**The Australian Journal of Mathematical Analysis and Applications**

AJMAA

**Volume 9**, Issue 1, Article 16, pp. 1-6, 2012

# A NOTE ON MIXED NORM SPACES OF ANALYTIC FUNCTIONS

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*Received 2 July, 2008; accepted 6 April, 2009; published April, 2012.*

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ABSTRACT. A direct and elementary proof of an estimate of Littlewood is given together with an application concerning the sharpness and strictness of some inclusions in mixed norm spaces of analytic functions.

*Key words and phrases:* Analytic function; Mixed norm space; Sharp inclusions.

2000 *Mathematics Subject Classification.* 30H05, 46E15.

ISSN (electronic): 1449-5910

## 1. INTRODUCTION

Let $\mathbb{D}$ be the unit disc in the complex plane and $\mathbb{T}$ its boundary. If $f(z) = f(re^{i\theta})$ is a measurable function in $\mathbb{D}$, then we write as usual

$$M_p(f;r) = \big\|f(r\cdot)\big\|_{L^p(\mathbb{T};dm)}, \quad 0 \le r < 1, \quad 0 < p \le \infty,$$

where $dm$ is the Lebesgue measure on $\mathbb{T}$. The collection of analytic functions $f(z)$, for which $\|f\|_{H^p} = \sup\limits_{0\le r<1} M_p(f;r) < +\infty$, is the usual Hardy space $H^p$. The quasi-normed space $H(p,q,\alpha)\big(0 < p,q \le \infty, \alpha > 0\big)$ is the set of those functions $f(z)$ analytic in the unit disc $\mathbb{D}$, for which the quasi-norm

$$\|f\|_{p,q,\alpha} = \begin{cases} \left(\displaystyle\int_0^1 (1-r)^{\alpha q-1} M_p^q(f;r)dr\right)^{1/q}, & 0 < q < \infty, \\ \sup\limits_{0\le r<1} (1-r)^{\alpha} M_p(f;r), & q = \infty, \end{cases}$$

is finite. If $(1-r)^{\alpha} M_p(f;r) = o(1)$ as $r \to 1^-$, then we write $f \in H_0(p,\infty,\alpha)$. For $p = q < \infty$ the spaces $H(p,q,\alpha)$ coincide with the well-known weighted Bergman spaces, while $q = \infty$ they are known as growth spaces, and $H_0(p,\infty,\alpha)$ corresponding "little" space.

The mixed norm spaces consisting of harmonic functions will be denoted by $h(p,q,\alpha)$. In [1] among others, some continuous inclusions of Hardy–Littlewood–Flett type in $h(p,q,\alpha)$ are proved in the context of functions $n$-harmonic in the unit polydisc of $\mathbb{C}^n$.

**Theorem 1.** *The following inclusions are continuous for any $\alpha,\beta \in \mathbb{R}$, $0 < p,q \le \infty$:*

$$\begin{array}{rll} (i) & h(p,q,\alpha) \subset h(p,q,\beta), & \beta > \alpha, \\ (ii) & h(p,q,\alpha) \subset h(p_0,q,\alpha), & 0 < p_0 < p \le \infty, \\ (iii) & h(p,q,\alpha) \subset h(p,q_0,\alpha), & 0 < q < q_0 \le \infty, \\ (iv) & h(p,q,\alpha) \subset h(p_0,q,\beta), & \beta \ge \alpha + 1/p - 1/p_0,\ 0 < p \le p_0 \le \infty, \\ (v) & h(p,q,\alpha) \subset h(p_0,q_0,\beta), & \beta > \alpha + 1/p,\ 0 < p_0, q_0 \le \infty, \\ (vi) & h(p,q,\alpha) \subset h(p,q_0,\beta), & \beta > \alpha,\ 0 < q_0 \le \infty, \\ (vii) & H^p \subset H\left(p_0, q, \frac{1}{p} - \frac{1}{p_0}\right), & 0 < p < p_0 \le \infty,\ 0 < p \le q \le \infty. \end{array}$$

*Of course, the inclusions (i), (ii) are obvious, while some others are much deeper, for instance, (iii), (iv) and (vii) which were originally proved by Hardy and Littlewood* [8, Th.31] *and Flett* [5, pp.755-756] *for functions analytic in the unit disc, see also* [4, Th.5.11], [6, Th.3.1], [9].

The purpose of this note is to prove that the inclusions (i)-(vii) for analytic functions are strict and best possible in a certain sense. See Theorem 2 below for the precise formulation.

