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Vol. 47, No. 5, 2012
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Real and Complex Analysis

ISSN 1068-3623, Journal of Contemporary Mathematical Analysis, 2012, Vol. 47, No. 5, pp. 209–220. © Allerton Press, Inc., 2012.
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Continuous Embeddings in Harmonic Mixed Norm Spaces on the Unit Ball in $\mathbb{R}^n$

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Received December 20, 2011

Abstract—In this paper continuous embeddings in spaces of harmonic functions with mixed norm on the unit ball in $\mathbb{R}^n$ are established, generalizing some Hardy-Littlewood embeddings for similar spaces of holomorphic functions in the unit disc. Differences in indices between the spaces of harmonic and holomorphic spaces are revealed. As a consequence an analogue of classical Fejér-Riesz inequality is obtained. Embeddings in the special case of Riesz systems are also established.

MSC2010 numbers : 31B05
DOI: 10.3103/S1068362312040012

Keywords: *Harmonic function; unit ball in $\mathbb{R}^n$; mixed norm space; Hardy space; embeddings; Fejér-Riesz inequality; Riesz system.*

1. INTRODUCTION

Let $B = B_n$ $(n \geq 2)$ be an open unit ball in $\mathbb{R}^n$ and $S = \partial B$ be its boundary, that is, the unit sphere. The integral averages of order p of a harmonic function $u(x) = u(r\zeta)$ on the sphere $|x| = r$ we denote by

$$M_p(u;r) = \big\|u(r\cdot)\big\|_{L^p(S;d\sigma)}, \quad 0 \leq r < 1, \quad 0 < p \leq \infty,$$

where $d\sigma$ is the $(n-1)$-dimensional normed spherical Lebesgue measure on S i.e. $\sigma(S) = 1$. The set of all harmonic functions on the unit ball B we denote by $h(B)$. Observe that the class of harmonic functions $u \in h(B)$ satisfying

$$\|u\|_{h^p} = \sup_{0<r<1} M_p(u;r) < +\infty,$$

is the classical Hardy space $h^p(B)$ on the unit ball B.

We define the mixed norm space $h(p,q,\alpha)\big(0 < p, q \leq \infty, \alpha \in \mathbb{R}\big)$ to be the space of those harmonic functions $u \in h(B)$ for which the pre-norm

$$\|u\|_{p,q,\alpha} = \begin{cases} \left(\displaystyle\int_0^1 (1-r)^{\alpha q-1} M_p^q(u;r)dr\right)^{1/q}, & 0 < q < \infty, \\ \displaystyle\sup_{0<r<1} (1-r)^{\alpha} M_p(u;r), & q = \infty. \end{cases}$$

is finite. If $(1-r)^\alpha M_p(u;r) = o(1)$ for $r \to 1^-$, then we say that the harmonic function $u(x)$ belongs to a small space $h_0(p,\infty,\alpha)$. Notice that for $p = q < \infty$ the spaces $h(p,q,\alpha)$ coincide with weighted Bergman classes, while for $q = \infty$ these spaces are referred as weighted Hardy spaces.

The mixed norm spaces $h(p,q,\alpha)$ and their subspaces consisting of holomorphic, pluriharmonic and harmonic functions in the disc, and on the ball in $\mathbb{C}^n$ or $\mathbb{R}^n$ were extensively discussed in the literature.

The mixed norm spaces for holomorphic in the unit disc functions were introduced by Hardy and Littlewood in [1], [2].

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Embeddings with different indices and relationships with usual Hardy classes $H^p(B_2)$ were the first results for mixed norm spaces. Some of the results in this area can be found in the monograph by Duren [3]. Flett [4], [5] has considerably extended, strengthened, and suggested simple proofs for a number of original results by Hardy and Littlewood. Later multidimensional extensions of results by Hardy and Littlewood, and Flett were obtained for domains in $\mathbb{C}^n$ and $\mathbb{R}^n$. The spaces $h(p,p,\alpha), h(p,q,\alpha)$ on the unit ball in $\mathbb{R}^n$ were considered in [6-11] (see also references therein). The spaces $h(p,q,\alpha)$ consisting of n-harmonic functions on polydisc in $\mathbb{C}^n$, were studied in papers [12], [13]. It is known (see [14]) that for $0<p<1$ the Hardy classes $h^p(B_n)$ have complicated structure and essentially differ from Hardy classes $h^p(B_n)$ with $p \geq 1$. Similar phenomenon we observe for spaces $h(p,q,\alpha)$. If $1 \leq p,q \leq \infty$, then $h(p,q,\alpha)$ are Banach spaces with norm $\|\cdot\|_{p,q,\alpha}$. If either $0<p<1$ or $0<q<1$, then $h(p,q,\alpha)$ are complete metric spaces with invariant metric $d(u,v) = \|u-v\|_{p,q,\alpha}^{\min\{p,q\}}$ and pre-norm $\|\cdot\|_{p,q,\alpha}$.

In this paper we establish Hardy-Littlewood type continuous embeddings for spaces $h(p,q,\alpha)$ on the ball in $\mathbb{R}^n$. The main result of the paper is the following theorem.

Theorem 1.1. *For any $\alpha,\beta \in \mathbb{R}$, $0<p,q\leq\infty$ the following embeddings are continuous:*

$$
\begin{array}{rll}
(\mathbf{i}) & h(p,q,\alpha) \subset h(p,q,\beta), & \beta > \alpha, \\
(\mathbf{ii}) & h(p,q,\alpha) \subset h(p_0,q,\alpha), & 0 < p_0 < p \leq \infty, \\
(\mathbf{iii}) & h(p,q,\alpha) \subset h(p,q_0,\alpha), & 0 < q < q_0 \leq \infty, \\
(\mathbf{iv}) & h(p,q,\alpha) \subset h(p_0,q,\beta), & \beta \geq \alpha + \dfrac{n-1}{p} - \dfrac{n-1}{p_0},\ p \leq p_0 \leq \infty, \\
(\mathbf{v}) & h(p,q,\alpha) \subset h(\infty,q_0,\beta), & \beta > \alpha + \dfrac{n-1}{p},\ 0 < q_0 \leq \infty, \\
(\mathbf{vi}) & h(p,q,\alpha) \subset h(p,q_0,\beta), & \beta > \alpha,\ 0 < q_0 \leq \infty, \\
(\mathbf{vii}) & h^p \subset h\left(p_0, \infty, \dfrac{n-1}{p} - \dfrac{n-1}{p_0}\right), & 0 < p < p_0 \leq \infty, \\
(\mathbf{viii}) & h^p \subset h\left(p_0, q, \dfrac{n-1}{p} - \dfrac{n-1}{p_0}\right), & 1 < p < p_0 \leq \infty,\ p \leq q \leq \infty, \\
(\mathbf{ix}) & h^p \subset h(p_0,q,\beta), & \beta > \dfrac{n-1}{p} - \dfrac{n-1}{p_0},\ 0 < p < p_0 \leq \infty. \\
(\mathbf{x}) & \textit{If } u \in h(p,q,\alpha),\ 0 < q < \infty, & \textit{then } u \in h_0(p,\infty,\alpha).
\end{array}
$$

2. PRELIMINARY RESULTS

To prove Theorem 1.1 we need a number of lemmas. First we introduce some notation.

