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# Subharmonicity and a Version of Riesz Theorem on Harmonic Conjugates

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*Dedicated to Professor Klaus Gürlebeck on the occasion of his 60th birthday*

**Abstract.** The aim of the paper is to prove a monogenic version of classical M. Riesz theorem on harmonic conjugates in the framework of quaternionic analysis in $\mathbb{R}^4$. Our proof is subharmonic and somewhat simpler than that for less general Riesz-Stein-Weiss systems of harmonic conjugate functions.

**Keywords.** Quaternionic analysis, monogenic function, subharmonic function, harmonic conjugates.

## 1. Introduction

The purpose of this paper is to study the harmonic conjugation in Hardy spaces $H^p$ in the framework of quaternionic analysis in $\mathbb{R}^4$. Earlier [2, 3, 20] for the same purposes, we used the well-known Sudbery integral formula [25] and another integral formula in $\mathbb{R}^3$ for the construction of harmonic conjugates of quaternion-valued functions in $\mathbb{R}^4$ or $\mathbb{R}^3$. Instead, in the present paper, we use subharmonicity and some related estimates to obtain a version of the M. Riesz theorem on harmonic conjugation in Hardy spaces in $\mathbb{R}^4$.

The problem of harmonic conjugates in the framework of quaternionic and Clifford analysis was studied by many authors. After Sudbery found an explicit integral formula ([25]) for conjugate harmonic functions in $\mathbb{R}^4$, a higher dimensional generalization of the mentioned formula is obtained in [5].

Brackx, Delanghe et al. in a series of papers (see [6, 4, 7, 8] and references therein) made a detailed investigation of harmonic conjugation in the general Clifford analysis setting.

We modify some arguments of Stein and Weiss [23, 22, 24], Kuran [17, 18], Coifman and Weiss [10], Essén [11], Li and Peng [19], Kheyfits and Tepper

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[16], and give a somewhat simpler proof for our monogenic version of the M. Riesz theorem than that for less general Riesz systems.

Let $B = B_4$ be the open unit ball in the 4-dimensional Euclidean space $\mathbb{R}^4$, and $S = S^3 = \partial B$ be its boundary, the unit sphere. We will work in $\mathbb{H} \cong \mathbb{R}^4$, the skew field of real quaternions. Each element of $\mathbb{H}$ can be written in the form $x = x_0\mathbf{e}_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \quad (x_0, x_1, x_2, x_3 \in \mathbb{R})$, where the system $\mathbf{e}_0 = 1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ forms a basis of $\mathbb{H}$, and $\mathbf{Sc}\, x = x_0, \mathbf{Vec}\, x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$. The corresponding multiplication rules are given by $\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -1,\ \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_3,\ \mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_1,\ \mathbf{e}_3\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_2$. The conjugate element to $x \in \mathbb{H}$ is defined by $\bar{x} = x_0 - x_1\mathbf{e}_1 - x_2\mathbf{e}_2 - x_3\mathbf{e}_3$, and so $x\bar{x} = \bar{x}x = |x|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$.

Let $D = \mathbf{e}_0\frac{\partial}{\partial x_0} + \mathbf{e}_1\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} + \mathbf{e}_3\frac{\partial}{\partial x_3}$ denote the Cauchy-Riemann-Fueter operator. A real-differentiable function $f = u_0\mathbf{e}_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$, is said to be (left) monogenic if $Df = 0$. A right monogenic function is defined by the equation $fD = 0$. It is well known (see e.g. [19]) that functions $f$ which are at the same time left and right monogenic, are exactly those ones whose conjugates $\overline{f}$ are Riesz systems, that is, $\operatorname{div}\overline{f} = 0,\ \operatorname{curl}\overline{f} = 0$. We only consider left monogenic functions in this paper.

We refer to [23, 22, 24, 21] for the general theory of Riesz systems of harmonic conjugate functions and to [5, 14, 13] for the general theory of quaternionic and Clifford analysis.

For a function $f(x) = f(r\zeta)$ in $B$ $(0 \le r < 1, \zeta \in S)$, its integral mean is defined by

$$M_p(f;r) = \|f(r\cdot)\|_{L^p(S,d\sigma)}, \qquad 0 \le r < 1, \quad 0 < p < \infty,$$

where $d\sigma$ is the normalized surface measure on $S$ so that $\sigma(S) = 1$. The monogenic Hardy space $H^p(B)$, $0 < p < \infty$, consists of all (left) monogenic functions $f$ in $B$, satisfying

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(f;r) < +\infty.$$

The corresponding (real) harmonic Hardy space in $B$ will be denoted by $h^p(B)$.

Recall the classical M. Riesz theorem (1927) on harmonic conjugates in the Hardy spaces over the unit disc $\mathbb{D}$.