## 2. ESTIMATES

Throughout the paper, the capital letters $C(\alpha,\beta,\dots), C_\alpha$ stand for different positive constants depending only on the parameters indicated. For $A, B > 0$ the notation $A \approx B$ denotes the two-sided estimate $C_1 A \le B \le C_2 A$ with some inessential positive constants $C_1$ and $C_2$ independent of the variable involved.

The estimates appearing in the next lemma were essentially proved by Littlewood in [10, pp.93-96], see also in [2], [3, p.14]. Such type inequalities are usually proved by means of growth estimates for Taylor coefficients, which were due to Faber and Littlewood [10, pp.93–96], [11, Ch.5,Th.2.31]. Below we give a direct and elementary proof of the estimates avoiding growth estimates for Taylor coefficients.

**Lemma 1.** *Suppose that* $\alpha,\ \beta \in \mathbb{R}$ *and*

$$J_{\alpha,\beta} = J_{\alpha,\beta}(r) := \int_{-\pi}^{\pi} \left|1 - re^{i\theta}\right|^{-\alpha-1} \left|\log \frac{e}{1-re^{i\theta}}\right|^{-\beta} d\theta.$$

*Then for all* $0 \le r < 1$

$$J_{\alpha,\beta} \approx \begin{cases} (1-r)^{-\alpha} \left(\log \dfrac{e}{1-r}\right)^{-\beta}, & \alpha > 0,\ \beta \in \mathbb{R}, \\ 1, & \alpha < 0,\ \beta \in \mathbb{R}, \end{cases} \tag{2.1}$$

$$J_{0,\beta} \approx \begin{cases} \left(\log \dfrac{e}{1-r}\right)^{1-\beta}, & \beta < 1, \\ 1, & \beta > 1, \\ \log \left(e \log \dfrac{e}{1-r}\right), & \beta = 1, \end{cases} \tag{2.2}$$

*where the involved constants* $C = C(\alpha,\beta) > 0$ *depend only on* $\alpha, \beta$.

*Proof.* It suffices to prove all the estimates only for all $r$ close enough to $1$, and moreover for all $z \in \mathbb{D}$ lying in a small neighborhood of $1$. For the expression $\left|1-re^{i\theta}\right| = \sqrt{(1-r)^2 + 4r\sin^2 \frac{\theta}{2}}$, we have the simple estimate

$$\frac{1}{\sqrt{2}} \left(1 - r + 2\sqrt{r}\frac{|\theta|}{\pi}\right) \le \left|1 - re^{i\theta}\right| \le 1 - r + |\theta|, \qquad z = re^{i\theta} \in \mathbb{D},$$

in particular,

$$\frac{1}{\pi}(1 - r + |\theta|) \le \left|1 - re^{i\theta}\right| \le 1 - r + |\theta|, \qquad \frac{1}{2} \le r < 1. \tag{2.3}$$

Define the ring sector $E := \left\{z = re^{i\theta} \in \mathbb{D} :\ \frac{9}{10} < r < 1,\ |\theta| < \frac{1}{2}\right\}$, so that $|1-z| < \dfrac{1}{2}$ $(z \in E)$, and the following inequalities are valid:

$$\begin{aligned} \left|\log \frac{1}{1-z}\right| &\le \log \frac{1}{|1-z|} + \frac{\pi}{2} \le 5 \log \frac{1}{|1-z|}, \qquad z \in E, \\ \left|\log \frac{1}{1-z}\right| &\ge \log \frac{1}{|1-z|} \ge \log \frac{1}{1-r+|\theta|} \ge \log \frac{5}{3} > \frac{1}{2}, \qquad z \in E. \end{aligned} \tag{2.4}$$

Assuming that $\dfrac{9}{10} < r < 1$ everywhere below and $\alpha > 0$, we begin with the proof of the first estimate in (2.1).