The letters $C(\alpha,\beta,\dots)$ and c_α with or without parameters will stand for positive constants that can vary from line to line, and will depend only on specified parameters $\alpha,\beta,\dots$. The notation $A \approx B$ for $A,B>0$ means that there are positive absolute constants c_1 and c_2, such that $c_1 A \leq B \leq c_2 A$. For any p, $1 \leq p \leq \infty$ by $p' = p/(p-1)$ we denote the conjugate index (also, we set $1/\infty = 0$ and $1/0 = +\infty$).

Lemma 2.1. *Let*

$$B_x = \left\{ y = \rho\xi \in \mathbb{R}^n : |y-x| < \frac{1-|x|}{2} \right\} \subset B. \tag{2.1}$$

be the ball with center at $x = r\zeta \in B$ and radius $\frac{1-r}{2}$. Then

$$1-|x| < |\xi - \rho x| < 3(1-|x|) \qquad \textit{for all} \quad y = \rho\xi \in B_x.$$

Proof. The first inequality is obvious. To prove the second inequality, we fix an arbitrary point $x = r\zeta \in B$. If $0 \le r \le \frac{1}{3}$, then the desired inequality is obvious because in this case $|\xi - \rho x| < 2$ and $1 - r \ge \frac{2}{3}$. Now let $\frac{1}{3} < r < 1$. Observe that the origin lies outside the ball B_x, and

$$\frac{1}{2}(1-r) < 1 - |y| < \frac{3}{2}(1-r), \quad \rho'' - \rho' = 1 - r, \quad 0 < \rho' < |y| < \rho'', \quad y = \rho\xi \in B_x,$$

where

$$\rho' = r - \frac{1-r}{2} = \frac{3r-1}{2}, \qquad \rho'' = r + \frac{1-r}{2} = \frac{1+r}{2}. \tag{2.2}$$

By $\theta \in (0, \pi/2)$ we denote the angle between vectors ζ and ξ. It is easy to see that

$$\cos\frac{\theta}{2} > \frac{\sqrt{2}}{2}, \quad \sin^2\frac{\theta}{2} < \frac{\sin^2\theta}{2}, \quad \sin\theta = \frac{1-r}{2r}.$$

Hence we have

$$\begin{aligned} |\xi - \rho x|^2 &= 1 + \rho r - 2\rho r\cos\theta = (1-\rho r)^2 + 4\rho r \sin^2\frac{\theta}{2} < \\ &< (1-\rho' r)^2 + 2\rho r\left(\frac{1-r}{2r}\right)^2 = \left(1 - \frac{3r-1}{2}r\right)^2 + \frac{\rho}{2r}(1-r)^2 = \\ &= (1-r)^2(1+3r/2)^2 + \frac{\rho}{2r}(1-r)^2 < \frac{31}{4}(1-r)^2 < 9(1-r)^2, \end{aligned}$$

and the result follows.

Lemma 2.2. *Each function $u \in h^p(B)$ with $0 < p \le \infty$ admits the estimate*

$$|u(x)| \le C(p,n)\|u\|_{h^p}(1-|x|)^{-(n-1)/p}, \qquad x \in B,$$

or, in terms of embeddings, $h^p \subset h\left(\infty, \infty, \frac{n-1}{p}\right)$.

Proof. For a fixed point $x \in B$ and a ball B_x (see (1)), we apply Fefferman-Stein inequality [15] $(0 < p < \infty)$ and use (2.2) to obtain

$$|u(x)|^p \le \frac{C(p,n)}{(1-r)^n}\int_{B_x}|u(y)|^p\,dy \le \frac{C(p,n)}{(1-r)^n}\int_{\rho'<|y|<\rho''}|u(y)|^p\rho^{n-1}d\rho d\sigma(\xi) \le \frac{C(p,n)}{(1-r)^n}\int_{\rho'}^{\rho''}M_p^p(u;\rho)d\rho.$$

If $0 \le r \le \frac{1}{3}$, then

$$|u(x)|^p \le \frac{C(p,n)}{(1-r)^n}\int_0^{\rho''}M_p^p(u;\rho)d\rho \le C(p,n)\sup_{0<\rho<2/3}M_p^p(u;\rho) \le C(p,n)\|u\|_{h^p}^p,$$

yielding the desired inequality.

For $\frac{1}{3} < r < 1$, taking into account $\frac{1}{2}(1-r) < 1-\rho < \frac{3}{2}(1-r)$, we can write

$$|u(x)|^p \le \frac{C(p,n)}{(1-r)^n}\int_{\rho'}^{\rho''}M_p^p(u;\rho)d\rho \le \frac{C(p,n)}{(1-r)^n}(\rho''-\rho')\sup_{\rho'<\rho<\rho''}M_p^p(u;\rho) \le \frac{C(p,n)}{(1-r)^{n-1}}\|u\|_{h^p}^p.$$

This completes the proof.

Lemma 2.3. *If $u \in h^p(B)$ ш $0 < p < p_0 \le \infty$, then*

$$M_{p_0}(u;r) \le C(p,p_0,n)\|u\|_{h^p}(1-r)^{-(n-1)\left(\frac{1}{p}-\frac{1}{p_0}\right)}, \qquad 0 \le r < 1,$$

or, in terms of embeddings, $h^p \subset h\left(p_0, \infty, \frac{n-1}{p} - \frac{n-1}{p_0}\right)$.