**Riesz Theorem.** *If a harmonic function $u_1$ in the unit disc $\mathbb{D}$ is in the Hardy space $h^p(\mathbb{D})$ for some $p, 1 < p < \infty$, then its harmonic conjugate $u_0$ is also in $h^p(\mathbb{D})$. Moreover, for the holomorphic function $f = u_0 + iu_1$ with $u_0(0) = 0$, there exists a constant $C_p$ depending only on $p$, such that*

$$M_p(f;r) \le C_p\, M_p(u_1;r), \qquad 0 \le r < 1.$$

The main result of this paper is the following monogenic version of the Riesz Theorem.

**Theorem 1.1.** *Let $f = u_0\mathbf{e}_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$ be a monogenic function in the unit ball $B$, $f_0 = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$, $u_0(0) = 0, 1 < p < \infty$. Then*

$$M_p(f;r) \le C_p\, M_p(f_0;r), \qquad 0 \le r < 1. \tag{1.1}$$

*The constant* $C_p$ *can be chosen as*

$$C_p = \left(\frac{4}{p-1}\right)^{1/p} \quad \textit{for} \quad 1 < p \leq 2,$$
$$C_p = |\lambda|^{-1}\left(A^{p/2} - (1-\lambda)^p\right)^{1/p} \quad \textit{for} \quad 2 < p < \infty,$$

*where* $A \geq 4p(p-1),\ 0 < |\lambda| \leq \min\left\{1, \frac{1}{16(p-2)}\right\}$.

*Remark* 1.2. In general, $f_0$ in (1.1) cannot be replaced by a function of type $f_{00} = u_2\mathbf{e}_2 + u_3\mathbf{e}_3$ containing fewer components than $f_0$. Indeed, inequality (1.1) fails, for example, for the monogenic function $f = x_1 + x_0\mathbf{e}_1$ with $f_{00} \equiv 0$.

*Remark* 1.3. For Riesz systems in $\mathbb{R}^n$, Theorem 1.1 was first proved by Kuran [17], see also Essén [11] (for $1 < p \leq 2$), Burkholder [9], Arcozzi [1].

## 2. Some Subharmonic Functions

In this section, we prove the subharmonicity of three important functions which are essential to the proof of our main theorem in Sec. 3. Everywhere below we assume $f = u_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 =: u_0 + f_0$, where $u_0 = \mathbf{Sc}\, f$, $f_0 = \mathbf{Vec}\, f = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$, $f_\lambda := \lambda u_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$ for $\lambda \in \mathbb{R}$.

**Lemma 2.1.** *If a function* $f : \mathbb{H} \longrightarrow \mathbb{H}$ *is monogenic in the unit ball* $B$*, then the following three functions are subharmonic in* $B$*:*

(a) $s_1 := |f|^p$, *for* $p \geq \frac{2}{3}$;

(b) $s_2 := A|f_0|^p - |f|^p$, *for* $1 < p \leq 2,\ A \geq \frac{4}{p-1}$;

(c) $s_3 := A|f_0|^2|f_\lambda|^{p-2} - |f_\lambda|^p$, *for* $p \geq 2,\ A \geq 4p(p-1)$, $|\lambda| \leq \min\left\{1, \frac{1}{16(p-2)}\right\}$.

*The exponent* $2/3$ *in (a) is sharp.*

*Remark* 2.2. Part (a) of Lemma 2.1 is proved by Stein and Weiss [23] for Riesz systems in $\mathbb{R}^n$. Lemma 2.1 entirely with other restrictions on $A$ and $\lambda$ is proved by Kuran [17, 18] again for Riesz systems in $\mathbb{R}^n$. Coifman and Weiss [10] proved Lemma 2.1 for more general Generalized Cauchy-Riemann Systems but without the determination of the subharmonicity exponent. Li and Peng [19] stated part (a) for two-sided monogenic functions in $\mathbb{R}^4$, that is, again for Riesz systems. Kheyfits and Tepper [16] obtained an octonion version of Lemma 2.1.

The equation $Df = 0$ is equivalent to the system

$$\begin{cases} \dfrac{\partial u_0}{\partial x_0} - \dfrac{\partial u_1}{\partial x_1} - \dfrac{\partial u_2}{\partial x_2} - \dfrac{\partial u_3}{\partial x_3} = 0 \\ \dfrac{\partial u_0}{\partial x_1} + \dfrac{\partial u_1}{\partial x_0} - \dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2} = 0 \\ \dfrac{\partial u_0}{\partial x_2} + \dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_2}{\partial x_0} - \dfrac{\partial u_3}{\partial x_1} = 0 \\ \dfrac{\partial u_0}{\partial x_3} - \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} + \dfrac{\partial u_3}{\partial x_0} = 0. \end{cases} \tag{2.1}$$

Define two associated matrices

$$M = \begin{pmatrix} \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_0} \end{pmatrix}, \; N = \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix}.$$

By means of the column vector $f^T = (u_0, u_1, u_2, u_3)^T$, system (2.1) can be written as the equation $Mf^T = 0$.

