Research Article

Mathematical Methods in the Applied Sciences

Received 22 April 2012 Published online 9 January 2013 in Wiley Online Library

(wileyonlinelibrary.com) DOI: 10.1002/mma.2709
MOS subject classification: 30G35; 31B05

On Riesz systems of harmonic conjugates in $\mathbb{R}^3$

J. Morais*†, K. Avetisyan and K. Gürlebeck

Communicated by S. Georgiev

In continuation of recent studies, we discuss two constructive approaches for the generation of harmonic conjugates to find null solutions to the Riesz system in $\mathbb{R}^3$. This class of solutions coincides with the subclass of monogenic functions with values in the reduced quaternions. Our first algorithm for harmonic conjugates is based on special systems of homogeneous harmonic and monogenic polynomials, whereas the second one is presented by means of an integral representation. Some examples of function spaces illustrating the techniques involved are given. More specifically, we discuss the (monogenic) Hardy and weighted Bergman spaces on the unit ball in $\mathbb{R}^3$ consisting of functions with values in the reduced quaternions. We end up proving the boundedness of the underlying harmonic conjugation operators in certain weighted spaces. Copyright © 2013 John Wiley & Sons, Ltd.

Keywords: quaternion analysis; spherical harmonics; Riesz system; monogenic functions; harmonic conjugates; Hardy space; Bergman space

1. Introduction

Quaternion analysis is a higher dimensional generalization of complex analysis theory to four dimensions. It involves the analysis of quaternion functions that are defined in open subsets of $\mathbb{R}^n$ ($n = 3, 4$) and that are solutions of generalized Cauchy–Riemann or Riesz systems. They are often called monogenic functions. Meanwhile quaternion analysis has become a well-established branch in mathematics and greatly successful in many different directions (including connections with boundary value problems and partial differential equations theory). A thorough treatment of this higher dimensional function theory is listed in the bibliography, for example, Gürlebeck and Sprößig [1,2], Kravchenko and Shapiro [3], Kravchenko [4], Shapiro and Vasilevski [5,6] or Sudbery [7].

Today, a central role in quaternion analysis theory plays the approximation of a monogenic function by monogenic polynomials. Earlier work, going back to Fueter [8–11], was done by means of the notion of hypercomplex variables. Half a century after Brackx, Delanghe, and Sommen [12] and Malonek [13] worked out those variables and succeed to develop a monogenic function by a local approximation (Taylor series) in terms of the so-called Fueter polynomials. Since then, this became a studied object of its own. Leutwiler [14], based on these polynomials, constructed a complete set of polynomial null solutions to the Riesz system in $\mathbb{R}^3$. In the following years, Delanghe generalized directly Leutwiler's results to arbitrary dimensions in the framework of a Clifford algebra [15]. The major difficulty of the approach followed from both authors lies exactly in the fact that the Fueter polynomials are, in general, not orthogonal with respect to the scalar inner product [16,17]. A key step in the evolution of this problem is the introduction of a more suitable basis. For a look at the literature of the topic from the perspective of the last years, the interested reader is referred to [14–28] and elsewhere.

In the meantime, Sudbery [7], Xu [29], Brackx, Delanghe, and Sommen [30], Brackx and Delanghe [31], Avetisyan, Gürlebeck and Sprößig [32], and Morais *et al.* [16,33,34] made significant contributions to the study of the interplay between the notions of harmonic conjugate and monogenic functions. For more on this subject, we refer the reader to [35–40]. The main point in the approach presented in [30,31] as well as Sudbery's formula [7] is the construction of harmonic conjugates in $\mathbb{R}^4$ 'function by function'. Namely, no effort has been devoted to the question to which function spaces these conjugate harmonics and the whole monogenic function belong. In [32], this question was studied for conjugate harmonics via Sudbery's formula in the scale of Bergman spaces. These results are, however, not applicable to functions with values in the reduced quaternions. A recent article [33] (cf. [34]) treats the problem of conjugate harmonicity also, proposing an algorithm for the generation of polynomial solutions to the Riesz system in $\mathbb{R}^3$; it uses a solid spherical monogenics expansion, that is, homogeneous monogenic polynomials which offer a refinement of the notion of solid spherical harmonics. The underlying spherical functions (cf. Sansone [41]) are beautiful and interesting in their own right, and they form a natural bridge between properties of the Legendre and Chebyshev polynomials. Working with such expansion, it becomes possible to overcome problems that lead in [30] and [31] to the necessity to solve a Poisson equation (resulting then in an existence theorem)

Centro de Investigação e Desenvolvimento em Matemática e Aplicações (CIDMA), Universidade de Aveiro, 3810-193 Aveiro, Portugal
**Correspondence to: J. Morais, Centro de Investigação e Desenvolvimento em Matemática e Aplicações (CIDMA), Universidade de Aveiro, 3810-193 Aveiro, Portugal.*
† E-mail: joao.pedro.morais@ua.pt

so that we can express explicitly the general form of a pair of conjugate harmonic functions. Without going into details, we point out that this method leads to the definition of certain bounded operators between spaces of harmonic and monogenic functions.

The present paper is organized as follows. After presenting some definitions and basic properties of quaternion analysis in Section 2, Section 3 reviews the algorithm proposed in [33], and it examines the possibility of setting up concrete a-priori criterions for the given harmonic function that ensure the existence of a 'unique' monogenic function. Besides this, we propose here yet another algorithm to the explicit construction of a pair of conjugate harmonic functions in $\mathbb{R}^3$ through its first coordinate, and we believe it is the simplest and shortest published so far (Section 4). Ultimately, we discuss the (monogenic) Hardy and weighted Bergman spaces on the unit ball in $\mathbb{R}^3$ consisting of functions with values in the reduced quaternions. In addition, we prove the boundedness of the underlying harmonic conjugation operators in the given weighted spaces.

2. Notation and definitions

This section fairly comprises some definitions and basic properties of quaternion analysis. In the presentation here,

$$\mathbb{H} := \{\mathbf{z} = z_0 + z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k},\ z_i \in \mathbb{R},\ i = 0,1,2,3\}$$

is the real quaternion algebra, where the imaginary units $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are subject to the multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1; \qquad \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

Evidently the real vector space $\mathbb{R}^4$ may be embedded in $\mathbb{H}$ by identifying the element $z := (z_0, z_1, z_2, z_3) \in \mathbb{R}^4$ with $\mathbf{z} := z_0 + z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k} \in \mathbb{H}$. Consider the subset $\mathscr{A} := \operatorname{span}_{\mathbb{R}}\{1, \mathbf{i}, \mathbf{j}\} \subset \mathbb{H}$, then the real vector space $\mathbb{R}^3$ may be embedded in $\mathscr{A}$ via the identification of $x := (x_0, x_1, x_2) \in \mathbb{R}^3$ with the reduced quaternion $\mathbf{x} := x_0 + x_1\mathbf{i} + x_2\mathbf{j} \in \mathscr{A}$. In the viewpoint, throughout the text, we will often use the symbol x to represent a point in $\mathbb{R}^3$ and $\mathbf{x}$ to represent the corresponding reduced quaternion. It should be noted, however, that $\mathscr{A}$ is a real vectorial subspace but not a subalgebra of $\mathbb{H}$. Like in the complex case, $\mathbf{Sc}(\mathbf{x}) = x_0$ and $\mathbf{Vec}(\mathbf{x}) = x_1\mathbf{i} + x_2\mathbf{j}$ define the scalar and vector parts of $\mathbf{x}$. The conjugate of $\mathbf{x}$ is the reduced quaternion $\overline{\mathbf{x}} = x_0 - x_1\mathbf{i} - x_2\mathbf{j}$; the norm $|\mathbf{x}|$ of $\mathbf{x}$ is defined by $|\mathbf{x}| = \sqrt{\mathbf{x}\overline{\mathbf{x}}} = \sqrt{\overline{\mathbf{x}}\mathbf{x}} = \sqrt{x_0^2 + x_1^2 + x_2^2}$, and it coincides with its corresponding Euclidean norm as a vector in $\mathbb{R}^3$. In the sequel, let B denote the 3D unit ball centered at the origin, and S its boundary. We say that

$$\mathbf{f} : B \subset \mathbb{R}^3 \longrightarrow \mathscr{A}, \qquad \mathbf{f}(x) = [\mathbf{f}(x)]_0 + [\mathbf{f}(x)]_1\mathbf{i} + [\mathbf{f}(x)]_2\mathbf{j}$$

is a reduced quaternion-valued function or, in other words, an $\mathscr{A}$-valued function, where $[\mathbf{f}]_l$ $(l = 0,1,2)$ are real-valued functions defined in B. Properties (like integrability, continuity or differentiability) of $\mathbf{f}$ are defined componentwise. For a real-differentiable $\mathscr{A}$-valued function $\mathbf{f}$ that has continuous first partial derivatives, the (reduced) quaternionic operators

$$D\mathbf{f} = \frac{\partial \mathbf{f}}{\partial x_0} + \mathbf{i}\frac{\partial \mathbf{f}}{\partial x_1} + \mathbf{j}\frac{\partial \mathbf{f}}{\partial x_2}, \qquad \text{and} \qquad \overline{D}\mathbf{f} = \frac{\partial \mathbf{f}}{\partial x_0} - \mathbf{i}\frac{\partial \mathbf{f}}{\partial x_1} - \mathbf{j}\frac{\partial \mathbf{f}}{\partial x_2}$$

are called, respectively, generalized and conjugate generalized Cauchy–Riemann operators on $\mathbb{R}^3$.

Remark 2.1
For a continuously real-differentiable scalar-valued function, the application of the operator D coincides with the usual gradient, ∇.

To make our definitions and get started, one simple notion is needed. Namely, a continuously real-differentiable $\mathscr{A}$-valued function $\mathbf{f}$ is said to be monogenic if $D\mathbf{f} = 0$, which is equivalent to the system

$$(\mathrm{R}) \quad \begin{cases} \dfrac{\partial [\mathbf{f}]_0}{\partial x_0} - \dfrac{\partial [\mathbf{f}]_1}{\partial x_1} - \dfrac{\partial [\mathbf{f}]_2}{\partial x_2} = 0 \\[2ex] \dfrac{\partial [\mathbf{f}]_0}{\partial x_1} + \dfrac{\partial [\mathbf{f}]_1}{\partial x_0} = 0, \quad \dfrac{\partial [\mathbf{f}]_0}{\partial x_2} + \dfrac{\partial [\mathbf{f}]_2}{\partial x_0} = 0, \quad \dfrac{\partial [\mathbf{f}]_1}{\partial x_2} - \dfrac{\partial [\mathbf{f}]_2}{\partial x_1} = 0 \end{cases}$$

or, in a more compact form:

$$\begin{cases} \operatorname{div} \overline{\mathbf{f}} &= 0 \\ \operatorname{curl} \overline{\mathbf{f}} &= 0. \end{cases}$$

A big step towards is the realization that any monogenic $\mathscr{A}$-valued function is two-sided monogenic. This means it satisfies simultaneously the equations $D\mathbf{f} = \mathbf{f}D = 0$.

We may point out that the 3-tuple $\overline{\mathbf{f}}$ is said to be a system of conjugate harmonic functions in the sense of Stein-Weiß [42, 43], and system (R) is called the Riesz system [44]; it is a historical precursor that generalizes the classical Cauchy–Riemann system in the plane. Following [14], the solutions of the system (R) are customary called (R)-solutions. The subspace of polynomial (R)-solutions of degree n will be denoted by $\mathscr{R}^+(B; \mathscr{A}; n)$. In [14], it is shown that the space $\mathscr{R}^+(B; \mathscr{A}; n)$ has dimension $2n + 3$. We further introduce the real-linear Hilbert space of square integrable $\mathscr{A}$-valued functions defined in B, which we denote by $L^2(B; \mathscr{A}; \mathbb{R})$. Also, $\mathscr{R}^+(B; \mathscr{A}) := L^2(B; \mathscr{A}; \mathbb{R}) \cap \ker D$ will denote the space of square integrable $\mathscr{A}$-valued monogenic functions defined in B.

