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Lacunary Series in Mixed Norm Spaces in the Disc

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Abstract—The paper establishes a necessary and sufficient condition under which a lacunary series belong to a mixed norm space of functions holomorphic in the unit disc. As a corollary, some sharp pointwise estimates are obtained for lacunary series.

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1. INTRODUCTION

Let $\mathbb{D}$ be the unit disc of the complex plane and let $\mathbb{T}$ be its boundary. By $H(\mathbb{D})$ we denote the set of functions holomorphic in $\mathbb{D}$. For a measurable in $\mathbb{D}$ function $f(z) = f(r\zeta)$, the integral means of the order p as usually are defined by

$$M_p(f;r) = \|f(r\cdot)\|_{L^p(\mathbb{T};dm)}, \quad 0 \le r < 1, \quad 0 < p \le \infty,$$

where dm is the Lebesgue measure on the circle $\mathbb{T}$. The family of holomorphic functions $f(z)$ for which $\|f\|_{H^p} = \sup\limits_{0<r<1} M_p(f;r) < +\infty$, is the usual Hardy space H^p. The quasinormed space $H(p,q,\alpha)$ $(0 < p, q \le \infty,\ \alpha > 0)$ is the set of those functions $f(z)$ holomorphic in $\mathbb{D}$, which possess finite quasinorms

$$\|f\|_{p,q,\alpha} = \begin{cases} \left(\displaystyle\int_0^1 (1-r)^{\alpha q-1} M_p^q(f;r)dr\right)^{1/q}, & 0 < q < \infty, \\ \sup\limits_{0<r<1} (1-r)^{\alpha} M_p(f;r), & q = \infty. \end{cases}$$

The mixed norm spaces $H(p,q,\alpha)$ are closely connected with many known functional spaces as the weighted Hardy spaces $(q = \infty)$, the analytic spaces of Besov, Sobolev, Bloch, Dirichlet etc., see [1], [2]. If $(1-r)^{\alpha} M_p(f;r) = o(1)$ as $r \to 1^-$, then it is said that a holomorphic function f belongs to the little space $H_0(p,\infty,\alpha)$.

Everywhere, the symbols $C(\alpha,\beta,\dots)$, c_α etc. will denote some positive constants which can be different in different formulas and depend only on the mentioned parameters $\alpha,\beta,\dots$. The symbol $A \approx B$ means that there exist some positive constants C_1 and C_2 (the values of which are not essential), such that $C_1|A| \le |B| \le C_2|A|$.

Lemma 1. *Let* $f \in H(p,q,\alpha)$ *for some* $0 < p \le \infty,\ 0 < q < \infty,\ \alpha > 0$, *and let* $f_\rho(z) = f(\rho z)$ *be an expanded function. Then* $\|f - f_\rho\|_{p,q,\alpha} = o(1)$ *as* $\rho \to 1^-$.

The proof is rather standard, and it can be found, for example, in [3, Proposition 2.3], see also [4] for $p = q$.

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Lemma 2. *Let $\alpha > 0$, $p > 0$, $a_k \geq 0$, $I_k = \{j \in \mathbb{N} : 2^k \leq j < 2^{k+1}\}$, $k = 1, 2, \ldots$. Then*

$$\int_0^1 (1-r)^{\alpha-1} \left(\sum_{k=1}^{\infty} a_k r^k \right)^p dr \approx \sum_{k=0}^{\infty} \frac{1}{2^{\alpha k}} \left(\sum_{j \in I_k} a_j \right)^p,$$

where the constants in two-sided estimates $C_i = C_i(p, \alpha)$, $i = 1, 2$ depend only on p and α.

Lemma 3. *Let $p > 0$, $a_k \geq 0$, $N \in \mathbb{N}$. Then*

$$\min\{1, N^{p-1}\} \left(\sum_{k=1}^{N} a_k^p \right) \leq \left(\sum_{k=1}^{N} a_k \right)^p \leq \max\{1, N^{p-1}\} \left(\sum_{k=1}^{N} a_k^p \right).$$

Lemma 2 is proved by M. Mateljević and M. Pavlović [5], while Lemma 3 is a consequence of Hölder's inequality.

A sequence of natural numbers $\{m_k\}_{k=0}^{\infty}$ is called lacunary (in Hadamard's sense), if there exists a constant $\lambda > 1$ such that $\frac{m_{k+1}}{m_k} \geq \lambda$ for all $k = 0, 1, 2, \ldots$ The corresponding power series is called lacunary. Paley's classical theorem characterizes the lacunary series in Hardy spaces.