By the estimates (2.3) and (2.4), we obtain

$$\begin{aligned} J_{\alpha,\beta} &= \left(\int_{|\theta|>1/2} + \int_{|\theta|<1/2}\right) \frac{d\theta}{\left|1 - re^{i\theta}\right|^{\alpha+1} \left|\log \frac{e}{1-re^{i\theta}}\right|^{\beta}} \\ &\approx C(\alpha,\beta) + C(\alpha,\beta) \int_0^{1/2} \frac{d\theta}{(1-r+\theta)^{\alpha+1} \left(\log \frac{1}{1-r+\theta}\right)^{\beta}} \\ &= C(\alpha,\beta) + C(\alpha,\beta) \int_{\log \frac{1}{3/2-r}}^{\log \frac{1}{1-r}} \frac{e^{\alpha t}}{t^{\beta}}\, dt \approx \int_1^{\log \frac{1}{1-r}} \frac{e^{\alpha t}}{t^{\beta}}\, dt. \end{aligned} \tag{2.5}$$

Here we have used the inequalities

$$0<\log\frac{5}{3}<\log\frac{1}{3/2-r}<\log 2,\qquad \frac{9}{10}<r<1.$$

Since by l'Hôspital rule,

$$\int_1^x \frac{e^{\alpha t}}{t^\beta}dt\sim\frac{e^x}{\alpha\, x^\beta}\qquad \text{as}\qquad x\to+\infty\quad(\alpha>0),$$

we conclude that

$$J_{\alpha,\beta}\approx\frac{e^{\alpha\log\frac{1}{1-r}}}{\left(\log\frac{1}{1-r}\right)^\beta}=\frac{1}{(1-r)^\alpha\left(\log\frac{1}{1-r}\right)^\beta}$$

for all $r$ sufficiently close to $1$. It proves the first inequality in (2.1). The second inequality in (2.1) when $\alpha<0$ follows from (2.5).

We now turn to the proof of (2.2) when $\alpha=0$.

**Case** $\beta<1$**.** Making use of the estimates (2.3) and (2.4), we deduce that

$$\begin{aligned}J_{0,\beta}&=\int_{|\theta|>1/2}+\int_{|\theta|<1/2}\approx C_\beta+C_\beta\int_0^{1/2}\frac{d\theta}{(1-r+\theta)\left(\log\frac{1}{1-r+\theta}\right)^\beta}\\&=C_\beta+C_\beta\left[\left(\log\frac{1}{1-r}\right)^{1-\beta}-\left(\log\frac{1}{3/2-r}\right)^{1-\beta}\right]\\&\approx\left(\log\frac{1}{1-r}\right)^{1-\beta},\end{aligned}\tag{2.6}$$

where we have used the inequalities

$$0<\left(\log\frac{5}{3}\right)^{1-\beta}<\left(\log\frac{1}{3/2-r}\right)^{1-\beta}<(\log 2)^{1-\beta}<\left(\frac{1}{2}\log\frac{1}{1-r}\right)^{1-\beta}$$

for all $\frac{9}{10}<r<1$.

**Case** $\beta=1$**.** In view of (2.3) and (2.4), we obtain for all $r$ close enough to $1$

$$\begin{aligned}J_{0,1}&\approx C+C\int_0^{1/2}\frac{d\theta}{(1-r+\theta)\left(\log\frac{1}{1-r+\theta}\right)}\\&=C+C\left[\log\left(\log\frac{1}{1-r}\right)-\log\left(\log\frac{1}{3/2-r}\right)\right]\\&\approx\log\left(\log\frac{1}{1-r}\right),\end{aligned}\tag{2.7}$$

where

$$\log\log\frac{5}{3}<\log\log\frac{1}{3/2-r}<\log\log 2<0,\qquad\frac{9}{10}<r<1.$$

**Case** $\beta>1$**.** Similarly to (2.5), we have

$$J_{0,\beta}\approx C_\beta+C\int_{\log\frac{1}{3/2-r}}^{\log\frac{1}{1-r}}\frac{1}{t^\beta}\,dt\approx C_\beta+C\int_1^{\log\frac{1}{1-r}}\frac{1}{t^\beta}\,dt\approx 1.\tag{2.8}$$

Combining (2.6)–(2.8), we obtain (2.2). This completes the proof. ∎

## 3. AN APPLICATION IN MIXED NORM SPACES

Define the following test function

$$F_{b,c}(z) := \big(1-z\big)^{-b}\left(\log\frac{e}{1-z}\right)^{-c}, \qquad z\in\mathbb{D},$$

where $b, c\in\mathbb{R}$. The functions $F_{b,c}$ are very useful as typical functions in many function spaces, see, for example, [2]-[7]. The next lemma gives exact information on $F_{b,c}$ to be in $H(p,q,\alpha)$ or $H_0(p,\infty,\alpha)$.