Proof. First observe that for $p_0 = \infty$ lemma is reduced to Lemma 2.2. Hence we assume that $0 < p < p_0 < \infty$. Then by Lemma 2.2

$$M_{p_0}^{p_0}(u;r) = \int_S |u(r\zeta)|^{p_0-p}|u(r\zeta)|^p d\sigma(\zeta) \le M_\infty^{p_0-p}(u;\rho)M_p^p(u;\rho) \le$$
$$\le C(p,p_0,n)\Big[\|u\|_{h^p}(1-r)^{-(n-1)/p}\Big]^{p_0-p} M_p^p(u;\rho) \le$$
$$\le C(p,p_0,n)\|u\|_{h^p}^{p_0}(1-r)^{-(n-1)p_0\left(\frac{1}{p}-\frac{1}{p_0}\right)},$$

and the result follows.

Lemma 2.4. *(See [7], [9], [16]) The following integral estimates hold:*

$$\int_S \frac{d\sigma(\xi)}{|\xi - x|^{\alpha+n-1}} \approx \begin{cases} \dfrac{1}{(1-|x|)^\alpha}, & \alpha > 0, \\ 1, & \alpha < 0, \\ \log\dfrac{e}{1-|x|}, & \alpha = 0, \end{cases} \qquad x \in B,$$

$$\int_0^1 \frac{(1-t)^{\alpha-1}}{(1-rt)^\beta}dt \approx \begin{cases} \dfrac{1}{(1-r)^{\beta-\alpha}}, & \beta > \alpha, \\ 1, & \beta < \alpha, \\ \log\dfrac{e}{1-r}, & \beta = \alpha, \end{cases} \qquad 0 \le r < 1,\ \alpha > 0.$$

In all the estimates the implicit constants depend only on parameters α, β, n.

The next lemma shows that the behavior of harmonic function $u(x)$ in a vicinity of origin essentially does not affect to the value of its pre-norm $\|u\|_{p,q,\alpha}$ to within equivalence.

Lemma 2.5. *For any $0 < p, q \le \infty$, $\alpha \in \mathbb{R}$ and for all $u \in h(B)$ the inequalities hold:*

$$\|u\|_{p,q,\alpha} \le C(p,q,\alpha,n)\left(\int_{1/2}^1 (1-r)^{\alpha q-1}M_p^q(u;r)dr\right)^{1/q}, \quad if \quad 0 < q < \infty,$$

$$\|u\|_{p,\infty,\alpha} \le C(p,\alpha,n) \sup_{1/2<r<1} (1-r)^\alpha M_p(u;r), \quad if \quad q = \infty.$$

Proof. It is enough to prove the truncated inequalities

$$\int_0^{1/2}(1-r)^{\alpha q-1}M_p^q(u;r)dr \le C(p,q,\alpha,n)\int_{1/2}^1 (1-r)^{\alpha q-1}M_p^q(u;r)dr, \quad \text{for} \quad 0 < q < \infty,$$

$$\sup_{0<r<1/2}(1-r)^\alpha M_p(u;r) \le C(p,\alpha,n)\sup_{1/2<r<1}(1-r)^\alpha M_p(u;r), \quad \text{for} \quad q = \infty.$$

If $1 \le p < \infty$, then the function $|u|^p$ is subharmonic, and the integral averages $M_p(u;r)$ monotonically increase with respect to r. Hence in this case the proof is evident. Now assume $0 < p < 1$ and consider three subcases.

Case $0 < p < 1$, $q = \infty$. Consider an arbitrary point x, $|x| = 3/4$ and a ball $B_{1/8}(x)$ with center at x and radius $1/8$. By Fefferman-Stein inequality [15] we have

$$|u(x)|^p \le \frac{C(p,n)}{(1/8)^n}\int_{B_{1/8}(x)}|u(y)|^p dy \le C(p,n)\int_{5/8<|y|<7/8}|u(y)|^p dy \le$$

$$\le C(p,n)\int_{5/8}^{7/8} M_p^p(u;\rho)\,d\rho \le C(p,n)\sup_{5/8<\rho<7/8} M_p^p(u;\rho) \le$$
$$\le C(p,\alpha,n)\sup_{1/2<\rho<1}(1-\rho)^{\alpha p}M_p^p(u;\rho), \qquad |x|=3/4.$$

Hence, using the maximum principle for subharmonic function $|u|$, we get

$$\sup_{0<\rho<1/2}(1-\rho)^{\alpha}M_p(u;\rho) \le \sup_{0<\rho<3/4} M_p(u;\rho) \le \sup_{|x|<3/4}|u(x)| =$$
$$= \max_{|x|=3/4}|u(x)| \le C(p,\alpha,n)\sup_{1/2<\rho<1}(1-\rho)^{\alpha}M_p(u;\rho),$$

yielding the result.

Case $0<p<1, 0<p\le q<\infty$**.** We use the inequality, which was proved before

$$|u(x)|^p \le C(p,n)\int_{5/8}^{7/8} M_p^p(u;\rho)\,d\rho, \qquad |x|=3/4. \tag{2.3}$$

Then applying Hölder inequality with indices $\frac{q}{p}$ and $\left(\frac{q}{p}\right)'$, we get

$$|u(x)|^p \le C(p,n)\int_{5/8}^{7/8} M_p^p(u;\rho)\,d\rho \le C(p,q,n)\left(\int_{5/8}^{7/8} M_p^q(u;\rho)\,d\rho\right)^{p/q}.$$

Hence for any point $x,\ |x|=3/4$, we have

$$|u(x)|^q \le C(p,q,n)\int_{5/8}^{7/8} M_p^q(u;\rho)\,d\rho \le C(p,q,\alpha,n)\int_{5/8}^{7/8}(1-\rho)^{\alpha q-1}M_p^q(u;\rho)\,d\rho \le$$
$$\le C(p,q,\alpha,n)\int_{1/2}^{1}(1-\rho)^{\alpha q-1}M_p^q(u;\rho)\,d\rho.$$

Using the maximum principle for subharmonic function $|u|$, we get

$$\int_0^{1/2}(1-\rho)^{\alpha q-1}M_p^q(u;\rho)\,d\rho \le C(q,\alpha)\sup_{0<r<1/2}M_p^q(u;\rho) \le C(q,\alpha)\sup_{|x|<1/2}|u(x)|^q$$
$$\le C(q,\alpha)\sup_{|x|<3/4}|u(x)|^q = C(q,\alpha)\max_{|x|=3/4}|u(x)|^q \le C(p,q,\alpha,n)\int_{1/2}^{1}(1-\rho)^{\alpha q-1}M_p^q(u;\rho)\,d\rho, \tag{2.4}$$

and the result follows.