In our next section, we review a suitable set of special monogenic polynomials, which forms a complete orthogonal system in $\mathscr{R}^+(B;\mathscr{A})$ in the sense of the scalar inner product

$$< \mathbf{f},\mathbf{g} >_{L^2(B;\mathscr{A};\mathbb{R})} := \int_B \mathbf{Sc}(\bar{\mathbf{f}}\,\mathbf{g})\,dV\,, \tag{1}$$

where dV denotes the volume measure of B normalized so that $V(B) = 1$. To simplify matters further, we shall remark that using the embedding of $\mathbb{R}$ in $\mathscr{A}$, the inner product of two scalar-valued functions $f, g : B \longrightarrow \mathbb{R}$ can also be written by using the inner product (1), and it will be denoted simply by $< f, g >_{L^2(B)}$.

For $\mathbf{f}(x) = \mathbf{f}(r\zeta)$ in B $(0 \le r < 1,\ \zeta \in S)$, its integral means are defined by

$$\mathscr{M}_p(\mathbf{f}; r) := \left(\int_S |\mathbf{f}(r\zeta)|^p \, d\sigma(\zeta) \right)^{1/p}, \quad 0 \le r < 1, \quad 0 < p < \infty, \tag{2}$$

where $d\sigma$ is the surface area measure on S normalized so that $\sigma(S) = 1$. We will also denote by $h(B;\mathbb{X})$ the set of harmonic functions on B with values in $\mathbb{X}$ ($\mathbb{X} = \mathbb{R}$ or $\mathscr{A}$). As usual, the Hardy spaces of monogenic or harmonic functions are defined as follows

$$h^p(B) = \{u \in h(B;\mathbb{R}) \text{ or } u \in h(B;\mathscr{A}) : \|u\|_{h^p(B)} < \infty\}$$

$$\mathscr{H}^p(B) = h^p(B) \cap \ker D.$$

The norm in the Hardy space of $\mathbf{f}$ in B is defined by

$$\|\mathbf{f}\|_{h^p(B)} := \sup_{0<r<1} \mathscr{M}_p(\mathbf{f}; r), \quad 1 \le p < \infty.$$

Unless otherwise stated, throughout this paper, the letters $C(\alpha, \beta, \ldots)$, C_p, and the like stand for positive different constants depending only on the parameters indicated not necessarily the same in each instance. For any $A, B > 0$, the notation $A \approx B$ denotes the two-sided estimate $c_1 A \le B \le c_2 A$ with some positive constants c_1 and c_2 independent of the variable involved. For any p so that $1 \le p < \infty$, we define the conjugate index p' as $p' = p/(p-1)$.

In order to state our results, we shall need some further notation. Let $1 < p < \infty$, and $\alpha > -1$. We set the weighted Bergman space of $\mathbf{f}$ on B by

$$L^p_\alpha(B) = \left\{ \mathbf{f} \text{ measurable in } B : \|\mathbf{f}\|^p_{L^p_\alpha(B)} := \int_B (1 - |x|)^\alpha |\mathbf{f}(x)|^p dV(x) < \infty \right\}.$$

Let the subspaces of $L^p_\alpha(B)$ consisting of harmonic or monogenic functions be

$$h^p_\alpha(B) = L^p_\alpha(B) \cap h(B), \qquad \text{and} \qquad \mathscr{H}^p_\alpha(B) = L^p_\alpha(B) \cap \ker D.$$

In polar coordinates, we have $dV(x) = 3r^2 dr d\sigma(\zeta)$. Therefore

$$\|\mathbf{f}\|_{L^p_\alpha(B)} := \left(3 \int_0^1 (1-r)^\alpha \mathscr{M}^p_p(\mathbf{f}; r)\, r^2\, dr \right)^{1/p}.$$

The norm of a monogenic function in the weighted Hardy space is defined by

$$\|\mathbf{f}\|_{h(p,\beta)(B)} := \sup_{0<r<1} (1-r)^\beta \mathscr{M}_p(\mathbf{f}; r), \qquad 1 \le p < \infty, \quad \beta > 0.$$

We define

$$h(p,\beta)(B) = \{u \in h(B;\mathbb{R}) \text{ or } u \in h(B;\mathscr{A}) : \|u\|_{h(p,\beta)(B)} < \infty\},$$

$$\mathscr{H}(p,\beta)(B) = h(p,\beta)(B) \cap \ker D.$$

It should be observed that for $\beta = 0$, we obviously come to the usual Hardy spaces h^p and $\mathscr{H}^p$.

Ultimately, we recall two definitions which will be needed through the text.

Definition 2.1 (see [7,45,46])
Let $\mathbf{f}$ be a continuously real-differentiable $\mathscr{A}$-valued function, $\left(\frac{1}{2}\overline{D}\right)\mathbf{f}$ is called hypercomplex derivative of $\mathbf{f}$.

Definition 2.2
An $\mathscr{A}$-valued monogenic function with an identically vanishing hypercomplex derivative is called a hyperholomorphic constant.

3. Special systems of homogeneous harmonic and monogenic polynomials

The material of this section is mainly inspired by the work of Cação [17,21,22] and Morais *et al.* [26]. The treatment is only introductory because we have not attempted to cover all the ongoing research. For more detailed information, we refer to Refs. [33,47], and [16].

The following constructions are based on the introduction of a standard system of spherical harmonics as considered, for example, in [41]. We use spherical coordinates, $x_0 = r\cos\theta$, $x_1 = r\sin\theta\cos\varphi$, $x_2 = r\sin\theta\sin\varphi$, where $0 < r < \infty$, $0 < \theta \leq \pi$, and $0 < \varphi \leq 2\pi$. In order to explain our standing position, we shall start by recalling a suitable set of homogeneous harmonic polynomials,

$$\left\{r^{n+1}U^l_{n+1}, r^{n+1}V^m_{n+1},\ l = 0,1,\ldots,n+1,\ m = 1,\ldots,n+1\right\}_{n\in\mathbb{N}_0} \tag{3}$$

($\mathbb{N}_0 = \{0,1,2,\ldots\}$) formed by the extensions in the ball of the spherical harmonics

$$U^l_{n+1}(\theta,\varphi) = P^l_{n+1}(\cos\theta)\cos(l\varphi) \quad (l = 0,\ldots,n+1), \qquad V^m_{n+1}(\theta,\varphi) = P^m_{n+1}(\cos\theta)\sin(m\varphi) \quad (m = 1,\ldots,n+1). \tag{4}$$

Here, P_{n+1} stands for the Legendre polynomial of degree $n+1$ and the functions P^l_{n+1}, where $l = 0,\ldots,n+1$, are the associated Legendre functions. In [17] and [20], a special $\mathbb{R}$-linear complete orthonormal system of $\mathscr{A}$-valued monogenic polynomials in the unit ball of $\mathbb{R}^3$ is explicitly constructed by applying the operator $\frac{1}{2}\overline{D}$ to the system (3). Restricting the resulting solid spherical monogenics to the surface of B, we do obtain a system of spherical monogenics, denoted by

$$\left\{\mathbf{X}^l_n,\ \mathbf{Y}^m_n : l = 0,\ldots,n+1,\ m = 1,\ldots,n+1\right\}_{n\in\mathbb{N}_0}. \tag{5}$$

Although $\mathbf{X}^0_n$ is built in terms of the Legendre polynomials whereas $\mathbf{X}^m_n$ are built in terms of the associated Legendre functions, we will still include the treatment of the first into the general case whenever this treatment remains the same.

In a recent paper [26], Morais *et al.* have shown that system (5) may be seen as a refinement of the conventional spherical harmonics, and correspondingly, it constitutes an extension of the role of the well-known Chebyshev and Legendre polynomials (resp. associated Legendre functions). More importantly, it can be explicitly constructed by using recurrence relations and preserves some basic properties in common with holomorphic z-powers. Despite the quite long bibliography that is related to these polynomials, for the fundamental references for the preceding arguments and explicit expressions of these special polynomials see Refs. [16,34], and [26]. We recall from [17,19] and [26] the following properties:

1. The functions $\mathbf{X}^{l,\dagger}_n := r^n\mathbf{X}^l_n$, and $\mathbf{Y}^{m,\dagger}_n := r^n\mathbf{Y}^m_n$ are homogeneous monogenic polynomials;
2. For each $n = 0,1,\ldots$, the polynomials $\mathbf{X}^{l,\dagger}_n$ $(l = 0,\ldots,n+1)$, and $\mathbf{Y}^{m,\dagger}_n$ $(m = 1,\ldots,n+1)$ form a complete orthogonal system in $\mathscr{R}^+(B;\mathscr{A})$, and their norms are explicitly given by

$$\|\mathbf{X}^{l,\dagger}_n\|_{L^2(B;\mathscr{A};\mathbb{R})} = \sqrt{(1+\delta_{l,0})\frac{\pi}{2}\frac{(n+1)}{(2n+3)}\frac{(n+1+l)!}{(n+1-l)!}}, \qquad \|\mathbf{Y}^{m,\dagger}_n\|_{L^2(B;\mathscr{A};\mathbb{R})} = \sqrt{\frac{\pi}{2}\frac{(n+1)}{(2n+3)}\frac{(n+1+m)!}{(n+1-m)!}},$$

   where $\delta_{l,0}$ denotes the Kronecker symbol;
3. For each $n = 0,1,\ldots$, the scalar parts of the polynomials $\mathbf{X}^{l,\dagger}_n$ and $\mathbf{Y}^{m,\dagger}_n$ form a complete orthogonal system in $L^2(B)$, and their norms are explicitly given by

$$\|\mathbf{Sc}(\mathbf{X}^{l,\dagger}_n)\|_{L^2(B)} = \frac{(n+1+l)}{\sqrt{2n+3}}\sqrt{(1+\delta_{l,0})\frac{\pi}{2}\frac{1}{(2n+1)}\frac{(n+l)!}{(n-l)!}}, \qquad \|\mathbf{Sc}(\mathbf{Y}^{m,\dagger}_n)\|_{L^2(B)} = \frac{(n+1+m)}{\sqrt{2n+3}}\sqrt{\frac{\pi}{2}\frac{1}{(2n+1)}\frac{(n+m)!}{(n-m)!}};$$

4. For $n \geq 1$, we have

$$\left(\frac{1}{2}\overline{D}\right)\mathbf{X}^{l,\dagger}_n = (n+l+1)\mathbf{X}^{l,\dagger}_{n-1} \quad (l = 0,\ldots,n), \qquad \left(\frac{1}{2}\overline{D}\right)\mathbf{Y}^{m,\dagger}_n = (n+m+1)\mathbf{Y}^{m,\dagger}_{n-1} \quad (m = 1,\ldots,n),$$

   that is, the hypercomplex differentiation of a basis function delivers a multiple of another basis function one degree lower;
5. The polynomials $\mathbf{X}^{n+1,\dagger}_n$ and $\mathbf{Y}^{n+1,\dagger}_n$ are hyperholomorphic constants.

From now on, we shall denote by $\mathbf{X}^{l,\dagger,*}_n$ and $\mathbf{Y}^{m,\dagger,*}_n$ the new normalized basis functions $\mathbf{X}^{l,\dagger}_n$, $\mathbf{Y}^{m,\dagger}_n$ in $L^2(B;\mathscr{A};\mathbb{R})$.