Theorem A (Paley [6, Chapter V, Theorem 8.20]). *Let $\{m_k\}_{k=0}^{\infty}$ be an arbitrary lacunary sequence and let $f(z)$ be a function holomorphic in $\mathbb{D}$, given by the convergent lacunary series $f(z) = \sum_{k=0}^{\infty} a_k z^{m_k}$. Then, for any p, $0 < p < \infty$, the function f belongs to the Hardy space H^p if and only if $\{a_k\} \in \ell^2$. Moreover, the corresponding norms are equivalent:*

$$\|f\|_{H^p} \approx \left(\sum_{k=0}^{\infty} |a_k|^2 \right)^{1/2}.$$

The lacunary series in classical functional spaces, as Bloch, Besov, Dirichlet spaces, Q-spaces, contemporarily are widely studied, see [7]-[17]. The recent work [18] of the author contains the proof of a version of the following theorem (for higher dimensions, see [19]), which characterizes the lacunary series in the Hardy-Bloch spaces.

Theorem B ([18], [19]) *Let $\{m_k\}_{k=0}^{\infty}$ be an arbitrary lacunary sequence, let $\alpha > 0$, and let $f(z)$ be a holomorphic function in $\mathbb{D}$, given by the convergent lacunary series $f(z) = \sum\limits_{k=0}^{\infty} a_k z^{m_k}$. Then, the following statements are equivalent:*

(a) $f(z) \in H(\infty, \infty, \alpha)$,

(b) $f(z) \in H(p, \infty, \alpha)$ *for some* $p \in (0, \infty)$,

(c) $f(z) \in H(p, \infty, \alpha)$ *for all* $p \in (0, \infty)$,

(d) $\sup\limits_{k \geq 0} \frac{|a_k|}{m_k^{\alpha}} < +\infty$.

Moreover, the corresponding norms are equivalent:

$$\|f\|_{\infty,\infty,\alpha} \approx \|f\|_{p,\infty,\alpha} \approx \sup_{k \geq 0} \frac{|a_k|}{m_k^{\alpha}}.$$

The main aim of this paper is to extend Theorem B to all values $q \in (0, \infty)$, i.e. to the case of mixed norm spaces $H(p, q, \alpha)$. As a corollary, we derive some sharp pointwise estimates for lacunary series from $H(p, q, \alpha)$.

2. THE MAIN RESULTS

Recently, Stević [14, 15] obtained a characterization of lacunary series and their derivatives in $H(p,q,\alpha)$, for finite values of p. In the below Theorems 1 and 2, we generalize and make more exact his results by application of the fractional integro-differentiation of arbitrary order.

Theorem 1. *Let $0<q<\infty$, $\alpha>0$, let $\{m_k\}_{k=0}^{\infty}$ be an arbitrary lacunary sequence, and $f(z)$ be a holomorphic in $\mathbb{D}$ function given by the convergent lacunary series $f(z)=\sum\limits_{k=0}^{\infty} a_k z^{m_k}$. Then, the following statements are equivalent:*

(a) $f(z)\in H(\infty,q,\alpha)$,

(b) $f(z)\in H(p,q,\alpha)$ *for some* $p\in(0,\infty)$,

(c) $f(z)\in H(p,q,\alpha)$ *for all* $p\in(0,\infty)$,

(d) $\sum\limits_{k=0}^{\infty} \frac{|a_k|^q}{m_k^{\alpha q}}<+\infty$.

Moreover, the corresponding norms are equivalent:

$$\|f\|_{\infty,q,\alpha}\approx\|f\|_{p,q,\alpha}\approx\left(\sum_{k=0}^{\infty}\frac{|a_k|^q}{m_k^{\alpha q}}\right)^{1/q}.$$

Proof: The implication (a)⇒(b) is obvious by the elementary embedding $H(\infty,q,\alpha)\subset H(p,q,\alpha)$. The implication (b)⇒(c) follows form the Paley inequalities of Theorem A, which asserts that $M_p(f;r)\approx M_s(f;r)$ for any $s,\ 0<s<\infty$. Let us prove the implication (c)⇒(d). By the condition (c), particularly we have $f(z)\in H(2,q,\alpha)$. Then, by Lemmas 2 and 3 we obtain