Case $0<q\le p<1$**.** Again we use the inequality (2.3), but now with q instead of p,

$$|u(x)|^q \le C(q,n)\int_{5/8}^{7/8} M_q^q(u;\rho)\,d\rho, \qquad |x|=3/4.$$

An application of Hölder inequality with indices $\frac{p}{q}$ and $\left(\frac{p}{q}\right)'$, yields

$$|u(x)|^q \le C(q,n)\int_{5/8}^{7/8}\int_S |u(\rho\xi)|^q d\sigma(\xi)\,d\rho \le C(q,n)\int_{5/8}^{7/8}\left(\int_S |u(\rho\xi)|^p d\sigma(\xi)\right)^{q/p} d\rho =$$
$$= C(q,n)\int_{5/8}^{7/8} M_p^q(u;\rho)\,d\rho \le C(q,\alpha,n)\int_{1/2}^{1}(1-\rho)^{\alpha q-1}M_p^q(u;\rho)\,d\rho.$$

The rest of the proof is as in (2.4). This completes the proof.

3. PROOF OF THEOREM 1.1

First observe that, in view of Lemma 2.5, it is enough to prove all the inequalities with norms and pre-norms outside some vicinity of the origin.

(i)-(ii). The embeddings (i) and (ii) are evident. Notice only that the embedding (ii) is evident because the averages $M_p(u;r)$ monotonically increase by p.

(iii) The proof of embedding (iii) we start with the case $q_0=\infty$, that is, we have to prove that $h(p,q,\alpha)\subset h(p,\infty,\alpha)$. Since for $1\le p\le\infty$ the averages $M_p(u;r)$ increase by r, the proof of embedding $h(p,q,\alpha)\subset h(p,\infty,\alpha)$ is trivial:

$$\|u\|^q_{p,q,\alpha}\ge\int_\rho^1(1-r)^{\alpha q-1}M_p^q(u;r)\,dr\ge C(\alpha,q)M_p^q(u;\rho)(1-\rho)^{\alpha q},\qquad 0<\rho<1.$$

Now assume $0<p<1$, and observe that it is enough to prove the inequality

$$\sup_{1/3<r<1}(1-r)^\alpha M_p(u;r)\le C(p,q,\alpha,n)\|u\|_{p,q,\alpha}.$$

Let x, $\frac13<|x|<1$ be a fixed point. As in the proof of Lemma 2.2, the Fefferman-Stein inequality [15] for ball $B_x(1)$, and Lemma 2.1 yield

$$|u(x)|^p\le\frac{C(p,n)}{(1-r)^n}\int_{B_x}|u(y)|^pdy\le C(p,n)\int_{\rho'<|y|<\rho''}\frac{|u(y)|^p}{|\xi-\rho x|^n}dy. \tag{3.1}$$

Integrating, and using Lemma 2.4 and the identity $|\xi-\rho r\zeta|=|\zeta-\rho r\xi|$, we obtain

$$M_p^p(u;r)\le C(p,n)\int_{\rho'<|y|<\rho''}|u(y)|^p\left(\int_S\frac{d\sigma(\zeta)}{|\xi-\rho r\zeta|^n}\right)dy\le$$
$$\le C(p,n)\int_{\rho'<|y|<\rho''}\frac{|u(y)|^p}{1-\rho r}dy\le\frac{C(p,n)}{1-r}\int_{\rho'}^{\rho''}M_p^p(u;\rho)\,d\rho.$$

Case $p<q<\infty$**.** By Hölder inequality with indices $\frac qp$ and $\left(\frac we haveqp\right)'=\frac{1}{1-p/q}$,

$$(1-r)M_p^p(u;r)\le C(p,n)\int_{\rho'}^{\rho''}M_p^p(u;\rho)\,d\rho\le C(p,n)\left(\int_{\rho'}^{\rho''}M_p^q(u;\rho)\,d\rho\right)^{p/q}(\rho''-\rho')^{1-p/q}.$$

This implies

$$M_p(u;r)\le\frac{C(p,n)}{(1-r)^{1/q}}\left(\int_{\rho'}^{\rho''}M_p^q(u;\rho)\,d\rho\right)^{1/q},\qquad\frac13<r<1.$$

Since $\frac12(1-r)<1-\rho<\frac32(1-r)$, we have

$$M_p(u;r)\le\frac{C(p,q,\alpha,n)}{(1-r)^{1/q}(1-r)^{(\alpha q-1)/q}}\left(\int_{\rho'}^{\rho''}(1-\rho)^{\alpha q-1}M_p^q(u;\rho)\,d\rho\right)^{1/q}\le\frac{C\|u\|_{p,q,\alpha}}{(1-r)^\alpha},$$

and the result follows.

Case $q\le p<1$**.** We use the inequality (3.1) with q instead of p,

$$|u(x)|^q\le C(q,n)\int_{\rho'<|y|<\rho''}\frac{|u(y)|^q}{|\xi-\rho x|^n}dy.$$

Replacing x by Qx, where Q is a linear orthogonal transformation $Q:\rho^n\lll\rho^n$, that is, $|Qx|=|x|$ for all $x\in\rho^n$, and taking into account that the spherical measure σ is invariant under rotations, meaning that $\sigma\big(Q(G)\big)=\sigma(G)$ for each Borel set $G\subset S$ and each orthogonal transformation Q, we obtain

$$|u\big(Qx)|^q\le C(q,n)\int_{\rho'}^{\rho''}\int_S\frac{|u(\rho\xi)|^q}{|\xi-\rho Qx|^n}\rho^{n-1}d\rho d\sigma(\xi).$$

The change of variable $\xi \mapsto Q\xi$ in the integral yields

$$|u(Qx)|^q \le C(q,n) \int_{\rho'}^{\rho''} \int_S \frac{|u(\rho Q\xi)|^q}{|\xi - \rho x|^n} \rho^{n-1} d\rho d\sigma(\xi).$$

Applying Minkowski inequality with index $\frac{p}{q} \ge 1$ and the identity with an integral over orthogonal group

$$M_p(F;|z|) = \left(\int |F(Qz)|^p dQ \right)^{1/p}, \qquad z \in B,$$

in view of Lemma 2.4, we find

$$M_p^q(u;r) \le C(q,n) \int_{\rho'}^{\rho''} M_p^q(u;\rho) \int_S \frac{d\sigma(\xi)}{|\xi - \rho x|^n} d\rho$$
$$\le C(q,n) \int_{\rho'}^{\rho''} \frac{M_p^q(u;\rho)}{1-\rho r} d\rho \le \frac{C(q,n)}{1-r} \int_{\rho'}^{\rho''} M_p^q(u;\rho)\, d\rho, \quad \frac{1}{3} < r < 1.$$