Based on Statement 2, we can easily write down the Fourier expansion of a square integrable $\mathscr{A}$-valued monogenic function. Furthermore, according to the fact that the polynomials $\mathbf{X}^{n+1,\dagger}_n$ and $\mathbf{Y}^{n+1,\dagger}_n$ are hyperholomorphic constants, in [48], we have proved that each $\mathbf{f} \in \mathscr{R}^+(B;\mathscr{A})$ can be decomposed in an orthogonal sum of a monogenic 'main part' of the function ($\mathbf{g}$) and a hyperholomorphic constant ($\mathbf{h}$). Putting these facts together, next we formulate a modified version of the aforementioned result, which happens to be the more suitable upon further studying the harmonic conjugacy problem. To really understand the aforementioned claims, we strongly recommend the reader to consult [48].

We formulate the result.

Lemma 3.1 (Orthogonal Fourier expansion)
Let $\mathbf{f} \in \mathscr{R}^+(B;\mathscr{A})$. The function $\mathbf{f}$ can be represented in the following way

$$\mathbf{f}(x) := \mathbf{g}(x) + \mathbf{h}(\underline{x})$$
$$= \sum_{n=0}^{\infty}\left(\mathbf{X}_n^{0,\dagger,*}(x)\, a_n^0 + \sum_{m=1}^{n}\left[\mathbf{X}_n^{m,\dagger,*}(x)\, a_n^m + \mathbf{Y}_n^{m,\dagger,*}(x)\, b_n^m\right]\right) + \sum_{n=0}^{\infty}\left[\mathbf{X}_n^{n+1,\dagger,*}(\underline{x})\, a_n^{n+1} + \mathbf{Y}_n^{n+1,\dagger,*}(\underline{x})\, b_n^{n+1}\right],$$

where for each $n \in \mathbb{N}_0$, a_n^l $(l = 0,\dots,n+1)$ and b_n^m $(m = 1,\dots,n+1)$ are the associated (real-valued) Fourier coefficients.

Alternatively, using Parseval's identity, $\mathbf{f}$ may be characterized by its coefficients in the following way:

Theorem 3.1 (Parseval identity)
The function $\mathbf{f}$ is a square integrable $\mathscr{A}$-valued monogenic function iff

$$\sum_{n=0}^{\infty}\left(\left(a_n^0\right)^2 + \left(a_n^{n+1}\right)^2 + \left(b_n^{n+1}\right)^2 + \sum_{m=1}^{n}\left[\left(a_n^m\right)^2 + \left(b_n^m\right)^2\right]\right) < \infty. \tag{6}$$

4. Generation of $\mathscr{A}$-valued monogenic functions by conjugate harmonics

We begin by recalling the notion of harmonic conjugates in the context of quaternion analysis.

Definition 4.1 (Conjugate harmonic functions)
Let U be a harmonic function defined in an open subset Ω of $\mathbb{R}^3$. A vector-valued harmonic function V in Ω is called conjugate harmonic to U if $\mathbf{f} := U + V$ is monogenic in Ω. The pair $(U; V)$ is called a pair of conjugate harmonic functions in Ω.

We recall from [33], an algorithm for the calculation of any $f \in \mathscr{R}^+(B;\mathscr{A})$ via conjugate harmonics. More details can be found in Ref. [34]. To begin with, we shall rephrase the idea behind this construction. In the sequel, assume U be a square integrable harmonic function defined in B. We start by considering the Fourier expansion of U with respect to the complete orthonormal system formed by the scalar parts of the aforementioned monogenic polynomials (see Statement 3 before Lemma 3.1). The next step is to replace the scalar parts of each polynomial by the full polynomial in order to get a Fourier series expansion with respect to the system

$$\left\{\mathbf{X}_n^{l,\dagger}, \mathbf{Y}_n^{m,\dagger} : l = 0,\dots,n+1,\ m = 1,\dots,n+1\right\}_{n\in\mathbb{N}_0}.$$

Doing so, we have to take into account that the full polynomials are not normalized, and we have to correct the coefficients of the series expansion. This results in an additional condition on the original Fourier coefficients of U. We shall encounter similar ideas in the works of Moisil in [49], and Stein and Weiß in [42]. However, our study was based on a special system of spherical monogenics (5), whereas Ref. [42] is based on the gradient of harmonic functions in the upper half-space which are radial in two variables. Therefore, the link is not immediate, and we shall not elaborate on it here.

Let us consider the matter in more detail [33].

Theorem 4.1 (Construction of a harmonic conjugate)
Let $U(x)$ be harmonic and square integrable in B given by

$$U = \sum_{n=0}^{\infty}\left[\frac{\mathbf{Sc}\left(\mathbf{X}_n^{0,\dagger}\right)}{\|\mathbf{Sc}\left(\mathbf{X}_n^{0,\dagger}\right)\|_{L^2(B)}}\, a_n^0 + \sum_{m=1}^{n}\left(\frac{\mathbf{Sc}\left(\mathbf{X}_n^{m,\dagger}\right)}{\|\mathbf{Sc}\left(\mathbf{X}_n^{m,\dagger}\right)\|_{L^2(B)}}\, a_n^m + \frac{\mathbf{Sc}\left(\mathbf{Y}_n^{m,\dagger}\right)}{\|\mathbf{Sc}\left(\mathbf{Y}_n^{m,\dagger}\right)\|_{L^2(B)}}\, b_n^m\right)\right],$$

where for each $n \in \mathbb{N}_0$, a_n^l $(l = 0,\dots,n)$ and b_n^m $(m = 1,\dots,n)$ are the associated Fourier coefficients. If, additionally, the Fourier coefficients satisfy the condition

$$\sum_{n=0}^{\infty}\left(\frac{2n+1}{n+1}\left(a_n^0\right)^2 + \sum_{m=1}^{n}\frac{(n+1)(2n+1)}{(n+1)^2 - m^2}\left[\left(a_n^m\right)^2 + \left(b_n^m\right)^2\right]\right) < \infty, \tag{7}$$

then the series

$$\sum_{n=0}^{\infty}\left[\frac{\mathbf{Vec}\left(\mathbf{X}_n^{0,\dagger}\right)}{\|\mathbf{Sc}\left(\mathbf{X}_n^{0,\dagger}\right)\|_{L^2(B)}}\, a_n^0 + \sum_{m=1}^{n}\left(\frac{\mathbf{Vec}\left(\mathbf{X}_n^{m,\dagger}\right)}{\|\mathbf{Sc}\left(\mathbf{X}_n^{m,\dagger}\right)\|_{L^2(B)}}\, a_n^m + \frac{\mathbf{Vec}\left(\mathbf{Y}_n^{m,\dagger}\right)}{\|\mathbf{Sc}\left(\mathbf{Y}_n^{m,\dagger}\right)\|_{L^2(B)}}\, b_n^m\right)\right] \tag{8}$$

is convergent and defines a square integrable vector-valued harmonic function V conjugate to U such that $\mathbf{f}(x) := U(x) + V(x)$ is an $\mathscr{A}$-valued monogenic function.

As in the complex case, such a construction is obtained step by step, where each elementary step exhibits the classical existence and uniqueness (up to a non-trivial hyperholomorphic constant) of a vector-valued function V conjugate to the scalar-valued U. To see this, observe that by adding any hyperholomorphic constant $\boldsymbol{\varphi}$ to V the resulting function $\tilde{V} := V + \boldsymbol{\varphi}$ is harmonic conjugate to U also. On the other hand, each monogenic $\mathscr{A}$-valued function with vanishing scalar part must be a hyperholomorphic constant.

Of course, if the previous series are finite sums then the functions U and V are polynomials. Then, it is clear that the partial expansion (8) makes always sense. In this special case, our approach covers the results obtained in [30] and [31].

By the direct construction of formula (8), we only get $2n+1$ homogeneous monogenic polynomials of degree n (i.e., the monogenic 'main part' of $\mathbf{f}$). However, because $\dim \mathscr{R}^+(B;\mathscr{A};n) = 2n+3$, adding two hyperholomorphic constants the necessary number of independent polynomials is achieved. In the remainder of this section, we study how the quality of U influences the quality of V, and then how U and V together define a suitable space for $\mathbf{f}$. Such a result will allow the definition of a continuous operator between spaces of harmonic and monogenic functions given by the construction of harmonic conjugates.

In the sequel, we introduce the Sobolev-type space $M_2^1(B;\mathscr{A};\mathbb{R})$ of all functions from $\mathscr{R}^+(B;\mathscr{A})$, whose hypercomplex derivatives also belong to $\mathscr{R}^+(B;\mathscr{A})$. For some information of this space, see [1]. We formulate the first result.

Theorem 4.2

Let U be harmonic and square integrable in B. If the absolute values of its Fourier coefficients a_n^l $(l = 0,\dots,n+1)$ and b_n^m $(m = 1,\dots,n+1)$ satisfy the condition (7) and, additionally, are less than $\frac{c}{(n+1)^{1+\alpha}}$ $(\alpha > 1)$ for some positive constant c, then there exists a monogenic function $\mathbf{f}$ such that $\mathbf{f} \in M_2^1(B;\mathscr{A};\mathbb{R})$ and $[\mathbf{f}]_0 = U$ in B.

Proof

Let $U \in L^2(B)$ be a harmonic function given as in Theorem 4.1. As described, we replace the scalar part of each polynomial by the full polynomial, and by introducing suitable correction factors, we can rewrite the obtained series as a series expansion with respect to the normalized full polynomials. We get

$$\mathbf{f} = \sum_{n=0}^{\infty}\left[\mathbf{X}_n^{0,\dagger,*}\sqrt{\frac{2n+1}{n+1}}\,a_n^0 + \sum_{m=1}^{n}\sqrt{\frac{(n+1)(2n+1)}{(n+1)^2-m^2}}\left(\mathbf{X}_n^{m,\dagger,*}a_n^m + \mathbf{Y}_n^{m,\dagger,*}b_n^m\right)\right].$$

By assumption on the coefficients, $\mathbf{f}$ belongs to $\mathscr{R}^+(B;\mathscr{A})$. We note that the hypercomplex derivative of $\mathbf{f}$ is again monogenic; hence, it remains to prove that $\left(\frac{1}{2}\overline{D}\right)\mathbf{f} \in L^2(B;\mathscr{A};\mathbb{R})$. Because the previous series is convergent in L^2, it converges uniformly to $\mathbf{f}$ in each compact subset of B. Also the series of all partial derivatives converge uniformly to the corresponding partial derivatives of $\mathbf{f}$ in compact subsets of B. Because the operator $\overline{D}$ is continuous, we may apply the hypercomplex derivative $\frac{1}{2}\overline{D}$ term by term to the series, and using Property 4 of the basis polynomials, it finally follows

$$\left(\frac{1}{2}\overline{D}\right)\mathbf{f} = \sum_{n=1}^{\infty}\sqrt{n(2n+3)}\left[\mathbf{X}_{n-1}^{0,\dagger,*}\,a_n^0 + \sum_{m=1}^{n}\left(\mathbf{X}_{n-1}^{m,\dagger,*}\,a_n^m + \mathbf{Y}_{n-1}^{m,\dagger,*}\,b_n^m\right)\right]. \tag{9}$$

On the right-hand side of the previous expression, we recognize the Fourier expansion of the function $\left(\frac{1}{2}\overline{D}\right)\mathbf{f}$ with respect to the orthonormal system $\left\{\mathbf{X}_{n-1}^{l,\dagger,*}, \mathbf{Y}_{n-1}^{m,\dagger,*} : l = 0,\dots,n,\ m = 1,\dots,n\right\}_{n\in\mathbb{N}_0}$. Having in mind the conditions of the L^2-convergence of (9), our task now is to find out if the series

$$\sum_{n=1}^{\infty} n(2n+3)\left(\left(a_n^0\right)^2 + \sum_{m=1}^{n}\left[\left(a_n^m\right)^2 + \left(b_n^m\right)^2\right]\right) \tag{10}$$

is convergent. By assumption, there exists a constant c such that the Fourier coefficients a_n^l $(l = 0,\dots,n)$ and b_n^m $(m = 1,\dots,n)$ satisfy

$$|a_n^l|, |b_n^m| < \frac{c}{(n+1)^{1+\alpha}}, \quad \alpha > 1, \quad l = 0,\dots,n,\ m = 1,\dots,n.$$