$$\begin{aligned}\|f\|^q_{2,q,\alpha}&=\int_0^1(1-r)^{\alpha q-1}\left(\int_{-\pi}^{\pi}|f(re^{i\vartheta})|^2\frac{d\vartheta}{2\pi}\right)^{q/2}dr=\\&=\int_0^1(1-r)^{\alpha q-1}\left(\int_{-\pi}^{\pi}\left|\sum_{k=0}^{\infty}a_kr^{m_k}e^{im_k\vartheta}\right|^2\frac{d\vartheta}{2\pi}\right)^{q/2}dr\geq\\&\geq C\int_0^1(1-r)^{\alpha q-1}\left(\sum_{k=0}^{\infty}|a_k|^2r^{2m_k}\right)^{q/2}dr\geq\\&\geq C\int_0^1(1-r)^{\alpha q-1}\left(\sum_{k=0}^{\infty}|a_k|^2r^{m_k}\right)^{q/2}dr\geq C\sum_{k=0}^{\infty}\frac{1}{2^{k\alpha q}}\left(\sum_{m_j\in I_k}|a_j|^2\right)^{q/2}\geq\\&\geq C\sum_{k=0}^{\infty}\frac{1}{2^{k\alpha q}}\left(\sum_{m_j\in I_k}|a_j|^q\right)\geq C\sum_{k=0}^{\infty}\left(\sum_{m_j\in I_k}\frac{|a_j|^q}{m_j^{\alpha q}}\right)=C\sum_{k=0}^{\infty}\frac{|a_k|^q}{m_k^{\alpha q}},\end{aligned}$$

where $C=C(p,q,\alpha,\lambda)$.

To prove the implication (d)⇒(a), we apply Lemmas 2 and 3, taking into account that the quantity of numbers m_j contained in the interval I_k does not exceed $N=1+[\log_\lambda 2]$, we obtain

$$\|f\|^q_{\infty,q,\alpha}=\int_0^1(1-r)^{\alpha q-1}M^q_\infty(f;r)dr$$

$$
\begin{aligned}
&= \int_0^1 (1-r)^{\alpha q-1}\left(\sup_{\vartheta\in(-\pi,\pi)}\left|\sum_{k=0}^{\infty} a_k r^{m_k} e^{im_k\vartheta}\right|\right)^q dr \\
&\le C\int_0^1 (1-r)^{\alpha q-1}\left(\sum_{k=0}^{\infty}|a_k| r^{m_k}\right)^q dr \\
&\le C\sum_{k=0}^{\infty}\frac{1}{2^{k\alpha q}}\left(\sum_{m_j\in I_k}|a_j|\right)^q \le C\sum_{k=0}^{\infty}\frac{1}{2^{k\alpha q}}\left(\sum_{m_j\in I_k}|a_j|^q\right) \\
&\le C\sum_{k=0}^{\infty}\left(\sum_{m_j\in I_k}\frac{|a_j|^q}{m_j^{\alpha q}}\right) = C\sum_{k=0}^{\infty}\frac{|a_k|^q}{m_k^{\alpha q}},
\end{aligned}
$$

where $C = C(q,\alpha,\lambda)$. This completes the proof of Theorem 1.
Define the Riemann-Liouville fractional integro-differentiation operator

$$
\mathcal{D}^{-\alpha} f(z) = \sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} a_k z^k, \quad z\in\mathbb{D},\ \alpha>0,
$$

$$
\mathcal{D}^{\alpha} f(z) = \sum_{k=0}^{\infty}\frac{\Gamma(k+1+\alpha)}{\Gamma(k+1)} a_k z^k, \quad z\in\mathbb{D},\ \alpha>0,
$$

see [2], [20, Section 22, (22.51)].

Theorem 2. *Let $0<q<\infty,\ \alpha>0,\ \beta\in\mathbb{R}$, let $\{m_k\}_{k=0}^{\infty}$ be an arbitrary lacunary sequence, and $f(z)$ be a holomorphic in $\mathbb{D}$ function, given by the convergent lacunary series $f(z)=\sum\limits_{k=0}^{\infty} a_k z^{m_k}$. Then the following statements are equivalent:*

(a) $\mathcal{D}^{\beta} f(z) \in H(\infty,q,\alpha)$,

(b) $\mathcal{D}^{\beta} f(z) \in H(p,q,\alpha)$ *for some* $p\in(0,\infty)$,

(c) $\mathcal{D}^{\beta} f(z) \in H(p,q,\alpha)$ *for all* $p\in(0,\infty)$;

(d) $\sum\limits_{k=0}^{\infty}\frac{|a_k|^q}{m_k^{(\alpha-\beta)q}} < +\infty$.