**Lemma 2.** *Suppose that* $b,\ c\in\mathbb{R},\ 0<p\le\infty, 0<q<\infty, \alpha>0$. *Then*
*(a)* $F_{b,c}$ *is in* $H(p,q,\alpha)$ *if and only if* $b<\alpha+\frac{1}{p}, c\in\mathbb{R}$ *or* $b=\alpha+\frac{1}{p}, c>\frac{1}{q}$.
*(b)* $F_{b,c}$ *is in* $H(p,\infty,\alpha)$ *if and only if* $b<\alpha+\frac{1}{p}, c\in\mathbb{R}$ *or* $b=\alpha+\frac{1}{p}, c\ge 0$.
*(c)* $F_{b,c}$ *is in* $H_0(p,\infty,\alpha)$ *if and only if* $b<\alpha+\frac{1}{p}, c\in\mathbb{R}$ *or* $b=\alpha+\frac{1}{p}, c>0$.

*Proof.* The results follow from corresponding estimates of Lemma 1,

$$M_p(F_{b,c};r)\approx(1-r)^{-b+1/p}\left(\log\frac{e}{1-r}\right)^{-c}, \qquad 0\le r<1,$$

if $1/p<b\le\alpha+1/p$. ∎

Lemma 2 enables us to prove the sharpness and strictness of the inclusions (i)-(vii) in Theorem 1.

**Theorem 2.** *Suppose that* $0<p,q,p_0,q_0\le\infty,\ \alpha,\beta>0$ *are arbitrary. Then:*

$(i)$ $H(p,q,\alpha)\subset H(p,q,\beta),\ \beta>\alpha,$ *is strict*.
$(ii)$ $H(p,q,\alpha)\subset H(p_0,q,\alpha),\ p_0<p,$ *is strict*.
$(iii)$ $H(p,q,\alpha)\subset H(p,q_0,\alpha),\ q<q_0,$ *is strict, and the inclusion* $H(p,q,\alpha)\subset H_0(p,\infty,\alpha)$ *is sharp in the sense that* $\alpha$ *on the right cannot be decreased*.
$(iv)$ $H(p,q,\alpha)\subset H(p_0,q,\beta),\ p\le p_0,$ *holds if and only if* $\beta\ge\alpha+\frac{1}{p}-\frac{1}{p_0}$.
$(v)$ $H(p,q,\alpha)\subset H(\infty,q_0,\beta),\ \beta>\alpha+1/p,\ q_0<q,$ *is strict and sharp in the sense that* $\beta$ *cannot be decreased*.
$(vi)$ $H(p,q,\alpha)\subset H(p,q_0,\beta),\ \beta>\alpha,\ q_0<q,$ *is strict and sharp in the sense that* $\beta$ *cannot be decreased*.
$(vii)$ $H^p\subset H(p_0,q,1/p-1/p_0),\ p<p_0,p\le q,$ *is sharp in the sense that it fails for* $p>q$.

*Proof.* (i) The inclusion (i) is strict because of the function $F_{\alpha+1/p,0}$ for $q<\infty$, and the function $F_{\beta+1/p,0}$ for $q=\infty$.

(ii) The strictness of the inclusion (ii) is proved by the examples $F_{\alpha+1/p,0}$ for $0<q<\infty$, and $F_{\alpha+1/p_0,0}$ for $q=\infty$.

(iii) The strictness of the inclusion (iii) is proved by the examples $F_{\alpha+1/p,0}$ for $q_0=\infty$, and $F_{\alpha+1/p,1/q}$ for $q_0<\infty$. The sharpness of the second inclusion in (iii) means that the inclusion $H(p,q,\alpha)\subset H_0(p,\infty,\alpha-\varepsilon)$ is false for any $0<p\le\infty, 0<q<\infty, 0<\varepsilon<\alpha$. The function $F_{\alpha+1/p-\varepsilon/2,0}(z)$ gives a corresponding example.

(iv) The statement (iv) is proved in [1, p.733].

(v)-(vi) The inclusions (v) and (vi) are strict because of the example $F_{\alpha+1/p,0}$. On the other hand, the inclusions (v) and (vi) are sharp for $q_0<q$ in the sense that $\beta$ cannot be decreased. The function $F_{\alpha+1/p,1/q_0}(z)$ gives a suitable example for both inclusions.

(vii) The inclusion (vii) is sharp in the sense that the condition $p \leq q$ is essential, that is for $p > q$ the inclusion (vii) fails. A corresponding example can be provided by the function $F_{1/p,\lambda}(z)$, where $1/p < \lambda < 1/q$. Indeed, $F_{1/p,\lambda}(z) \in H^p$, but $F_{1/p,\lambda}(z)$ is not in $H(p_0, q, 1/p - 1/p_0)$, by Lemma 2. ∎

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