Substituting in the expression (10), after few straightforward computations, we may show that

$$\|\mathbf{f}\|^2_{L^2(B;\mathscr{A};\mathbb{R})} < \sum_{n=1}^{\infty}\frac{6c^2}{(n+1)^{2\alpha-1}}.$$

The series on the right-hand side is convergent, because by assumption $\alpha > 1$. Consequently, the series (10) is convergent. This means that $\left(\frac{1}{2}\overline{D}\right)\mathbf{f} \in L^2(B;\mathscr{A};\mathbb{R})$, which proves our statement. □

Lemma 4.1

Let $\alpha > -1$. Suppose that $\mathbf{f}$ is an arbitrary $\mathscr{A}$-valued monogenic function on B, and let it have the expansion $\mathbf{f}(x) = \sum_{n=0}^{\infty}\mathbf{P}_n(x)$ where

$$\mathbf{P}_n(x) = \frac{\mathbf{X}_n^{0,\dagger}}{\|\mathbf{Sc}\left(\mathbf{X}_n^{0,\dagger}\right)\|_{L^2(B)}}\,a_n^0 + \sum_{m=1}^{n}\left(\frac{\mathbf{X}_n^{m,\dagger}}{\|\mathbf{Sc}\left(\mathbf{X}_n^{m,\dagger}\right)\|_{L^2(B)}}\,a_n^m + \frac{\mathbf{Y}_n^{m,\dagger}}{\|\mathbf{Sc}\left(\mathbf{Y}_n^{m,\dagger}\right)\|_{L^2(B)}}\,b_n^m\right).$$

Then

$$\|\mathbf{f}\|_{L^2_\alpha(B)} \approx \left(\sum_{n=0}^{\infty} \frac{1}{(n+1)^{\alpha+1}} \|\mathbf{P}_n\|^2_{L^2(S;\mathscr{A};\mathbb{R})} \right)^{1/2}. \tag{11}$$

Proof

In view of the homogeneity and orthogonality of the polynomials $\mathbf{P}_n$ for each $n = 0, 1, \ldots$, and by splitting $|\mathbf{f}(x)|^2$ into $\mathbf{f}(x)\,\overline{\mathbf{f}(x)}$, and integrating term by term, it follows easily that

$$(\mathscr{M}_2(\mathbf{f};r))^2 = \int_S |\mathbf{f}(r\zeta)|^2 \, \mathrm{d}\sigma(\zeta) = \sum_{n=0}^{\infty} r^{2n} \|\mathbf{P}_n\|^2_{L^2(S;\mathscr{A};\mathbb{R})}, \qquad 0 \le r < 1. \tag{12}$$

It immediately leads to a Hardy norm in $h^2(B)$ or $\mathscr{H}^2(B)$

$$\|\mathbf{f}\|_{h^2(B)} = \|\mathbf{f}\|_{L^2(S)} = \left(\sum_{n=0}^{\infty} \|\mathbf{P}_n\|^2_{L^2(S;\mathscr{A};\mathbb{R})} \right)^{1/2}. \tag{13}$$

In order to get an equivalent norm in the Bergman spaces $h^2_\alpha(B)$ or $\mathscr{H}^2_\alpha(B)$, we integrate (12) on the interval $(0, R)$ for every $0 < R < 1$,

$$3 \int_0^R (1-r)^\alpha (\mathscr{M}_2(\mathbf{f};r))^2 r^2 \, \mathrm{d}r = 3 \sum_{n=0}^{\infty} \|\mathbf{P}_n\|^2_{L^2(S;\mathscr{A};\mathbb{R})} \int_0^R (1-r)^\alpha \, r^{2n+2} \, \mathrm{d}r.$$

By the Stirling's formula, it follows that

$$\int_0^1 (1-r)^\alpha \, r^{2n+2} \, \mathrm{d}r = \Gamma(\alpha+1) \frac{\Gamma(2n+3)}{\Gamma(\alpha+2n+4)} \sim \frac{\Gamma(\alpha+1)}{2^{\alpha+1}} \frac{1}{(n+1)^{\alpha+1}}$$

as $n \to \infty$. By letting R approach 1^-, we obtain (11). □

Remark 4.1

For Clifford algebra-valued functions expanded into spherical harmonics, the Hardy norm (13) as well as the unweighted Bergman $L^2_0(B)$-norm are obtained in [50]. For scalar-valued harmonic functions in the unit ball in $\mathbb{R}^n$, the equivalence of (11) is obtained in [51].

Remark 4.2

Let $U \in h(B;\mathbb{R})$ be a square integrable harmonic function and let also $U \in h^2(B)$. It is natural to ask whether the $\mathscr{A}$-valued monogenic function $\mathbf{f}(x) := U(x) + V(x)$ constructed in Theorem 4.1 belongs to $\mathscr{H}^2(B)$. As we may now prove, for some positive constant C, the expected inequality

$$\|\mathbf{f}\|_{\mathscr{H}^2(B)} \le C \|U\|_{h^2(B)} \tag{14}$$

fails. In fact, for the particular case

$$U(x) = \sum_{n=0}^{\infty} \frac{\mathbf{Sc}\left(\mathbf{X}_n^{n,\dagger}\right)}{\|\mathbf{Sc}\left(\mathbf{X}_n^{n,\dagger}\right)\|_{L^2(B)}} a_n^n,$$

so that $a_n^n = (n+1)^{-3/2}$, it follows

$$\|U(x)\|^2_{h^2(B)} = \sum_{n=0}^{\infty} (2n+3) \left(a_n^n\right)^2 < \infty,$$

but

$$\|[\mathbf{f}]_0\|^2_{\mathscr{H}^2(B)} = \left\| \sum_{n=0}^{\infty} \frac{\mathbf{X}_n^{n,\dagger}}{\|\mathbf{Sc}\left(\mathbf{X}_n^{n,\dagger}\right)\|_{L^2(B)}} a_n^n \right\|^2_{\mathscr{H}^2(B)} = \sum_{n=0}^{\infty} (n+1)(2n+3) \left(a_n^n\right)^2 = \infty,$$

which contradicts (14).

We now proceed to prove the boundedness of the underlying harmonic conjugation operators in some given weighted spaces.

Theorem 4.3
Let $U \in h(B;\mathbb{R})$ be a square integrable harmonic function. The operator $U \longmapsto \mathbf{f}$ is bounded from the harmonic Hardy space $h^2(B)$ into the (unweighted) Bergman space $\mathscr{H}_0^2(B)$. Moreover, it holds

$$\|\mathbf{f}\|_{L_0^2(B)} \leq \|U\|_{h^2(B)}.$$

Proof
For technical reasons, we set $\mathbf{f}(x) = \sum_{n=0}^{\infty} \mathbf{P}_n(x)$, where $\mathbf{P}_n$ is given as in the previous lemma. A first straightforward computation and Lemma 4.1 show that

$$\begin{aligned} \|\mathbf{f}\|^2_{L_0^2(B)} &= \sum_{n=0}^{\infty} \frac{(2n+1)(2n+3)}{(n+1)^2}\left(a_n^0\right)^2 + \sum_{m=1}^{n} \frac{(2n+1)(2n+3)}{(n+1)^2-m^2}\left[\left(a_n^m\right)^2 + \left(b_n^m\right)^2\right] \\ &\leq \sum_{n=0}^{\infty}(2n+3)\left[\left(a_n^0\right)^2 + \sum_{m=1}^{n}\left(a_n^m\right)^2 + \left(b_n^m\right)^2\right] \\ &= \|U\|^2_{h^2(B)}. \end{aligned}$$

□

Theorem 4.4
Let $U \in h(B;\mathbb{R})$ be a square integrable harmonic function. The operator $U \longmapsto \mathbf{f}$ is bounded from the weighted Bergman space $h_\alpha^2(B)$ $(\alpha > -1)$ into $\mathscr{H}^2_{\alpha+1}(B)$. Moreover, it holds

$$\|\mathbf{f}\|_{L^2_{\alpha+1}(B)} \leq \|U\|_{L^2_\alpha(B)}.$$

The previous relation is sharp in the sense that the exponent $\alpha+1$ on the left-hand side cannot be replaced by any smaller one, and the operator $U \longmapsto \mathbf{f}$ is unbounded from $h^2_\alpha(B)$ to $\mathscr{H}^2_{\alpha+1-\varepsilon}(B)$ for any $\varepsilon > 0$.

Proof
Let $\mathbf{P}_n$ be given as in the previous theorem. For the proof of the inequality, a straightforward computation and Lemma 4.1 show that

$$\begin{aligned} \|\mathbf{f}\|^2_{L^2_{\alpha+1}(B)} &= \sum_{n=0}^{\infty} \frac{1}{(n+1)^{\alpha+2}}\left[\frac{(2n+1)(2n+3)}{n+1}\left(a_n^0\right)^2 + \sum_{m=1}^{n}\frac{(2n+1)(2n+3)}{(n+1)^2-m^2}\left[\left(a_n^m\right)^2+\left(b_n^m\right)^2\right]\right] \\ &\leq \sum_{n=0}^{\infty}\frac{2n+3}{(n+1)^{\alpha+1}}\left[\left(a_n^0\right)^2 + \sum_{m=1}^{n}\left(a_n^m\right)^2 + \left(b_n^m\right)^2\right] \\ &= \|U\|^2_{L^2_\alpha(B)}. \end{aligned}$$

The second part of the proof then consists of looking for a counterexample of the inequality

$$\|\mathbf{f}\|_{L^2_{\alpha+1-\varepsilon}(B)} \leq C(\alpha,\varepsilon)\|U\|_{L^2_\alpha(B)} \tag{15}$$

for some real positive constant $C(\alpha,\varepsilon)$. Let ε be arbitrarily chosen and fixed so that $0 < \varepsilon < 1$, and consider the example

$$U(x) := \sum_{n=0}^{\infty} \frac{\mathbf{Sc}\left(\mathbf{X}_n^{n,\dagger}\right)}{\|\mathbf{Sc}\left(\mathbf{X}_n^{n,\dagger}\right)\|_{L^2(B)}}\, a_n^n,$$

where $a_n^n = (n+1)^{(\alpha-1-\varepsilon)/2}$. Direct computations show that

$$\begin{aligned} \|U\|^2_{L^2_\alpha(B)} &= \sum_{n=0}^{\infty}\frac{2n+3}{(n+1)^{\alpha+1}}\left(a_n^n\right)^2 \leq 3\sum_{n=0}^{\infty}\frac{1}{(n+1)^{\alpha}}\left(a_n^n\right)^2 < \infty \\ \|[\mathbf{f}]_0\|^2_{L^2_{\alpha+1-\varepsilon}(B)} &= \sum_{n=0}^{\infty}\frac{2n+3}{(n+1)^{\alpha+1-\varepsilon}}\left(a_n^n\right)^2 \geq 2\sum_{n=0}^{\infty}\frac{1}{(n+1)^{\alpha-\varepsilon}}\left(a_n^n\right)^2 = \infty, \end{aligned}$$

which contradicts (15). □

Remark 4.3
The Hardy space $\mathscr{H}^2(B)$ may be considered as the limiting case of the Bergman space $\mathscr{H}^2_\alpha(B)$ as α approaches -1^+. So, if we identify $\mathscr{H}^2_{-1}(B)$ with $\mathscr{H}^2(B)$, then Theorem 4.4 can be viewed, respectively, as a generalization of Theorem 4.3.