Since $\frac{1}{2}(1-r) < 1-\rho < \frac{3}{2}(1-r)$, the arguments of the previous case yield

$$(1-r)^\alpha M_p(u;r) \le C \left(\int_{\rho'}^{\rho''} (1-\rho)^{\alpha q - 1} M_p^q(u;\rho)\, d\rho \right)^{1/q} \le C\|u\|_{p,q,\alpha}. \tag{3.2}$$

This completes the proof of the embedding $h(p,q,\alpha) \subset h(p,\infty,\alpha)$. Now observe that the general case $0 < q < q_0 < \infty$ with a finite q_0 can be reduced to the discussed case via the simple chain of inequalities:

$$\|u\|_{p,q_0,\alpha}^{q_0} \le \|u\|_{p,\infty,\alpha}^{q_0-q} \|u\|_{p,q,\alpha}^q \le C\|u\|_{p,q,\alpha}^{q_0-q}\|u\|_{p,q,\alpha}^q = C\|u\|_{p,q,\alpha}^{q_0}.$$

(iv) First we prove the embedding (iv) for $p_0 = \infty$, that is, the embedding $h(p,q,\alpha) \subset h\left(\infty, q, \alpha + \frac{n-1}{p}\right)$.

For $0 < p < q < \infty$, as in Lemma 2.2 and (iii) we have

$$M_\infty^p(u;r) \le \frac{C(p,n)}{(1-r)^n} \int_{\rho'}^{\rho''} M_p^p(u;\rho)\, d\rho \le \frac{C(p,n)}{(1-r)^{n-1}} \int_{\rho'}^{\rho''} M_p^p(u;\rho) \frac{d\rho}{1-\rho},$$

By Hölder inequality with indices q/p and $(q/p)'$ we get

$$M_\infty^q(u;r) \le \frac{C(p,q,n)}{(1-r)^{(n-1)q/p}} \left(\int_{\rho'}^{\rho''} M_p^p(u;\rho) \frac{d\rho}{1-\rho} \right)^{q/p} \le \frac{C(p,q,n)}{(1-r)^{(n-1)q/p}} \int_{\rho'}^{\rho''} M_p^q(u;\rho) \frac{d\rho}{1-\rho}.$$

Integrating with a weight and then applying Fubini's theorem we obtain

$$\|u\|_{\infty,q,\alpha+(n-1)/p}^q = \int_0^1 (1-r)^{(\alpha + \frac{n-1}{p})q-1} M_\infty^q(u;r)\, dr$$
$$\le C(p,q,n) \int_0^1 (1-r)^{\alpha q - 1} \left(\int_{\rho'}^{\rho''} M_p^q(u;\rho) \frac{d\rho}{1-\rho} \right) dr$$
$$= C(p,q,n) \int_0^1 M_p^q(u;\rho) \left(\int_{2\rho-1}^{(2\rho+1)/3} (1-r)^{\alpha q-1} dr \right) \frac{d\rho}{1-\rho} = C(p,q,\alpha,n)\|u\|_{p,q,\alpha}^q.$$

We omit the proof in the case where $0 < p < q = \infty$ because it is simple and can be carried out as in Lemma 2.2. The case $0 < q \le p \le \infty$ can be proved similarly. The embedding $h(p,q,\alpha) \subset h\left(\infty, q, \alpha + \frac{n-1}{p}\right)$ is proved. Thus, the desired embedding

$$h(p,q,\alpha) \subset h\left(p_0, q, \alpha + \frac{n-1}{p} - \frac{n-1}{p_0}\right), \qquad 0 < p \le p_0 \le \infty,$$

is proved for the boundary case $p_0 = \infty$, and for $p_0 = p$ it is evident. According to a version of Riesz-Thorin interpolation theorem for quasinormed spaces (see [17]–[19]), we obtain the desired embedding for all $0 < p \le p_0 \le \infty$.

(v) The embedding $h(p, q, \alpha) \subset h(\infty, q_0, \beta),\ \beta > \alpha + \frac{n-1}{p}$, $0 < q_0 \le \infty$, follows from the embedding $h(p, q, \alpha) \subset h(\infty, \infty, \alpha + \frac{n-1}{p})$ and the inequality

$$|u(x)| \le C(p, q, \alpha, n) \frac{\|u\|_{p,q,\alpha}}{(1-r)^{\alpha + \frac{n-1}{p}}}, \qquad x = r\zeta \in B.$$

Indeed, for $0 < q_0 < \infty$ we have by Lemma 2.4

$$\|u\|^{q_0}_{\infty,q_0,\beta} = \int_0^1 (1-r)^{\beta q_0 - 1} M^{q_0}_{\infty}(u; r) dr \le$$

$$\le C \|u\|^{q_0}_{p,q,\alpha} \int_0^1 \frac{(1-r)^{\beta q_0 - 1}}{(1-r)^{\alpha q_0 + (n-1) q_0 / p}} dr \le C(p, q, \alpha, \beta, q_0, n) \|u\|^{q_0}_{p,q,\alpha}.$$

(vi) It is enough to prove the embedding (vi) for the most broader class $h(p, \infty, \alpha)$, that is, the embedding:

$$h(p, \infty, \alpha) \subset h(p, q_0, \beta), \qquad \beta > \alpha,\ 0 < q_0 \le \infty,$$

Assume $0 < q_0 < \infty$, and use the embedding (iii) and Lemma 2.4 to obtain

$$\|u\|^{q_0}_{p,q_0,\beta} = \int_0^1 (1-r)^{\beta q_0 - 1} M^{q_0}_p(u; r) dr \le C \|u\|^{q_0}_{p,\infty,\alpha} \int_0^1 \frac{(1-r)^{\beta q_0 - 1}}{(1-r)^{\alpha q_0}} dr \le C(p, \alpha, \beta, q_0, n) \|u\|^{q_0}_{p,\infty,\alpha}.$$

(vii) The embedding (vii) is proved in Lemma 2.3.

(viii) In view of embedding (iii) it is enough to prove the embedding (viii) for $q = p$, that is, for the most narrower class. So we have to prove the embedding

$$h^p \subset h\left(p_0, p, \frac{n-1}{p} - \frac{n-1}{p_0}\right), \qquad 1 < p < p_0 \le \infty. \tag{3.3}$$

Observe also that the case $p_0 = \infty$ follows from the other cases with finite $p_0 < \infty$, because by embedding (iv)

$$h^p \subset h\left(p_0, p, \frac{n-1}{p} - \frac{n-1}{p_0}\right) \subset h\left(\infty, p, \frac{n-1}{p}\right).$$

Thus, it remains to prove the embedding (3.3) for finite $1 < p < p_0 < \infty$.