Moreover, the corresponding norms are equivalent:

$$
\|\mathcal{D}^{\beta} f\|_{\infty,q,\alpha} \approx \|\mathcal{D}^{\beta} f\|_{p,q,\alpha} \approx \left(\sum_{k=0}^{\infty}\frac{|a_k|^q}{m_k^{(\alpha-\beta)q}}\right)^{1/q}.
$$

Proof: The power expansion of the function $\mathcal{D}^{\beta} f(z)$ is also lacunary. Therefore, we can apply Theorem 1 to each of the functions

$$
\mathcal{D}^{\beta} f(z) = \sum_{k=0}^{\infty}\frac{\Gamma(m_k+1+\beta)}{\Gamma(m_k+1)} a_k z^{m_k},
$$

$$
\mathcal{D}^{-\beta} f(z) = \sum_{k=0}^{\infty}\frac{\Gamma(m_k+1)}{\Gamma(m_k+1+\beta)} a_k z^{m_k}.
$$

Further, Stirling's formula asserts that for $\beta > 0$

$$\frac{\Gamma(m_k+1+\beta)}{\Gamma(m_k+1)} \sim m_k^{\beta} \quad \text{as} \quad k \to \infty.$$

Consequently,

$$\|\mathcal{D}^{\beta} f\|_{p,q,\alpha}^q \approx \sum_{k=0}^{\infty} \left(\frac{\Gamma(m_k+1+\beta)}{\Gamma(m_k+1)}\right)^q \frac{|a_k|^q}{m_k^{\alpha q}} \approx \sum_{k=0}^{\infty} \frac{|a_k|^q}{m_k^{(\alpha-\beta)q}}.$$

Remark 1. *It is easy to see that Theorem 2 remains true if we replace the Riemann-Liouville operator $\mathcal{D}^{\beta}$ by Hadamard's fractional operator*

$$\mathcal{F}^{\alpha} f(z) = \sum_{k=0}^{\infty} (1+k)^{\alpha} a_k z^k, \quad z \in \mathbb{D},\ \alpha \in \mathbb{R}.$$

Remark 2. *Substituting $\beta - \alpha$ for α, one can be convinced that Theorem 2 covers all Besov spaces and generalizes the preceding analogous results from [7, 8, 13, 17].*

Theorems A and B permit analogous generalizations with fractional derivatives. We omit their proofs.

Theorem 3. *Let $\{m_k\}_{k=0}^{\infty}$ be an arbitrary lacunary sequence, $\beta \in \mathbb{R}$, and let $f(z)$ be a holomorphic in $\mathbb{D}$ function given by the convergent lacunary series $f(z) = \sum_{k=0}^{\infty} a_k z^{m_k}$. Then, for any p, $0 < p < \infty$, the function $\mathcal{D}^{\beta} f$ belongs to the Hardy class H^p if and only if $\{m_k^{\beta} a_k\} \in \ell^2$. Moreover, the corresponding norms are equivalent:*

$$\|\mathcal{D}^{\beta} f\|_{H^p} \approx \left(\sum_{k=0}^{\infty} m_k^{2\beta} |a_k|^2\right)^{1/2}.$$

Theorem 4. *Let $\{m_k\}_{k=0}^{\infty}$ be an arbitrary lacunary sequence, $\alpha > 0$, $\beta \in \mathbb{R}$, and let $f(z)$ be a holomorphic in $\mathbb{D}$ function given by the convergent lacunary series $f(z) = \sum_{k=0}^{\infty} a_k z^{m_k}$. Then, the following statements are equivalent:*

(a) *$\mathcal{D}^{\beta} f(z) \in H(\infty, \infty, \alpha)$,*

(b) *$\mathcal{D}^{\beta} f(z) \in H(p, \infty, \alpha)$ for some $p \in (0, \infty)$,*

(c) *$\mathcal{D}^{\beta} f(z) \in H(p, \infty, \alpha)$ for all $p \in (0, \infty)$,*

(d) *$\sup\limits_{k \geq 0} \frac{|a_k|}{m_k^{\alpha-\beta}} < +\infty$.*

Moreover, the corresponding norms are equivalent:

$$\|\mathcal{D}^{\beta} f\|_{\infty,\infty,\alpha} \approx \|\mathcal{D}^{\beta} f\|_{p,\infty,\alpha} \approx \sup_{k \geq 0} \frac{|a_k|}{m_k^{\alpha-\beta}}.$$