5. Construction of a Riesz system by its first component

If we want to make the previous results more precise, then we need a-priori criteria for the given function U that ensures the convergence of the constructed series for the monogenic function in $L^2(B;\mathscr{A};\mathbb{R})$ or in another space. Of course, the additional assumption (7) in Theorem 4.1 is such a criterion, but it is not well applicable in practice because it remains open if there is a known function space that is defined exactly by these conditions.

In the present section, we introduce an alternative algorithm to the explicit construction of a 'unique' pair of conjugate harmonic functions in $\mathbb{R}^3$, which will allow us to answer such important questions in the forthcoming sections.

Next, we formulate the result.

Theorem 5.1 (Construction of a harmonic conjugate)
Let U be a scalar-valued harmonic function defined in B. Define

$$[V(x)]_1 := -x_0 \int_0^1 \frac{\partial U(\rho x_0, x_1, x_2)}{\partial x_1}\, d\rho + W(x_1, x_2), \tag{16}$$

where the function $W(x_1,x_2)$ is chosen so that $\Delta_{(x_1,x_2)} W = \frac{\partial^2 U(0,x_1,x_2)}{\partial x_0\, \partial x_1}$, and

$$[V(x)]_2 := \int_0^1 \left[-\begin{vmatrix} x_0 & x_2 \\ \frac{\partial U(tx)}{\partial x_0} & \frac{\partial U(tx)}{\partial x_2} \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ \frac{\partial [V(tx)]_1}{\partial x_1} & \frac{\partial [V(tx)]_1}{\partial x_2} \end{vmatrix} \right] dt. \tag{17}$$

Then the function $\mathbf{f} := U + [V]_1\mathbf{i} + [V]_2\mathbf{j}$ is monogenic in B. Moreover, the most general monogenic function $\mathbf{g}$ having U as its scalar part is given by

$$\mathbf{g}(x) = \mathbf{f}(x) + \varphi(x_1, x_2),$$

where $\varphi(x_1,x_2)$ is a hyperholomorphic constant.

Proof
We should check that the function $\mathbf{f} = U + [V]_1\mathbf{i} + [V]_2\mathbf{j}$ satisfies the (R)-system. On account of the assumption about the functions U and $[V]_1$, we have

$$[V(x)]_1 = -\int_0^{x_0} \frac{\partial U(t, x_1, x_2)}{\partial x_1}\, dt + W(x_1, x_2), \qquad x \in B,$$

so that

$$\frac{\partial [V]_1}{\partial x_0} = -\frac{\partial U}{\partial x_1}. \tag{18}$$

A direct computation shows that $[V(x)]_1$ is harmonic in B. Indeed,

$$\begin{aligned} \Delta_x [V(x)]_1 &= \frac{\partial^2 [V(x)]_1}{\partial x_0^2} + \Delta_{(x_1,x_2)}[V(x)]_1 \\ &= -\frac{\partial^2 U(x)}{\partial x_0 \partial x_1} - \int_0^{x_0} \frac{\partial}{\partial x_1} \Delta_{(x_1,x_2)} U(t,x_1,x_2)\, dt + \Delta_{(x_1,x_2)} W(x_1,x_2) \\ &= -\frac{\partial^2 U(x)}{\partial x_0 \partial x_1} + \int_0^{x_0} \frac{\partial^3 U(t,x_1,x_2)}{\partial x_0^2\, \partial x_1}\, dt + \Delta_{(x_1,x_2)} W(x_1,x_2) \\ &= -\frac{\partial^2 U(x)}{\partial x_0 \partial x_1} + \frac{\partial^2 U(x)}{\partial x_0 \partial x_1} - \frac{\partial^2 U(0,x_1,x_2)}{\partial x_0 \partial x_1} + \Delta_{(x_1,x_2)} W(x_1,x_2) \\ &= 0. \end{aligned}$$

Let us now define the function

$$F(x) := \int_0^{x_2} \left(\frac{\partial U(0,0,t)}{\partial x_0} - \frac{\partial [V(0,0,t)]_1}{\partial x_1} \right) dt + \int_0^{x_1} \frac{\partial [V(0,t,x_2)]_1}{\partial x_2} dt - \int_0^{x_0} \frac{\partial U(t,x_1,x_2)}{\partial x_2} dt. \tag{19}$$

We immediately find all three partial derivatives of F:

$$\frac{\partial F(x)}{\partial x_0} = -\frac{\partial U(x_0,x_1,x_2)}{\partial x_2}. \tag{20}$$

Using (18), we get

$$\begin{aligned}\frac{\partial F(x)}{\partial x_1} &= \frac{\partial[V(0,x_1,x_2)]_1}{\partial x_2} - \int_0^{x_0} \frac{\partial^2 U(t,x_1,x_2)}{\partial x_1 \partial x_2}\,dt \\ &= \frac{\partial[V(0,x_1,x_2)]_1}{\partial x_2} + \int_0^{x_0} \frac{\partial^2 [V(t,x_1,x_2)]_1}{\partial x_0 \partial x_2}\,dt \\ &= \frac{\partial[V(0,x_1,x_2)]_1}{x_2} + \left.\frac{\partial[V(t,x_1,x_2)]_1}{\partial x_2}\right|_0^{x_0} = \frac{\partial[V(x_0,x_1,x_2)]_1}{\partial x_2}.\end{aligned} \tag{21}$$

Moreover, using (18) and harmonicity of U and $[V]_1$, we obtain

$$\begin{aligned}\frac{\partial F(x)}{\partial x_2} &= \frac{\partial U(0,0,x_2)}{\partial x_0} - \frac{\partial[V(0,0,x_2)]_1}{\partial x_1} + \int_0^{x_1} \frac{\partial^2[V(0,t,x_2)]_1}{\partial x_2^2}\,dt - \int_0^{x_0} \frac{\partial^2 U(t,x_1,x_2)}{\partial x_2^2}\,dt \\ &= \frac{\partial U(0,0,x_2)}{\partial x_0} - \frac{\partial[V(0,0,x_2)]_1}{\partial x_1} - \int_0^{x_1} \left(\frac{\partial^2[V(0,t,x_2)]_1}{\partial x_0^2} + \frac{\partial^2[V(0,t,x_2)]_1}{\partial x_1^2}\right) dt \\ &\quad + \int_0^{x_0} \left(\frac{\partial^2 U(t,x_1,x_2)}{\partial x_0^2} + \frac{\partial^2 U(t,x_1,x_2)}{\partial x_1^2}\right) dt \\ &= \frac{\partial U(0,0,x_2)}{\partial x_0} - \frac{\partial[V(0,0,x_2)]_1}{\partial x_1} - \int_0^{x_1} \left(-\frac{\partial^2 U(0,t,x_2)}{\partial x_0 \partial x_1} + \frac{\partial^2[V(0,t,x_2)]_1}{\partial x_1^2}\right) dt \\ &\quad + \int_0^{x_0} \left(\frac{\partial^2 U(t,x_1,x_2)}{\partial x_0^2} - \frac{\partial^2[V(t,x_1,x_2)]_1}{\partial x_0 \partial x_1}\right) dt \\ &= \frac{\partial U(0,0,x_2)}{\partial x_0} - \frac{\partial[V(0,0,x_2)]_1}{\partial x_1} + \left.\left(\frac{\partial U(0,t,x_2)}{\partial x_0} - \frac{\partial[V(0,t,x_2)]_1}{\partial x_1}\right)\right|_0^{x_1} + \left.\left(\frac{\partial U(t,x_1,x_2)}{\partial x_0} - \frac{\partial[V(t,x_1,x_2)]_1}{\partial x_1}\right)\right|_0^{x_0} \\ &= \frac{\partial U(x_0,x_1,x_2)}{\partial x_0} - \frac{\partial[V(x_0,x_1,x_2)]_1}{\partial x_1}.\end{aligned}$$

Thus,

$$\frac{\partial F}{\partial x_2} = \frac{\partial U}{\partial x_0} - \frac{\partial[V]_1}{\partial x_1}. \tag{22}$$

Finally, we observe that functions $[V]_2$ and F coincide. Indeed, (19) can be written as a curve-line integral

$$F(x) = \int_{(0,0,0)}^{(x_0,x_1,x_2)} \left[-\frac{\partial U(\xi)}{\partial x_2}d\xi_0 + \frac{\partial[V(\xi)]_1}{\partial x_2}d\xi_1 + \left(\frac{\partial U(\xi)}{\partial x_0} - \frac{\partial[V(\xi)]_1}{\partial x_1}\right)d\xi_2\right]. \tag{23}$$

We note that this curve-line integral is path independent in view of the conditions (20), (21), (22), meaning that

$$\nabla F = \left(-\frac{\partial U}{\partial x_2}, \frac{\partial[V]_1}{\partial x_2}, \frac{\partial U}{\partial x_0} - \frac{\partial[V]_1}{\partial x_1}\right), \qquad \text{and} \qquad \operatorname{curl} \nabla F = 0.$$

Therefore, we may choose the integration path in (23) as segments parallel to the coordinate axes. Next, by a suitable change of variables in (23), we obtain

$$\begin{aligned}F(x) &= \int_0^1 \left[-x_0\frac{\partial U(tx)}{\partial x_2} + x_1\frac{\partial[V(tx)]_1}{\partial x_2} + x_2\left(\frac{\partial U(tx)}{\partial x_0} - \frac{\partial[V(tx)]_1}{\partial x_1}\right)\right] dt \\ &= \int_0^1 \left[-x_0\frac{\partial U(tx)}{\partial x_2} + x_2\frac{\partial U(tx)}{\partial x_0} + x_1\frac{\partial[V(tx)]_1}{\partial x_2} - x_2\frac{\partial[V(tx)]_1}{\partial x_1}\right] dt\end{aligned} \tag{24}$$

which coincides with (17). Therefore, $F(x) \equiv [V(x)]_2$ in B. Thus, any $\mathscr{A}$-valued function $\mathbf{f}$ so that $\mathbf{f} := U + [V]_1\mathbf{i} + [V]_2\mathbf{j}$ is a special solution to the (R)-system. Let now $\mathbf{g}$ be the most general monogenic function so that $[\mathbf{g}]_0 = U$. On account of the assumption about $\mathbf{f}(x)$, it follows that

$$2U(x) = \mathbf{g}(x) + \overline{\mathbf{g}(x)} = \mathbf{f}(x) + \overline{\mathbf{f}(x)},$$

and consequently, $\mathbf{f}(x) - \mathbf{g}(x) + (\overline{\mathbf{f}(x) - \mathbf{g}(x)}) = 0$. This implies $[\mathbf{f}(x) - \mathbf{g}(x)]_0 = 0$, for all $x \in B$. Because $\mathbf{f}(x) - \mathbf{g}(x)$ is monogenic in B, it is then clear that the difference $\mathbf{f}(x) - \mathbf{g}(x)$ reduces to a hyperholomorphic constant φ so that $[\varphi]_0 = 0$. Thus, it follows that

$$\mathbf{g}(x) = \mathbf{f}(x) + \varphi(x_1,x_2), \quad \text{for all } x \in B,$$

and this concludes the proof. □

6. Harmonic conjugates in weighted monogenic Hardy spaces

In this section, we discuss the weighted (monogenic) Hardy space on the unit ball of $\mathbb{R}^3$ consisting of functions with values in the reduced quaternions.

The following lemma can be found, for example, in [52, pp. 251].

