \in S$,

$$\|f_\zeta\|_{H^p(U)}^p \asymp |f(0)|^p + \int_U |f_\zeta(\lambda)|^{p-2}|f'_\zeta(\lambda)|^2(1-|z|)dm(\lambda), \tag{15}$$

where dm is the area Lebesgue measure on the unit disk U. Note that

$$f'_\zeta(\lambda) = \lambda^{-1}\mathcal{R}f(\lambda\zeta) \quad \text{and} \quad \mathcal{R}(f_\zeta) = (\mathcal{R}f)_\zeta, \qquad \lambda \in U, \, \zeta \in S. \tag{16}$$

We integrate (15) over the sphere S, making use of (16) and the formula (see, e.g., [1])

$$\int_{\mathbb{C}^n} g(w)|w|^{-2n}dV(w) = n\int_S \left(\int_{\mathbb{C}} g(z\zeta)|z|^{-2}dm(z)\right) d\sigma(\zeta), \tag{17}$$

to obtain

$$\|f\|_{H^p}^p \asymp |f(0)|^p + \int_B |f(w)|^{p-2}|\mathcal{R}f(w)|^2(1-|w|)dV(w), \tag{18}$$

which coincides with (14).

On the other hand, the inequality

$$|f(0)|^p + \int_B |\mathcal{R}f(z)|^p(1-|z|)^{p-1}dV(z) \le C\|f\|_{H^p}^p, \qquad 2 \le p < \infty, \tag{19}$$

is well-known, see, e.g., [1] and also [11] for a more general result. Therefore, by the Riesz–Thorin interpolation theorem (see, e.g., [28]) the inequalities (19) and (18), that is the inequality (12) for $q = 2$ and $q = p$ imply the inequality (12) for all $2 \le q \le p$.

Passing now to the proof of (13) we use a similar interpolation result. Namely, if a function g is in $L^{q_1}(d\mu) \cap L^{q_2}(d\mu)$ for $0 < q_1 < q_2 < \infty$, then $g \in L^q(d\mu)$ for all q, $q_1 \le q \le q_2$, and furthermore there exists a number $\theta \in (0,1)$ such that

$$\|g\|_{L^q} \le \|g\|_{L^{q_1}}^{1-\theta}\|g\|_{L^{q_2}}^{\theta}. \tag{20}$$

To prove (20) we choose θ such that $1/q = (1-\theta)/q_1 + \theta/q_2$, and then apply Hölder's inequality with indices $q_2/(q\theta) > 1$ and $(q_2/(q\theta))' = q_2/(q_2 - q\theta)$.

We use also an inequality converse to (19), see [1, 2, 11],

$$\|f\|_{H^p}^p \le C|f(0)|^p + C\int_B |\mathcal{R}f(z)|^p(1-|z|)^{p-1}dV(z), \qquad 0 < p \le 2. \tag{21}$$

Consider now three cases. If $0 < q < p < 2$, then choosing

$$g(z) = \frac{\mathcal{R}f(z)}{f(z)}(1-|z|), \qquad d\mu = |f(z)|^p\frac{dV(z)}{1-|z|},$$

$$\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{2}, \qquad \text{that is} \qquad \theta = \frac{2(p-q)}{p(2-q)},$$

by (19), (20), (14), we obtain

$$\|f\|_{H^p}^p \le C_p \mathcal{L}_{p,p}(f) = C_p \|g\|_{L^p(d\mu)}^p \le C_p \|g\|_{L^q(d\mu)}^{p(1-\theta)} \|g\|_{L^2(d\mu)}^{p\theta}$$

$$= C_p \big(\mathcal{L}_{p,q}(f)\big)^{p(1-\theta)/q} \big(\mathcal{L}_{p,2}(f)\big)^{p\theta/2}$$

$$\le C_p \big(\mathcal{L}_{p,q}(f)\big)^{p(1-\theta)/q} \|f\|_{H^p}^{p^2\theta/2}.$$

Thus,

$$\|f\|_{H^p}^{p(1-p\theta/2)} \le C_p \big(\mathcal{L}_{p,q}(f)\big)^{p(1-\theta)/q},$$

or

$$\|f\|_{H^p}^p \le C_p \mathcal{L}_{p,q}(f).$$

If $0 < q \le 2 = p$, then we may pass in the last inequality to the limit as $p \to 2-$ because the constant C_p in view of (20) is bounded in $p \in (q, 2)$.