Theorem 5. *Let $\{m_k\}_{k=0}^{\infty}$ be an arbitrary lacunary sequence, $\alpha > 0$, $\beta \in \mathbb{R}$, and let $f(z)$ be a holomorphic function in $\mathbb{D}$, given by the convergent lacunary series $f(z) = \sum_{k=0}^{\infty} a_k z^{m_k}$. Then, the following statements are equivalent:*

(a) $\mathcal{D}^\beta f(z) \in H_0(\infty,\infty,\alpha)$,

(b) $\mathcal{D}^\beta f(z) \in H_0(p,\infty,\alpha)$ *for some* $p \in (0,\infty)$,

(c) $\mathcal{D}^\beta f(z) \in H_0(p,\infty,\alpha)$ *for all* $p \in (0,\infty)$,

(d) $\lim\limits_{k\to\infty} \frac{|a_k|}{m_k^{\alpha-\beta}} = 0$.

Remark 3. *It should be mentioned, that Theorem 3 includes the case of the holomorphic Hardy-Sobolev spaces [21] containing the holomorphic functions with derivatives from H^p. Theorems 4 and 5 cover all Bloch weighted spaces and little Bloch spaces, by generalizing the preceding results from [7, 8, 10, 13, 17, 22].*

3. POINTWISE ESTIMATES OF LACUNARY SERIES

It is well-known, that any function $f(z) \in H(p,q,\alpha)$ satisfies the pointwise estimate

$$|f(z)| \le C(p,q,\alpha)\frac{\|f\|_{p,q,\alpha}}{(1-|z|)^{\alpha+1/p}}, \quad z \in \mathbb{D}. \tag{3.1}$$

The inequality (3.1) follows from the Hardy-Littlewood estimate

$$M_\infty(f;\rho) \le C(1-\rho)^{-1/p}M_p(f;\rho), \quad 0<\rho<1,$$

which can be found in [1, Theorem 5.9] or [3, Proposition 2.1]. Indeed, it suffices to raise to degree $q<\infty$ of the last inequality and integrate it along the interval $(r,1)$ with weight $(1-\rho)^{\alpha q-1}$:

$$\int_r^1 (1-\rho)^{q/p+\alpha q-1}M_\infty^q(f;\rho)d\rho \le C\int_r^1 (1-\rho)^{\alpha q-1}M_p^q(f;\rho)d\rho, \quad 0<r<1.$$

By monotonicity of the quantity $M_\infty(f;\rho)$, we obtain

$$M_\infty^q(f;r)\int_r^1 (1-\rho)^{q/p+\alpha q-1}d\rho \le C\|f\|_{p,q,\alpha}^q,$$

which immediately implies (3.1).

In (3.1), the degree $\alpha+1/p$ is the best possible for arbitrary functions. Indeed, the embedding $H(p,q,\alpha) \subset H(\infty,\infty,\alpha+1/p-\varepsilon)$ is not true for any small $\varepsilon>0$. One can verify that the function

$$g(z) = \big(1-z\big)^{-(\alpha+1/p)}\left(\log\frac{e}{1-z}\right)^{-2/q}, \quad z \in \mathbb{D},$$

belongs to $H(p,q,\alpha)$, but not to $H(\infty,\infty,\alpha+1/p-\varepsilon)$.

The next theorem shows that the lacunary series from $H(p,q,\alpha)$ have slower growth near the boundary, than any functions from $H(p,q,\alpha)$.

Theorem 6. *Let $0<p,q\le\infty$, $\alpha>0$, let $\{m_k\}_{k=0}^\infty$ be an arbitrary lacunary sequence, and $f(z)$ be a function of $H(p,q,\alpha)$, given by the convergent lacunary series $f(z)=\sum\limits_{k=0}^\infty a_k z^{m_k}$. Then,*

$$|f(z)| \le C(\lambda,p,q,\alpha)\frac{\|f\|_{p,q,\alpha}}{(1-|z|)^{\alpha}}, \quad z \in \mathbb{D}, \tag{3.2}$$

where the degree α can not be decreased.