We use the method developed in [5]. Consider an operator T and a measure ν defined by

$$Tu(r) := (1-r)^{-(n-1)/p_0} M_{p_0}(u; r), \qquad d\nu(r) := (n-1)(1-r)^{n-2} dr.$$

Since $T(u_1 + u_2)(r) \le T(u_1)(r) + T(u_2)(r)$, the operator T is sublinear. By Lemma 2.3 a positive constant $C_0 = C_0(p, p_0, n)$ can be found to satisfy

$$Tu(r) = (1-r)^{-(n-1)/p_0} M_{p_0}(u; r) \le C_0 (1-r)^{-(n-1)/p} \|u\|_{h^p}, \qquad 0 < r < 1.$$

Hence for any fixed $s > 0$ we have

$$\{0 < r < 1;\ Tu(r) > s\} \subset \left\{0 < r < 1;\ C_0 (1-r)^{-(n-1)/p} \|u\|_{h^p} > s\right\} =$$

$$= \left\{0 < r < 1;\ 1 - \left(C_0\, s^{-1} \|u\|_{h^p}\right)^{p/(n-1)} < r\right\} \subset (\lambda, 1),$$

where $\lambda = \max\left\{0; 1 - \left(C_0\, s^{-1} \|u\|_{h^p}\right)^{p/(n-1)}\right\}$. It is clear that for small $s > 0$ we have $\lambda = 0$, and for all sufficiently large $s > 0$

$$\nu\Big(\{0 < r < 1;\ Tu(r) > s\}\Big) \le \nu(\lambda, 1) = C_0^p\, s^{-p} \|u\|^p_{h^p}.$$

This means that the function $Tu(r)$ belongs to the weak space $L^p_{weak}(d\nu)$, that is, the operator T is of weak type (p,p) for all $1<p<\infty$. By Marcinkevich interpolation theorem (see, e.g., [18], [19]) T is an operator of strong type (p,p) for all $1<p<\infty$,

$$C\|u\|^p_{h^p} \ge \|Tu\|^p_{L^p(d\nu)} = (n-1)\int_0^1 |Tu(r)|^p(1-r)^{n-2}dr =$$
$$= (n-1)\int_0^1 (1-r)^{p(n-1)(1/p-1/p_0)-1}M^p_{p_0}(u;r)dr = (n-1)\|u\|^p_{p_0,p,(n-1)(1/p-1/p_0)},$$

which completes the proof of embedding (viii). As an addendum notice that in the limiting case $p=1$ the embedding (viii) $h^1\subset h\big(p_0,1,(n-1)(1-1/p_0)\big)$, $1<p_0\le\infty$, is invalid. The corresponding counterexample is Poisson kernel.

(ix) The embedding (ix) is a combination of the embeddings (vi) and (iv). Indeed, for any $\alpha>0$ we have

$$h^p\subset h(p,q,\alpha)\subset h\left(p_0,q,\alpha+\frac{n-1}{p}-\frac{n-1}{p_0}\right).$$

(x) The statement (x) can be obtained as a limiting case of inequality (3.2), as $r\to 1^-$. This completes the proof of Theorem 1.1.

4. APPLICATIONS OF THEOREM 1.1

Consider the harmonic vectors $U=(u_1,\dots,u_n)$ on the ball B, where $u_j(x)\in h(B)$ are the gradients of some harmonic function. Denote by $\mathcal{H}(B)$ the set of Riesz systems on the ball B, that is, the class of harmonic vectors $U=(u_1,\dots,u_n)$ on B for each of which some function $h(x)\in h(B)$ can be found to satisfy $U=\nabla h$ on B.

The Hardy classes consisting of vector-functions $U\in\mathcal{H}(B)$ we denote by $\mathcal{H}^p(B)$. In the case where for a vector-function $U\in\mathcal{H}(B)$ the mixed "norm" $\|U\|_{p,q,\alpha}<+\infty$ is finite, we will write $U\in\mathcal{H}(p,q,\alpha)$.

Denote by $S^p=S^p(B)$ the class of subharmonic functions $w(x)$ on the ball B satisfying Hardy condition:

$$\|w\|_{S^p}=\sup_{0<r<1}\left(\int_S |w(r\zeta)|^p d\sigma(\zeta)\right)^{1/p}<+\infty,\qquad 0<p<\infty.$$

As it was mentioned in the proof of Theorem 1.1, the Poisson kernel provides a counterexample showing that the embedding (viii) in Theorem 1.1 for harmonic functions is not valid for $p=1$. Meanwhile, the embedding (viii) in Theorem 1.1 is true for all $0<p<\infty$ for holomorphic functions in the unit disc (see [1], [3, Theorem 5.11]), and for holomorphic functions on the polydisc in $\mathbb{C}^n$ (see [13]). The introduced above Riesz systems $U\in\mathcal{H}(B)$ in ρ^n in some way replace holomorphic functions on ρ^2, and for these systems the embeddings stated in Theorem 1.1 in some cases have broader fields of admissible values of the parameters.

Corollary 4.1. *For $\frac{n-2}{n-1}<p<p_0\le\infty$, $p\le q\le\infty$ the continuous embedding holds:*

$$\mathcal{H}^p\subset\mathcal{H}\left(p_0,q,\frac{n-1}{p}-\frac{n-1}{p_0}\right).$$

Proof. The case $p>1$ follows from the similar embedding (viii) for scalar-valued harmonic functions. Let $\frac{n-2}{n-1}<p=q\le 1$ and $U=(u_1,\dots,u_n)\in\mathcal{H}^p(B)$. Denote $\lambda=\lambda_n:=\frac{n-2}{n-1}$. Then, according to a result by Stein and Weiss (see [18], [19]), $W:=|U|^\lambda$ and $W^{p/\lambda}=|U|^p$ are subharmonic functions. Therefore

$$M^{p/\lambda}_{p/\lambda}(W;r)=M^p_p(U;r),\qquad M^{1/\lambda}_{p_0/\lambda}(W;r)=M_{p_0}(U;r),\qquad p_0>\lambda.$$

Since W is a subharmonic function from Hardy class $S^{p/\lambda}$, W possesses a harmonic majorant $v(x) \in h^{p/\lambda}$ on B with equivalent Hardy norms (see [4]):

$$W(x) \le v(x), \quad x \in B, \qquad \|v\|_{h^{p/\lambda}} \le \|W\|_{S^{p/\lambda}}.$$