Lemma 6.1
Let $w(x)$ be a nonnegative subharmonic function in B, and

$$\mathscr{M}(w;r) = \int_S w(r\xi)\,\mathrm{d}\sigma(\xi), \qquad 0 \le r < 1.$$

If $\mathscr{M}(w;r)$ is bounded on $0 \le r < 1$, then $w(x)$ has a harmonic majorant $u(x) \in h^1(B)$ on B so that

$$w(x) \le u(x),\ x \in B, \qquad \text{and} \qquad \|u\|_{h^1(B)} \le C \sup_{0<r<1} \mathscr{M}(w;r).$$

Lemma 6.2
Let $1 \le p < \infty$, $\alpha > -1$, $\beta > 0$, m be a positive integer and $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{N}_0^3$. Then for all $\mathscr{A}$-valued harmonic functions f

$$\|f\|_{h(p,\beta)(B)} \approx \sum_{|\lambda|<m} \left|\partial^\lambda f(0)\right| + \sum_{|\lambda|=m} \left\|\partial^\lambda f\right\|_{h(p,\beta+m)(B)}, \tag{25}$$

$$\|f\|_{L^p_\alpha(B)} \approx \sum_{|\lambda|<m} \left|\partial^\lambda f(0)\right| + \sum_{|\lambda|=m} \left\|\partial^\lambda f\right\|_{L^p_{\alpha+pm}(B)}, \tag{26}$$

where ∂^λ denotes the partial differential operator of the order $|\lambda| = \lambda_0 + \lambda_1 + \lambda_2$ with respect to x_0, x_1, x_2. In particular,

$$\|f\|_{h(p,\beta)(B)} \approx |f(0)| + \|\nabla f\|_{h(p,\beta+1)(B)}, \tag{27}$$

$$\|f\|_{L^p_\alpha(B)} \approx |f(0)| + \|\nabla f\|_{L^p_{\alpha+p}(B)}. \tag{28}$$

The involved constants depend on the parameters p, m, α, β only.

Proof
For the proof of properties (25) and (27), see [32, Lemma 5]. The proof of (26) and (28) can be performed in the same manner. □

We now briefly recall some basic facts about the Poisson kernel and its related facts, which will be used to estimate the size of some integrals that appear in forthcoming proofs.

Lemma 6.3 (see [53])
Let Ω be a bounded domain in $\mathbb{R}^3$ with C^2-boundary $\partial\Omega$, and let $P_\Omega(x,y)$ be the Poisson kernel for Ω. Then

$$P_\Omega(x,y) \approx \frac{\mathrm{dist}(x,\partial\Omega)}{|x-y|^3}, \qquad x \in \Omega,\ y \in \partial\Omega.$$

For any fixed $\rho, r \in (0,1)$, we also consider the following bounded domain in $\mathbb{R}^3$:

$$E_{\rho,r} := \left\{ x = (x_0,x_1,x_2) \in \mathbb{R}^3 : \frac{x_0^2}{\rho^2 r^2} + \frac{x_1^2}{r^2} + \frac{x_2^2}{r^2} < 1 \right\},$$

which denotes the inner domain of the oblate spheroid $\partial E_{\rho,r}$.

Now we estimate the size of the Poisson kernel for $E_{\rho,r}$.

Lemma 6.4
Let $P_{E_{\rho,r}}(x,y)$ be the Poisson kernel for $E_{\rho,r}$. Then

$$P_{E_{\rho,r}}(x,y) \approx \frac{\mathrm{dist}(x,\partial E_{\rho,r})}{|x-y|^3}, \qquad x \in E_{\rho,r},\ y \in \partial E_{\rho,r},$$

in particular,

$$P_{E_{\rho,r}}(0,y) \approx \frac{\rho\, r}{|y|^3}, \qquad y \in \partial E_{\rho,r}.$$

Before we prove the main theorem, we state two more lemmas.

Lemma 6.5
For any $\alpha > 0$, and $\beta > 1$, it holds

$$\int_0^1 \frac{t^{\alpha-1}}{(1-tr)^{\beta}}\,\mathrm{d}t \sim \frac{1}{(\beta-1)(1-r)^{\beta-1}} \qquad \text{as} \quad r \to 1^-.$$

Proof
The proof is straightforward. □

Lemma 6.6 (see [54])
Let $w = w(x_1, x_2)$ be a nonnegative superharmonic function in the unit disk $\mathbb{D} := \{x_1^2 + x_2^2 < 1\}$, and $\gamma > -1$, $0 < p < 2 + \gamma$. Then for any point $a \in \mathbb{D}$

$$\|w\|_{L^p_\gamma(\mathbb{D})} \le C(p,\gamma,a)\, w(a).$$

Now, we are ready to formulate and prove the main result of this section.

Theorem 6.1
Let U be a scalar-valued harmonic function defined in B. Let also $W(x_1, x_2)$ be a solution of the equation

$$\Delta_{(x_1,x_2)} W = \frac{\partial^2 U(0,x_1,x_2)}{\partial x_0\, \partial x_1}, \tag{29}$$

such that $W(a)$ is finite for some point $a = (a_1, a_2)$, $a_1^2 + a_2^2 < 1$. If $U \in h(p,\beta)(B)$ for some $\beta > 0$ and $1 < p < \infty$, then there exist a monogenic function $\mathbf{f}$ so that $\mathbf{f} \in \mathscr{H}(p,\beta)(B)$ and $[\mathbf{f}]_0 = U$ in B, and a constant $C(p,\beta,a) < \infty$ such that

$$\|\mathbf{f}\|_{\mathscr{H}(p,\beta)(B)} \le C(p,\beta,a)\left(\|U\|_{h(p,\beta)(B)} + |W(a)|\right).$$

Proof
Given a real-valued harmonic function U, we use Theorem 5.1 to construct $\mathbf{f} = U + [V]_1\mathbf{i} + [V]_2\mathbf{j}$ where the coordinates $[V]_1$ and $[V]_2$ are defined by (16) and (17). For any point $x = r\eta \in B$, by Theorem 5.1 it follows

$$\begin{aligned} |[V]_1(x)| &\le |x_0| \int_0^1 \left| \frac{\partial U(\rho x_0, x_1, x_2)}{\partial x_1} \right| \mathrm{d}\rho + |W(x_1,x_2)| \\ &=: \widetilde{V}_1(x) + |W(x_1,x_2)|. \end{aligned} \tag{30}$$

We use Minkowski's inequality to estimate

$$\mathscr{M}_p(\widetilde{V}_1; r) \le \int_0^1 \left(\int_{|x|=r} |x_0|^p \left| \frac{\partial U(\rho x_0, x_1, x_2)}{\partial x_1} \right|^p \mathrm{d}\sigma \right)^{1/p} \mathrm{d}\rho.$$

Denote by $h(y)$ the smallest harmonic majorant of the subharmonic function $\left|\frac{\partial U(y)}{\partial x_1}\right|^p$ in the ball $B_{\sqrt{r}} = \{x \in \mathbb{R}^3 : |x| < \sqrt{r}\}$, then

$$\left| \frac{\partial U(y)}{\partial x_1} \right|^p \le h(y), \qquad y \in B_{\sqrt{r}}.$$

A direct computation shows that

$$\begin{aligned} \mathscr{M}_p(\widetilde{V}_1; r) &\le \int_0^1 \left(\int_{|x|=r} |x_0|^p\, h(\rho x_0, x_1, x_2)\, \mathrm{d}\sigma \right)^{1/p} \mathrm{d}\rho \\ &\le r \int_0^1 \left(\int_{|x|=r} h(\rho x_0, x_1, x_2)\, \mathrm{d}\sigma \right)^{1/p} \mathrm{d}\rho \\ &= r \int_0^1 \left(\int_{\partial E_{\rho,r}} h(y)\, \mathrm{d}\sigma(y) \right)^{1/p} \mathrm{d}\rho. \end{aligned}$$

We now write the Poisson integral representation of h in the spheroid $E_{\rho,r} \subset B_{\sqrt{r}}$ and estimate it at the origin by using Lemma 6.4:

$$h(x) = \int_{\partial E_{\rho,r}} P_{E_{\rho,r}}(x,y)\, h(y)\, \mathrm{d}\sigma(y).$$

Then, we obtain

$$h(0) = \int_{\partial E_{\rho,r}} P_{E_{\rho,r}}(0,y)\, h(y)\, \mathrm{d}\sigma(y) \;\ge\; C \int_{\partial E_{\rho,r}} \frac{\rho\, r}{|y|^3}\, h(y)\, \mathrm{d}\sigma(y) \;\ge\; C\frac{\rho}{r^2} \int_{\partial E_{\rho,r}} h(y)\, \mathrm{d}\sigma(y).$$

With these calculations at hand, we get

$$\mathscr{M}_p(\widetilde{V}_1;r) \le r\int_0^1 \left(\int_{\partial E_{\rho,r}} h(y)\,d\sigma(y)\right)^{1/p} d\rho \le C\,r\int_0^1 \left(\frac{r^2}{\rho}\,h(0)\right)^{1/p} d\rho = C_p\,r^{1+2/p}(h(0))^{1/p}.$$

By the mean-value equality for harmonic functions and by using Lemma 6.1, we obtain

$$\begin{aligned}\mathscr{M}_p(\widetilde{V}_1;r) &\le C_p\,r^{1+2/p}\int_0^1\left(\frac{1}{|S_{\sqrt{\rho r}}|}\int_{S_{\sqrt{\rho r}}} h\,d\sigma\right)^{1/p} d\rho\\ &= C_p\,r^{1+2/p}\int_0^1\left(\frac{1}{\rho r}\mathscr{M}_1(h;\sqrt{\rho r})\right)^{1/p} d\rho\\ &= C_p\,r^{1+1/p}\int_0^1 \frac{1}{\rho^{1/p}}\mathscr{M}_1^{1/p}(h;\sqrt{\rho r})\,d\rho\\ &\le C_p\,r^{1+1/p}\int_0^1 \frac{1}{\rho^{1/p}}\mathscr{M}_p\left(\frac{\partial U}{\partial x_1};\sqrt{\rho r}\right) d\rho.\end{aligned} \tag{31}$$

The next estimation is due to Lemma 6.5

$$\begin{aligned}\mathscr{M}_p(\widetilde{V}_1;r) &\le C_p\,r^{1+1/p}\int_0^1 \frac{(1-\sqrt{\rho r})^{\beta+1}\mathscr{M}_p\left(\frac{\partial U}{\partial x_1};\sqrt{\rho r}\right)}{\rho^{1/p}\,(1-\sqrt{\rho r})^{\beta+1}}\,d\rho\\ &\le C(p,\beta)\,r^{1+1/p}\left\|\frac{\partial U}{\partial x_1}\right\|_{h(p,\beta+1)(B)}\int_0^1 \frac{1}{\rho^{1/p}\,(1-\rho r)^{\beta+1}}\,d\rho\\ &\le C(p,\beta)\left\|\frac{\partial U}{\partial x_1}\right\|_{h(p,\beta+1)(B)}\frac{1}{(1-r)^{\beta}}.\end{aligned}$$

Therefore, by Lemma 6.2

$$(1-r)^{\beta}\,\mathscr{M}_p(\widetilde{V}_1;r) \le C\left\|\frac{\partial U}{\partial x_1}\right\|_{h(p,\beta+1)(B)} \le C\,\|\nabla U\|_{h(p,\beta+1)(B)} \le C\|U\|_{h(p,\beta)(B)}, \qquad 0<r<1.$$

The last term in (30) can be estimated by means of Lemma 6.6 as follows. It is well known (see, e.g. [55]) that the solution $W(x_1,x_2)$ of the Poisson equation (29) in $\mathbb{D}$ with vanishing boundary values on the unit circle $\partial\mathbb{D}$ is the Green potential of $\frac{\partial^2 U(0,x_1,x_2)}{\partial x_0\,\partial x_1}$. By splitting the function $\frac{\partial^2 U(0,x_1,x_2)}{\partial x_0\,\partial x_1}$ into its positive and negative parts, we come to $W = W^+ - W^-$, where $W^+ = \max\{W,0\}$ and $W^- = \max\{-W,0\}$ are nonnegative superharmonic functions in $\mathbb{D}$. By Lemma 6.6, it follows

$$\|W^+\|_{L_p^p(\mathbb{D})} \le C(p,a)\,W^+(a), \qquad \|W^-\|_{L_p^p(\mathbb{D})} \le C(p,a)\,W^-(a),$$

and hence

$$\|W\|_{L_p^p(\mathbb{D})} \le \|W^+\|_{L_p^p(\mathbb{D})} + \|W^-\|_{L_p^p(\mathbb{D})} \le C(p,a)\,|W(a)|.$$