If $0 < q \le 2 < p$, then choosing θ, satisfying

$$\frac{1}{2} = \frac{1-\theta}{q} + \frac{\theta}{p}, \qquad \text{that is} \qquad \theta = \frac{p(2-q)}{2(p-q)},$$

by (21), (20), (14), we obtain

$$\|f\|_{H^p}^p \le C \mathcal{L}_{p,2}(f) = C \|g\|_{L^2(d\mu)}^2 \le C \|g\|_{L^q(d\mu)}^{2(1-\theta)} \|g\|_{L^p(d\mu)}^{2\theta}$$

$$= C \big(\mathcal{L}_{p,q}(f)\big)^{2(1-\theta)/q} \big(\mathcal{L}_{p,p}(f)\big)^{2\theta/p}$$

$$\le C \big(\mathcal{L}_{p,q}(f)\big)^{2(1-\theta)/q} \|f\|_{H^p}^{2\theta}.$$

Thus,

$$\|f\|_{H^p}^{p-2\theta} \le C \big(\mathcal{L}_{p,q}(f)\big)^{2(1-\theta)/q},$$

or

$$\|f\|_{H^p}^{p(1/2-\theta/p)} \le C \big(\mathcal{L}_{p,q}(f)\big)^{(1-\theta)/q}.$$

In all cases (13) follows.

The next result is an analogue and consequence of Theorem 1.

Theorem 5. *Let $\alpha > -1$, $0 < q \le p < \infty$, $f \in H(B)$. Then*

$$|f(0)|^p + \int_B |f(z)|^{p-q} |\mathcal{R}f(z)|^q (1-|z|)^{\alpha+q} dV(z) \asymp \|f\|_{p,\alpha}^p. \tag{22}$$

Proof. For $n = 1$, Theorem 1 asserts that

$$\int_U |f(z)|^p (1-|z|)^\alpha dm(z) \asymp |f(0)|^p + \int_U |f(z)|^{p-q} |f'(z)|^q (1-|z|)^{\alpha+q} dm(z).$$

Apply it to the slice function $f_\zeta(z) = f(z\zeta)$, $z \in U$, $\zeta \in S$,

$$\int_U |f_\zeta(z)|^p (1-|z|)^\alpha dm(z) \asymp |f(0)|^p + \int_U |f_\zeta(z)|^{p-q} |f'_\zeta(z)|^q (1-|z|)^{\alpha+q} dm(z). \tag{23}$$

Using (16) and (17) we integrate (23) over the sphere and obtain (22).

From Theorems 1 and 5 the following corollary follows.

Corollary 1. *Suppose* $\alpha > -1$, $0 < q \le p < \infty$ *and* $f \in H(B)$. *Then*

$$\begin{aligned} \|f\|^p_{p,\alpha} &\asymp |f(0)|^p + \int_B |f(z)|^{p-q} |\mathcal{R}f(z)|^q (1-|z|)^{\alpha+q} dV(z) \\ &\asymp |f(0)|^p + \int_B |f(z)|^{p-q} |\nabla f(z)|^q (1-|z|)^{\alpha+q} dV(z). \end{aligned}$$

Corollary 2. *Suppose* $0 < q \le p < \infty$, $\alpha > -1$ *and* $f \in H(B)$, *then the following relationship holds*

$$||f||^p_{p,\alpha} \asymp |f(0)|^p + \int_0^1 M_p^{p-q}(f,r) M_p^q(\mathcal{R}f,r)(1-r)^{\alpha+q} dr.$$

Proof. The case $p = q$ follows from Theorem 5. Hence assume that $q < p$. From the proof of Theorem 2 for $g = \mathcal{R}f$ and Theorem 5, we obtain that

$$\begin{aligned} ||f||^p_{p,\alpha} &\le C\left(|f(0)|^p + \int_B |f(z)|^{p-q} |\mathcal{R}f(z)|^q (1-|z|)^{\alpha+q} dV(z)\right) \\ &\le C\left(|f(0)|^p + \int_0^1 M_p^{p-q}(f,r) M_p^q(\mathcal{R}f,r)(1-r)^{\alpha+q} dr\right). \end{aligned}$$

The reverse inequality, follows by applying Hölder's inequality with exponents $p/(p-q)$ and p/q to the last integral, and by Theorem 5 for $p = q$.

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