Hence, denoting $p_1 = \frac{p}{\lambda} > 1$, $p_{01} = \frac{p_0}{\lambda} > 1$, and using the already proved case, we can write

$$\|U\|^p_{p_0,p,(n-1)/p-(n-1)/p_0} = \int_0^1 (1-r)^{p(n-1)(1/p-1/p_0)-1} M^p_{p_0}(U;r)dr =$$
$$= \int_0^1 (1-r)^{p_1(n-1)(1/p_1-1/p_{01})-1} M^{p_1}_{p_{01}}(W;r)dr = \|W\|^{p_1}_{p_{01},p_1,(n-1)/p-(n-1)/p_0}$$
$$\le \|v\|^{p_1}_{p_{01},p_1,(n-1)/p-(n-1)/p_0} \le \|v\|^{p_1}_{h^{p_1}} \le C\|W\|^{p_1}_{S^{p_1}} = C\|U\|^p_{\mathcal{H}^p}.$$

The Riesz systems can be generalized as follows. The gradient of order $m \in \mathbb{N}$ for a sufficiently smooth function $v(x)$, denoted by $\nabla^m v$, is defined to be a vector-function the components of which are the partial derivatives $\partial^\gamma v$, $\gamma = (\gamma_1, \dots, \gamma_n)$ of order $|\gamma| = m$, arranged in some fixed order. Denote by $\mathcal{H}_m(B)$, $m \in \mathbb{N}$, the class of harmonic vectors $U = (u_1, \dots, u_n)$ on the ball B, for each of which a function $h(x) \in h(B)$ can be found to satisfy $U = \nabla^m h$ on B. The Hardy space and the mixed norm space consisting of functions $U \in \mathcal{H}_m(B)$ we denote by $\mathcal{H}^p_m$ and $\mathcal{H}_m(p,q,\alpha)$, respectively.

Corollary 4.2. *For $m \in \mathbb{N}$, $\frac{n-2}{m+n-2} < p < p_0 \le \infty$, $p \le q \le \infty$ the continuous embedding holds:*

$$\mathcal{H}^p_m \subset \mathcal{H}_m\left(p_0, q, \frac{n-1}{p} - \frac{n-1}{p_0}\right).$$

Proof is similar to that of Corollary 4.1. We need only define $\lambda = \lambda_{m,n} := \frac{n-2}{m+n-2}$ (instead of λ_n), and use the fact that $|U|^\lambda$, $U \in \mathcal{H}_m(B)$ is a subharmonic function, according to a result by Calderon and Zygmund [20].

In the special case where $p_0 = \infty$, $1 < p = q < \infty$, the embedding (viii) leads to a strengthened version of Fejér-Riesz inequality.

Corollary 4.3. *If $1 < p < \infty$, then for all $u(x) \in h(B)$*

$$\int_0^1 (1-r)^{n-2} \sup_{\zeta \in S} |u(r\zeta)|^p \, dr \le C(p,n)\|u\|^p_{h^p}, \tag{4.1}$$

or, in terms of embeddings, $h^p \subset h\left(\infty, p, \frac{n-1}{p}\right)$.

The classical Fejér-Riesz inequality can be found, for instance, in monograph [3]. For some, different from (4.1), versions of Fejér-Riesz inequality on the ball B, we refer the papers [21], [22], [8], [9]. Notice also that the result of Corollary 3 contains an answer to question by Stević posed in [9, p.209].

Although the result of Corollary 4.3 is invalid in the limiting case $p = 1$ (according to the example of Poisson kernel), nevertheless, as it follows from Corollaries 4.1 and 4.2, for Riesz system and for some other special harmonic vectors the inequality (4.1) remains valid also for some $p \le 1$.

Corollary 4.4. *If $\frac{n-2}{n-1} < p < \infty$, then for all $U \in \mathcal{H}(B)$*

$$\int_0^1 (1-r)^{n-2} \sup_{\zeta \in S} \big|U(r\zeta)\big|^p \, dr \le C(p,n)\big\|U\big\|^p_{\mathcal{H}^p}, \tag{4.2}$$

or, in terms of embeddings, $\mathcal{H}^p \subset \mathcal{H}\left(\infty, p, \frac{n-1}{p}\right)$.

Corollary 4.5. *If $m \in \mathbb{N}$, $\frac{n-2}{m+n-2} < p < \infty$, then for all $U \in \mathcal{H}_m(B)$ we have (4.2) and the corresponding embedding: $\mathcal{H}^p_m \subset \mathcal{H}_m\left(\infty, p, \frac{n-1}{p}\right)$.*

The most significant difference between the spaces $h(p,q,\alpha)$ in ρ^n and similar spaces of holomorphic functions in the unit disc is the fact that for some $\alpha \le 0$, $0 < p < 1$ the harmonic spaces $h(p,q,\alpha)$ are non-trivial, that is, they contain non-zero functions. This phenomenon was discussed in [14], [13].

Now we are going to find conditions in terms of indices under which the spaces $h(p,q,\alpha)$ are trivial. Notice that an appropriate example of non-trivial function of the space $h(p,q,\alpha)$ for suitable $\alpha \le 0$, $0 < p < 1$ is the Poisson kernel: $P(x) = P(x,\xi) = \frac{1-|x|^2}{|\xi - x|^n}$, $x \in B$, $\xi \in S$. The next lemma contains exact conditions under which the Poisson kernel belongs to the spaces $h(p,q,\alpha)$.

Lemma 4.1. *For* $0 < p \le \infty$, $0 < q < \infty$, $\alpha \in \mathbb{R}$ *the following assertions hold:*

(a) *If* $0 < p < \dfrac{n-1}{n}$, *then*
$$P(x) \in h(p,q,\alpha) \iff \alpha > -1;$$
$$P(x) \in h(p,\infty,\alpha) \iff \alpha \ge -1;$$
$$P(x) \in h_0(p,\infty,\alpha) \iff \alpha > -1;$$

(b) *If* $\dfrac{n-1}{n} < p \le \infty$, *then*
$$P(x) \in h(p,q,\alpha) \iff \alpha > (n-1)(1-1/p);$$
$$P(x) \in h(p,\infty,\alpha) \iff \alpha \ge (n-1)(1-1/p);$$
$$P(x) \in h_0(p,\infty,\alpha) \iff \alpha > (n-1)(1-1/p);$$

(c) *If* $p = \dfrac{n-1}{n}$, *then*
$$P(x) \in h(p,q,\alpha) \iff \alpha > -1;$$
$$P(x) \in h(p,\infty,\alpha) \iff \alpha > -1;$$
$$P(x) \in h_0(p,\infty,\alpha) \iff \alpha > -1.$$

Proof follows from Lemma 2.4 and the sharp estimates of Poisson kernel

$$M_p(P;r) \approx \begin{cases} 1-r, & \text{for} \quad 0 < p < \dfrac{n-1}{n}, \\ (1-r)^{-(n-1)(1-1/p)}, & \text{for} \quad \dfrac{n-1}{n} < p \le \infty, \\ (1-r)\left(\log \dfrac{e}{1-r}\right)^{1/p}, & \text{for} \quad p = \dfrac{n-1}{n}, \end{cases} \qquad 0 \le r < 1.$$

The next theorem contains exact conditions under which the spaces $h(p,q,\alpha)$ are trivial, that is, they contain only zero function.