Proof: By theorem 1, for any function $f \in H(p,q,\alpha)$ with $0 < q < \infty$

$$\sum_{k=0}^{\infty} \frac{|a_k|^q}{m_k^{\alpha q}} \le C\|f\|_{p,q,\alpha}^q \quad \text{and} \quad \frac{|a_k|^q}{m_k^{\alpha q}} \le C\|f\|_{p,q,\alpha}^q, \quad k \ge 0,$$

or

$$|a_k| \le C\|f\|_{p,q,\alpha} m_k^{\alpha} \quad \text{for all} \quad k \ge 0, \tag{3.3}$$

where $C = C(\lambda, p, q, \alpha)$ is independent of f. For $q = \infty$ and a function $f \in H(p, \infty, \alpha)$, the last inequality immediately follows from Theorem B. Thus, the lacunary expansion of $f \in H(p,q,\alpha)$ with $0 < q \le \infty$ satisfies the estimate (3.3). Consequently,

$$|f(z)| \le \sum_{k=0}^{\infty} |a_k||z|^{m_k} \le C\|f\|_{p,q,\alpha} \sum_{k=0}^{\infty} m_k^{\alpha}|z|^{m_k}$$

$$= C\|f\|_{p,q,\alpha} \sum_{k=0}^{\infty} \sum_{m_j \in I_k} m_j^{\alpha}|z|^{m_j}, \tag{3.4}$$

where $I_k = \{i \in \mathbb{N} : 2^k \le i < 2^{k+1}\}$. The number of integers m_j in I_k does not exceed $N = 1 + [\log_\lambda 2]$. Therefore, the inner sum of (3.4) can be estimated as follows:

$$\sum_{m_j \in I_k} m_j^{\alpha}|z|^{m_j} \le N 2^{(k+1)\alpha}|z|^{2^k}.$$

This means that by (3.4)

$$|f(z)| \le CN\|f\|_{p,q,\alpha} \sum_{k=0}^{\infty} 2^{(k+1)\alpha}|z|^{2^k} \approx \frac{\|f\|_{p,q,\alpha}}{(1-|z|)^{\alpha}}.$$

The last estimate can be found, for instance, in [1, p. 66].

Now, let us show that in (3.2) the degree α is the best possible. Suppose there is a degree $\beta,\ 0 < \beta < \alpha$, such that for any lacunary series $f \in H(p,q,\alpha)$ there exists a constant $C > 0$ such that

$$|f(z)| \le \frac{C\|f\|_{p,q,\alpha}}{(1-|z|)^{\beta}}, \quad z \in \mathbb{D}, \tag{3.5}$$

i.e. $f \in H(\infty, \infty, \beta)$. Choosing γ such that $\beta < \gamma < \alpha$, we define the counterexample

$$f_0(z) = \sum_{k=0}^{\infty} 2^{k\gamma} z^{2^k}, \quad z \in \mathbb{D}.$$

By Theorems B and 1, the function $f_0(z)$ belongs to $H(p,q,\alpha)$. On the other hand, $f_0(z) \notin H(\infty,\infty,\beta)$. This contradicts (3.5) and completes the proof.

Though we can not decrease the degree α in (3.2), we can refine the estimates (3.2) in the following sense.

Theorem 7. *Let $0 < p \le \infty$, $0 < q < \infty$, $\alpha > 0$, let $\{m_k\}_{k=0}^{\infty}$ be an arbitrary lacunary sequence, and $f(z)$ be a function of $H(p,q,\alpha)$, given by the convergent lacunary series $f(z) = \sum_{k=0}^{\infty} a_k z^{m_k}$. Then,*

$$f(z) = o\left(\frac{1}{(1-|z|)^{\alpha}}\right) \quad as \quad |z| \to 1^-. \tag{3.6}$$

Proof: By Lemma 1, for any $\varepsilon > 0$ we can choose $\rho_0 \in (0,1)$ close enough to 1, so that

$$\|f - f_\rho\|_{p,q,\alpha} < \varepsilon \quad \text{for all} \quad \rho \in (\rho_0, 1),$$

besides, $(1-r)^\alpha < \varepsilon$ for all $r \in (\rho_0, 1)$. By Theorem 6, the function $f - f_\rho \in H(p,q,\alpha)$ admits the estimate

$$|f(z) - f_\rho(z)| \le C(\lambda, p, q, \alpha) \frac{\|f - f_\rho\|_{p,q,\alpha}}{(1-|z|)^\alpha}, \quad z \in \mathbb{D}.$$

For a fixed $\rho \in (\rho_0, 1)$, we obtain

$$\begin{aligned} (1-|z|)^\alpha |f(z)| &\le (1-r)^\alpha |f(z) - f_\rho(z)| + (1-r)^\alpha |f_\rho(z)| \le \\ &\le C\|f - f_\rho\|_{p,q,\alpha} + C_\rho (1-r)^\alpha < C\varepsilon \end{aligned}$$

for all $r \in (\rho_0, 1)$. The proof of Theorem 7 is complete.
Note that by Theorem 6 the degree α can not be decreased in (3.6).

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