Because the integral means $\mathscr{M}_p(W^{\pm};r)$ of the superharmonic functions W^+ and W^- are decreasing with respect to r, whence

$$\sup_{1/2<r<1}(1-r)^{\beta}\,\mathscr{M}_p(W^{\pm};r) \le C\|W^{\pm}\|_{L_p^p(\mathbb{D})} \le C(p,\beta,a)\,W^{\pm}(a)$$

for any $\beta > 0$. Thus,

$$\sup_{1/2<r<1}(1-r)^{\beta}\,\mathscr{M}_p(W;r) \le C(p,\beta,a)\,|W(a)|, \tag{32}$$

and this conclusion is true not only for $W(x_1,x_2)$ but also for an extension $W(x_0,x_1,x_2)$ in the unit ball B. Therefore, (30), (32) and (32) together yield

$$\|[V]_1\|_{h(p,\beta)(B)} \le C(p,\beta)\sup_{1/2<r<1}(1-r)^{\beta}\,\mathscr{M}_p([V]_1;r) \le C(p,\beta,a)\left(\|U\|_{h(p,\beta)(B)} + |W(a)|\right). \tag{33}$$

The second coordinate $[V]_2$ can now be estimated by using (24) as follows

$$\begin{aligned}|[V]_2(x)| &\le \int_0^1\left(|x_0|\left|\frac{\partial U(tx)}{\partial x_2}\right| + |x_2|\left|\frac{\partial U(tx)}{\partial x_0}\right| + |x_1|\left|\frac{\partial [V(tx)]_1}{\partial x_2}\right| + |x_2|\left|\frac{\partial [V(tx)]_1}{\partial x_1}\right|\right) dt\\ &\le \sqrt{2}\int_0^1 \left(|\nabla U(tx)| + |\nabla [V(tx)]_1|\right) dt.\end{aligned}$$

We use Minkowski's inequality to estimate

$$\mathscr{M}_p([V]_2;r) \le C\int_0^1 \mathscr{M}_p(\nabla U;tr)\,dt + C\int_0^1 \mathscr{M}_p(\nabla[V]_1;tr)\,dt. \tag{34}$$

Hence,

$$\begin{aligned}\mathscr{M}_p([V]_2;r) &\le C\sup_{0<t<1}(1-tr)^{\beta+1}\mathscr{M}_p(\nabla U;tr)\int_0^1\frac{dt}{(1-tr)^{\beta+1}} + C\sup_{0<t<1}(1-tr)^{\beta+1}\mathscr{M}_p(\nabla[V]_1;tr)\int_0^1\frac{dt}{(1-tr)^{\beta+1}}\\ &\le C(1-r)^{-\beta}(\|\nabla U\|_{h(p,\beta+1)(B)} + \|\nabla[V]_1\|_{h(p,\beta+1)(B)}).\end{aligned}$$

Therefore, by Lemma 6.2 and (33)

$$\begin{aligned}\|[V]_2\|_{h(p,\beta)(B)} &\le C\,\|\nabla U\|_{h(p,\beta+1)(B)} + C\,\|\nabla[V]_1\|_{h(p,\beta+1)(B)}\\ &\le C\|U\|_{h(p,\beta)(B)} + C\|[V]_1\|_{h(p,\beta)(B)}\\ &\le C(p,\beta,a)(\|U\|_{h(p,\beta)(B)} + |W(a)|).\end{aligned}$$

This completes the proof of the theorem. □

7. Harmonic conjugates in weighted monogenic Bergman spaces

In the present section, we shall see that a similar result to Theorem 6.1 can also be obtained for weighted Bergman spaces $\mathscr{H}^p_\alpha(B)$ for any range $\alpha > -1$. The proof of this result is based on Theorem 5.1 along with some well-known inequalities.

To begin with, we state the following version of the Hardy inequality [56, pp. 490].

Lemma 7.1
If $1 \le p < \infty$, $\gamma < -1 < \alpha$, and $h(r) \ge 0$, then

$$\int_0^1 (1-r)^\alpha\, r^\gamma \left(\int_0^r h(t)dt\right)^p dr \;\le\; C\int_0^1 (1-r)^{\alpha+p}\, r^{\gamma+p}\, h^p(r)dr,$$

where the constant C depends only on the parameters p, α, γ.

In the next lemma, we present a useful estimate on weights.

Lemma 7.2
Let $1 \le p < \infty$, and $\gamma < -1 < \alpha$. Then for all $u \in h(B)$, there exists a constant $C(p,\alpha,\gamma) < \infty$ such that

$$\left(\int_0^1 (1-r)^\alpha \mathscr{M}_p^p(u;r)\, r^\gamma\, dr\right)^{1/p} \le C(p,\alpha,\gamma)\|u\|_{L^p_\alpha(B)}.$$

Proof
The result immediately follows from the subharmonicity of $|u|^p$ and monotonicity of the integral means $\mathscr{M}_p(u;r)$ with regard to r. Moreover, the inequality is also valid for $0 < p < 1$, but we do not consider this case in the present paper, cf. [32, Lemma 4]. □

Our main tool in this section is the following theorem.

Theorem 7.1
Let U be a scalar-valued harmonic function in B. Let also $W(x_1,x_2)$ be a solution of the equation $\Delta_{(x_1,x_2)}W = \frac{\partial^2 U(0,x_1,x_2)}{\partial x_0\,\partial x_1}$, such that $W(a)$ is finite for some point $a = (a_1,a_2)$, $a_1^2 + a_2^2 < 1$. If $U \in h^p_\alpha(B)$ for some $\alpha > -1$ and $1 < p < \infty$, then there exist a monogenic function $\mathbf{f}$ so that $\mathbf{f} \in \mathscr{H}^p_\alpha(B)$ and $[\mathbf{f}]_0 = U$ in B, and a constant $C(p,\alpha,a) < \infty$ such that

$$\|\mathbf{f}\|_{L^p_\alpha(B)} \le C(p,\alpha,a)\left(\|U\|_{L^p_\alpha(B)} + |W(a)|\right).$$

Proof
As in Theorem 6.1, given a scalar-valued harmonic function U, by Theorem 5.1, we construct $\mathbf{f} = U + [V]_1\mathbf{i} + [V]_2\mathbf{j}$ where its coordinates $[V]_1$ and $[V]_2$ are defined by (16) and (17). We use (30) and (31), while $|[V]_1(x)| \le \widetilde{V}_1(x) + |W(x_1,x_2)|$,

$$\begin{aligned}\mathscr{M}_p(\widetilde{V}_1;r) &\le C_p\, r^{1+1/p}\int_0^1 \frac{1}{\rho^{1/p}}\mathscr{M}_p\left(\frac{\partial U}{\partial x_1};\sqrt{\rho r}\right)d\rho\\ &= C_p\, r^{2/p}\int_0^r \frac{1}{t^{1/p}}\mathscr{M}_p\left(\frac{\partial U}{\partial x_1};\sqrt{t}\right)dt.\end{aligned}$$

Raise both sides of the previous expression to the power p and integrate. We now use Lemma 7.2 to obtain

$$\begin{aligned}\|\widetilde{V}_1\|^p_{L^p_\alpha(B)} &\leq C\int_0^1 (1-r)^\alpha \mathscr{M}_p^p(\widetilde{V}_1;r)\mathrm{d}r\\ &\leq C\int_0^1 (1-r)^\alpha r^2\left(\int_0^r t^{-1/p}\mathscr{M}_p\left(\frac{\partial U}{\partial x_1};\sqrt{t}\right)\mathrm{d}t\right)^p \mathrm{d}r\\ &\leq C\int_0^1 (1-r)^\alpha r^{-1-\delta}\left(\int_0^r t^{-1/p}\mathscr{M}_p\left(\frac{\partial U}{\partial x_1};\sqrt{t}\right)\mathrm{d}t\right)^p \mathrm{d}r.\end{aligned}$$

Here $\delta > 0$ can be chosen arbitrarily, and we choose $\delta = \frac{p-1}{2}$ in order to apply the Hardy inequality of Lemma 7.1. Hence, using also Lemma 7.2, it follows that

$$\begin{aligned}\|\widetilde{V}_1\|^p_{L^p_\alpha(B)} &\leq C\int_0^1 (1-r)^{\alpha+p}\, r^{p-1-\delta}\left(r^{-1/p}\mathscr{M}_p\left(\frac{\partial U}{\partial x_1};\sqrt{r}\right)\right)^p \mathrm{d}r\\ &= C\int_0^1 (1-r)^{\alpha+p}\, r^{(p-1)/2-1}\mathscr{M}_p^p\left(\frac{\partial U}{\partial x_1};\sqrt{r}\right)\mathrm{d}r\\ &\leq C(p,\alpha)\left\|\frac{\partial U}{\partial x_1}\right\|^p_{L^p_{\alpha+p}(B)}.\end{aligned}$$

Therefore, by Lemma 6.2, we obtain

$$\|\widetilde{V}_1\|_{L^p_\alpha(B)} \leq C(p,\alpha)\,\|\nabla U\|_{L^p_{\alpha+p}(B)} \leq C(p,\alpha)\|U\|_{L^p_\alpha(B)}. \tag{35}$$

The term $W(x_1,x_2)$ in (30) can be estimated as in Theorem 6.1. By Lemma 6.6, for nonnegative superharmonic functions $W^\pm$

$$\|W^\pm\|_{L^p_p(\mathbb{D})} \leq C(p,\alpha,a)\, W^\pm(a).$$

Because the integral means $\mathscr{M}_p(W^\pm;r)$ of the superharmonic functions W^+ and W^- are decreasing with respect to r,

$$\|W^\pm\|_{L^p_p(\mathbb{D})} \geq \left(2\int_0^r (1-t)^p\mathscr{M}_p^p(W^\pm;t)t\,\mathrm{d}t\right)^{1/p} \geq C_p\,\mathscr{M}_p(W^\pm;r)$$

for all $\frac{1}{2} < r < 1$. It follows that for any $\alpha > -1$

$$\left(\int_{1/2}^1 (1-r)^\alpha\mathscr{M}_p^p(W^\pm;r)r\,\mathrm{d}r\right)^{1/p} \leq C(p,\alpha)\,\|W^\pm\|_{L^p_p(\mathbb{D})} \leq C(p,\alpha,a)\,W^\pm(a). \tag{36}$$

Moreover, inequality (36) remains valid for an extension $W(x_0,x_1,x_2)$ of $W(x_1,x_2)$ to the ball B. Recalling that $|[V]_1(x)| \leq \widetilde{V}_1(x) + |W(x_1,x_2)|$, we then obtain from (35) and (36)

$$\begin{aligned}\|[V]_1\|_{L^p_\alpha(B)} &\leq C(p,\alpha)\left(\int_{1/2}^1 (1-r)^\alpha\mathscr{M}_p^p([V]_1;r)\,\mathrm{d}r\right)^{1/p}\\ &\leq C(p,\alpha,a)\left(\|U\|_{L^p_\alpha(B)} + |W(a)|\right).\end{aligned} \tag{37}$$

Now, we proceed the estimations for the second coordinate $[V]_2$. By (34), and using (37) and Lemma 6.2, we finally get

$$\mathscr{M}_p([V]_2;r) \leq C\int_0^1 \mathscr{M}_p(\nabla U;tr)\,\mathrm{d}t + C\int_0^1 \mathscr{M}_p(\nabla[V]_1;tr)\,\mathrm{d}t,$$

and

$$\begin{aligned}\|[V]_2\|_{L^p_\alpha(B)} &\leq C\,\|\nabla U\|_{L^p_{\alpha+p}(B)} + C\,\|\nabla[V]_1\|_{L^p_{\alpha+p}(B)}\\ &\leq C\|U\|_{L^p_\alpha(B)} + C\|[V]_1\|_{L^p_\alpha(B)}\\ &\leq C(p,\alpha,a)\left(\|U\|_{L^p_\alpha(B)} + |W(a)|\right).\end{aligned}$$

This completes the proof of our statement. □

Acknowledgements

This work was supported by *FEDER* funds through *COMPETE*–Operational Programme Factors of Competitiveness ('Programa Operacional Factores de Competitividade') and by Portuguese funds through the *Center for Research and Development in Mathematics and Applications* (University of Aveiro) and the Portuguese Foundation for Science and Technology ('FCT–Fundação para a Ciência e a Tecnologia'), within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690. Partial support from the Foundation for Science and Technology (FCT) via the post-doctoral grant SFRH/BPD/66342/2009 is also acknowledged by the first named author. The second author wishes to thank DAAD and the Bauhaus-Universität Weimar, Institut für Mathematik/Physik for their support.