Theorem 4.1. *The following assertions hold:*

- *If* $1 \le p \le \infty$, $0 < q < \infty$, *then* $h(p,q,\alpha) = \{0\} \iff \alpha \le 0;$
- *If* $1 \le p \le \infty$, *then* $h(p,\infty,\alpha) = \{0\} \iff \alpha < 0;$
- *If* $1 \le p \le \infty$, *then* $h_0(p,\infty,\alpha) = \{0\} \iff \alpha \le 0;$
- *If* $0 < p < \dfrac{n-1}{n}$, $0 < q < \infty$, *then* $h(p,q,\alpha) = \{0\} \iff \alpha \le -1;$
- *If* $0 < p < \dfrac{n-1}{n}$, *then* $h(p,\infty,\alpha) = \{0\} \iff \alpha < -1;$
- *If* $0 < p < \dfrac{n-1}{n}$, *then* $h_0(p,\infty,\alpha) = \{0\} \iff \alpha \le -1;$
- *If* $\dfrac{n-1}{n} \le p < 1$, $0 < q < \infty$, *then* $h(p,q,\alpha) = \{0\} \iff \alpha \le (n-1)(1-1/p);$
- *If* $\dfrac{n-1}{n} < p < 1$, *then* $h(p,\infty,\alpha) = \{0\} \iff \alpha < (n-1)(1-1/p);$
- *If* $\dfrac{n-1}{n} \le p < 1$, *then* $h_0(p,\infty,\alpha) = \{0\} \iff \alpha \le (n-1)(1-1/p);$

- *If* $p = \dfrac{n-1}{n}$, *then* $$h(p,\infty,\alpha) = \{0\} \iff \alpha \le -1.$$

Proof. For $1 \le p \le \infty$ the result follows from monotonicity by r of the integral averages $M_p(u;r)$. The assertions for classes $h_0(p,\infty,\alpha)$ were proved in Theorem 2.11 from [14]. In the remaining cases $0 < p < 1$, and using the embeddings (iii), (vi), (x) from Theorem 1.1 $h(p,q,\alpha) \subset h_0(p,\infty,\alpha)$, the proof can be reduced to [14] and Lemma 2.6.

The main results of this paper were reported on the international conference Harmonic Analysis and Approximations, V, 2011, Tsaghkadzor, Armenia, dedicated to the 75 anniversary of academician N.H.Arakelyan.

REFERENCES

1. G.H.Hardy and J.E. Littlewood, "Some properties of fractional integrals (II)", Math. Z., **34**, 403–439, 1932.
2. G.H. Hardy and J.E. Littlewood, "Theorems concerning mean values of analytic or harmonic functions", Quart. J. Math. (Oxford), **12**, 221–256, 1941.
3. P. Duren, *Theory of H^p Spaces* (Academic Press, New York, London, 1970).
4. T.M. Flett, "Inequalities for the pth mean values of harmonic and subharmonic functions with $p \le 1$", Proc. London Math. Soc., **20**, 249–275, 1970.
5. T.M. Flett, "On the rate of growth of mean values of holomorphic and harmonic functions", Proc. London Math. Soc., **20**, 749–768, 1970.
6. M. Pavlović, "Decompositions of L^p and Hardy spaces of polyharmonic functions", J. Math. Anal. Appl., **216**, 499–509, 1997.
7. M. Jevtić and M. Pavlović, "Harmonic Bergman functions on the unit ball in ρ^n", Acta Math. Hungar., **85**, 81–96, 1999.
8. S. Stević, "On harmonic Hardy spaces and area integrals, J. Math. Soc. Japan, **56**, 339–347, 2004.
9. S. Stević, "On harmonic function spaces. II", J. Comput. Anal. Appl., **10**, 205–228, 2008.
10. A.I. Petrosyan, "On weighted classes of harmonic functions in the unit ball of ρ^n", Complex Variables Theory Appl., **50**, 953–966, 2005.
11. A.I. Petrosyan, "On weighted harmonic Bergman spaces", Demonstratio Math., **41**, 73–83, 2008.
12. K. Avetisyan, "Continuous inclusions and Bergman type operators in n-harmonic mixed norm spaces on the polydisc", J. Math. Anal. Appl., **291**, 727–740, 2004.
13. K. Avetisyan, "Weighted integrals and Bloch spaces of n-harmonic functions on the polydisc", Potential Analysis, **29**, 49–63, 2008.
14. A.B. Aleksandrov, "On boundary decay in the mean of harmonic functions", St. Petersburg Math. J., **7**, 507–542, 1996.
15. C. Fefferman and E.M. Stein, "H^p spaces of several variables", Acta Math., **129**, 137–193, 1972.
16. C.W. Liu and J.H. Shi, "Invariant mean-value property and $\mathcal{M}$-harmonicity in the unit ball of ρ^n", Acta Math. Sinica, **19**, 187–200, 2003.
17. T. Holmstedt, "Interpolation of quasi-normed spaces", Math. Scand., **26**, 177–199, 1970.
18. E.M. Stein, *Singular Integrals and Differentiability Properties of Functions* (Princeton Univ. Press, Princeton, New Jersey 1970).
19. E.M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces* (Princeton Univ. Press, Princeton, New Jersey 1971).
20. A.P. Calderon and A. Zygmund, "On higher gradients of harmonic functions", Studia Math., **24**, 211–226, 1964.
21. N. Du Plessis, "Spherical Fejér–Riesz theorems", J. London Math. Soc., **31**, 386–391, 1956.
22. Y. Sagher, "On the Fejér–F. Riesz inequality in L^p", Studia Math., **61**, 269–278, 1977.