References

1. Gürlebeck K, Sprössig W. *Quaternionic Analysis and Elliptic Boundary Value Problems*. Akademie Verlag: Berlin, 1989.
2. Gürlebeck K, Sprössig W. *Quaternionic Calculus for Engineers and Physicists*. John Wiley and Sons: Chichester, 1997.
3. Kravchenko V, Shapiro M. *Integral Representations for Spatial Models of Mathematical Physics*, Research Notes in Mathematics. Pitman Advanced Publishing Program: London, 1996.
4. Kravchenko V. *Applied Quaternionic Analysis*, Research and Exposition in Mathematics, Vol. 28. Lemgo: Heldermann Verlag, 2003.
5. Shapiro M, Vasilevski NL. Quaternionic ψ-hyperholomorphic functions, singular operators and boundary value problems I ψ-hyperholomorphic function theory. *Complex Variables, Theory and Application* 1995; **27**(1):17–46.
6. Shapiro M, Vasilevski NL. Quaternionic ψ-hyperholomorphic functions, singular operators and boundary value problems II algebras of singular Integral operators and riemann type boundary value problems. *Complex Variables, Theory and Application* 1995; **27**(1):67–96.
7. Sudbery A. Quaternionic analysis. *Mathematical Proceedings of the Cambridge Philosophical Society* 1979; **85**:199–225.
8. Fueter R. Analytische Funktionen einer Quaternionenvariablen. *Commentarii Mathematici Helvetici* 1932; **4**:9–20.
9. Fueter R. Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen Variablen. *Commentarii Mathematici Helvetici* 1935; **7**:307–330.
10. Fueter R. Über die analytische Darstellung der regulären Funktionen einer Quaternionenvariablen. *Commentarii Mathematici Helvetici* 1935; **8**:371–378.
11. Fueter R. *Functions of a Hyper Complex Variable*, Lecture notes written and supplemented by E. Bareiss. Math. Inst. Univ: Zürich, Fall Semester, 1949.
12. Brackx F, Delanghe R, Sommen F. *Clifford Analysis*. Pitman Publishing: Boston-London-Melbourne, 1982.
13. Malonek H. Power series representation for monogenic functions in $\mathbb{R}^{m+1}$ based on a permutational product. *Complex Variables, Theory and Application* 1990; **15**(3):181–191.
14. Leutwiler H. Quaternionic analysis in $\mathbb{R}^3$ versus its hyperbolic modification. In *NATO Science Series II. Mathematics, Physics and Chemistry*, Vol. 25. Kluwer Academic Publishers: Dordrecht, Boston, London, 2001.
15. Delanghe R. On homogeneous polynomial solutions of the Riesz system and their harmonic potentials. *Complex Variables and Elliptic Equations* 2007; **52**(10–11):1047–1062.
16. Morais J. Approximation by homogeneous polynomial solutions of the Riesz system in $\mathbb{R}^3$. *Ph.D. thesis*, Bauhaus-Universität Weimar, Geschwister-Scholl-Straße 8 99423 Weimar, 2009.
17. Cação I. Constructive approximation by monogenic polynomials. *Ph.D. diss.*, Universidade de Aveiro, Campus Universitário de Santiago, 3810-193 Aveiro, Portugal, 2004.
18. Bock S, Gürlebeck K. On a generalized Appell system and monogenic power series. *Mathematical Methods in the Applied Sciences* 2009; **33**(4):394–411.
19. Cação I, Gürlebeck K, Bock S. In *Methods of Complex and Clifford Analysis*, Son LH *et al.* (eds). SAS International Publications: Delhi, 2005; 241–260.
20. Cação I, Gürlebeck K, Bock S. On derivatives of spherical monogenics. *Complex Variables and Elliptic Equations* 2006; **51**(8–11):847–869.
21. Cação I, Malonek H. Remarks on some properties of monogenic polynomials. In *Special Volume of Wiley-VCH*, Simos TE, Psihoyios G, Tsitouras Ch (eds). ICNAAM: New York, 2006; 596–599.
22. Cação I. Complete orthonormal sets of polynomial solutions of the Riesz and Moisil-Teodorescu systems in $\mathbb{R}^3$. *Numerical Algorithms* 2010; **55**(2–3):191–203.
23. Delanghe R, Lávička R, Soucek V. On polynomial solutions of generalized Moisil-Theodoresco systems and Hodge-de Rham systems. *Advances in Applied Clifford Algebras* 2011; **21**(3):521–530.
24. Lávicka R. Generalized Appell property for the Riesz system in dimension 3. *AIP Conference Proceedings* 2011; **1389**:291–294.
25. Lávicka R. Complete orthogonal Appell systems for spherical monogenics. *Complex Analysis and Operator Theory* 2012; **6**(2):477–489.
26. Morais J, Gürlebeck K. Real-Part Estimates for solutions of the Riesz system in $\mathbb{R}^3$. *Complex Variables and Elliptic Equations* 2012; **57**(5):505–522.
27. Morais J, Le HT. Orthogonal Appell systems of monogenic functions in the cylinder. *Mathematical Methods in the Applied Sciences* 2011; **34**(12):1472–1486.
28. Morais J, Le HT, Sprößig W. On some constructive aspects of monogenic function theory in $\mathbb{R}^4$. *Mathematical Methods in the Applied Sciences* 2011; **34**(14):1685–1706.
29. Xu Z. Boundary Value Problems and Function Theory for Spin-Invariant Differential Operators. *Ph.D thesis*, Ghent University, Sint-Pietersnieuwstraat 25, B - 9000 Ghent, Belgium, 1989.
30. Brackx F, Delanghe R, Sommen F. On conjugate harmonic functions in Euclidean space. *Mathematical Methods in the Applied Sciences* 2002; **25**(16–18):1553–1562.
31. Brackx F, Delanghe R. On harmonic potential fields and the structure of monogenic functions. *Zeitschrift für Analysis und ihre Anwendungen* 2003; **22**(2):261–273.
32. Avetisyan K, Gürlebeck K, Sprößig W. Harmonic conjugates in weighted Bergman spaces of quaternion-valued functions. *Computational Methods and Function Theory* 2009; **9**(2):593–608.
33. Gürlebeck K, Morais J. On orthonormal polynomial solutions of the Riesz system in $\mathbb{R}^3$. *Recent Advances in Computational and Applied Mathematics* 2011:143–158. DOI: 10.1007/978.90.481.9981.56.

34. Gürlebeck K, Morais J. On the construction of harmonic conjugates in the context of quaternionic analysis. *AIP Conference Proceedings* 2011; **1281**:1496–1499.
35. Brackx F, Van Acker N. A conjugate Poisson kernel in Euclidean space - MAPLE procedures for explicit calculation. *Simon Stevin* 1993; **67**(1–2):3–14.
36. Brackx F, De Knock B, De Schepper H, Eelbode D. On the interplay between the Hilbert transform and conjugate harmonic functions. *Mathematical Methods in the Applied Sciences* 2006; **29**(12):1435–1450.
37. Brackx F, De Schepper H. Conjugate harmonic functions in Euclidean space: a spherical approach. *Computational Methods and Function Theory* 2006; **6**(1):165–182.
38. Constales D. A conjugate harmonic to the Poisson kernel in the unit ball of $\mathbb{R}^n$. *Simon Stevin* 1988; **62**(3-4):289–291.
39. Shapiro M. On the conjugate harmonic functions of M. Riesz-E. Stein-G. Weiss. In *Topics in Complex Analysis, Differential Geometry and Mathematical Physics*, Dimiev S *et al.* (eds), Third International Workshop on Complex Structures and Vector Fields, St. Konstantin, Bulgaria, August 23–29, 1996. World Scientific: Singapore, 1997; 8–32.
40. Xu Z, Chen J, Zhang W. A harmonic conjugate of the Poisson kernel and a boundary value problem for monogenic functions in the unit ball of $\mathbb{R}^n$ $(n \geq 2)$. *Simon Stevin* 1990; **64**(2):187–201.
41. Sansone G. *Orthogonal Functions*, Pure and Applied Mathematics, Vol. IX. Interscience Publishers: New York, 1959.
42. Stein EM, Weiß G. On the theory of harmonic functions of several variables. Part I , The theory of H^p spaces. *Acta Math* 1960; **103**:25–62.
43. Stein EM, Weiß G. Generalization of the Cauchy-Riemann equations and representations of the rotation group. *American Journal of Mathematics* 1968; **90**:163–196.
44. Riesz M. *Clifford numbers and spinors*, Vol. 38. Inst. Phys. Sci. and Techn. Lect. Ser.: Maryland, 1958.
45. Gürlebeck K, Malonek H. A hypercomplex derivative of monogenic functions in $\mathbb{R}^{n+1}$ and its applications. *Complex Variables* 1999; **39**(3):199–228.
46. Mitelman I, Shapiro M. Differentiation of the Martinelli-Bochner integrals and the notion of hyperderivability. *Mathematische Nachrichten* 1995; **172**(1):211–238.
47. Gürlebeck K, Morais J. On Bohr's phenomenon in the context of Quaternionic analysis and related problems. In *Algebraic Structures In Partial Differential Equations Related To Complex And Clifford Analysis*, Le Hung Son, Tutschke W (eds). Ho Chi Minh City University of Education Press: Vietnam, 2010; 9–24.
48. Gürlebeck K, Morais J. Bohr Type Theorems for Monogenic Power Series. *Computational Methods and Function Theory* 2009; **9**(2):633–651.
49. Moisil G. Sur la généralisation des fonctions conjuguées. *Atti della Accademia Nazionale dei Lincei, Rendiconti, VI. Serie* 1931; **14**:401–408.
50. Bernstein S, Gürlebeck K, Reséndis LF, Tovar LM. Dirichlet and Hardy spaces of harmonic and monogenic functions. *Zeitschrift für Analysis und ihre Anwendungen* 2005; **24**:763–789.
51. Petrosyan AI. On weighted harmonic Bergman spaces. *Demonstratio Math* 2008; **41**:73–83.
52. Flett TM. Inequalities for the pth mean values of harmonic and subharmonic functions with $p \leq 1$. *Proceedings of the London Mathematical Society* 1970; **20**:249–275.
53. Krantz S. Calculation and estimation of the Poisson kernel. *Journal of Mathematical Analysis and Applications* 2005; **302**:143–148.
54. Zhao S. On the weighted L^p-integrability of nonnegative $\mathscr{M}$-superharmonic functions. *Proceedings of the American Mathematical Society* 1992; **115**:677–685.
55. Gilbarg D, Trudinger NS. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag: Berlin, Heidelberg, N.Y, 1983.
56. Flett TM. Mean values of power series. *Pacific Journal of Mathematics* 1968; **25**:463–494.