It should be noted that unlike Riesz systems, the matrix $N$ is not symmetric and has nonzero trace and multiple eigenvalues $\alpha_0 \pm i\sqrt{\alpha_0^2 + \alpha_1^2 + \alpha_2^2}$. Therefore the approach of [23, 10, 24] does not work here in monogenic function case.

*Proof of Lemma* 2.1. (a) Denote by $Z = Z(f)$ the zero set of the function $f$, and also $B_+ := B \setminus Z$ that is an open set in $B$. It is enough to prove that $\Delta s_1(x) = \Delta|f(x)|^p \geq 0$ at any point $x \in B_+$ (at the other points the subharmonicity is trivial). We begin with the well-known identity for the Laplacian (see, e.g., [23], [22, Ch.7])

$$\begin{aligned} \Delta|f(x)|^p &= p|f(x)|^{p-4}\left[|f|^2|\nabla f|^2 + \frac{p-2}{4}\left|\nabla\left(|f|^2\right)\right|^2\right] \\ &= p|f(x)|^{p-4}\left[|f|^2|\nabla f|^2 + (p-2)\sum_{j=0}^{3}\left(f \cdot \frac{\partial f}{\partial x_j}\right)^2\right] =: p|f|^{p-4}E(x), \end{aligned} \tag{2.2}$$

where the dot denotes the Euclidean inner product and $E(x)$ is the expression in square brackets. Now, our goal is to show that $E(x) \geq 0$ at the points $x \in B_+$. Fix an arbitrary point $x' \in B_+$. Without loss of generality, we may assume that $|f(x')| = 1$. Moreover, $\Delta$ and $|\nabla|$ are invariant under rotations of axes. Therefore, we can choose a new system of axes $(y_0, y_1, y_2, y_3)$ such that at the point $y'$ corresponding to $x'$, we have

$$f(y') = \mathbf{e}_0 \quad \text{and} \quad \frac{\partial u_j}{\partial y_k}(y') = 0 \quad \text{for all} \quad j \neq k, \; 1 \leq j, k \leq 3. \tag{2.3}$$

To this end, first we can choose the $y_0$-axis parallel to $f(y') = \mathbf{e}_0$, then by a suitable rotation of the other three axes, we can diagonalize the submatrix

$$N' = \begin{pmatrix} \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & \alpha_0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix}.$$

The diagonalization is possible since the matrix $N'$ has three different eigenvalues $\alpha_0$ and $\alpha_0 \pm i\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$.

Thus, we need to prove that $E(y') \geq 0$. First, calculate the inner product in (2.2), $\left(f \cdot \frac{\partial f}{\partial x_j}\right)^2 (y') = \left(\frac{\partial u_0(y')}{\partial x_j}\right)^2$. Second, with the use of (2.1) and (2.3), the first term in the brackets in (2.2) can be transformed into

$$\begin{aligned} |f(y')|^2 |\nabla f(y')|^2 &= \sum_{k=0}^{3} \left| \frac{\partial f(y')}{\partial y_k} \right|^2 = \sum_{k=0}^{3} \sum_{j=0}^{3} \left( \frac{\partial u_j(y')}{\partial y_k} \right)^2 \\ &= \sum_{k=1}^{3} \sum_{j=1}^{3} \left( \frac{\partial u_j(y')}{\partial y_k} \right)^2 + \sum_{k=0}^{3} \left( \frac{\partial u_0(y')}{\partial y_k} \right)^2 + \sum_{j=1}^{3} \left( \frac{\partial u_j(y')}{\partial y_0} \right)^2 \\ &= \sum_{k=0}^{3} \left( \frac{\partial u_k(y')}{\partial y_k} \right)^2 + 2 \sum_{k=1}^{3} \left( \frac{\partial u_0(y')}{\partial y_k} \right)^2. \end{aligned}$$

Inserting this into the expression of $E(y')$, we then estimate it from below,

$$\begin{aligned} E(y') &= \sum_{k=0}^{3} \left( \frac{\partial u_k(y')}{\partial y_k} \right)^2 + 2 \sum_{k=1}^{3} \left( \frac{\partial u_0(y')}{\partial y_k} \right)^2 + (p-2) \sum_{j=0}^{3} \left( \frac{\partial u_0(y')}{\partial x_j} \right)^2 \\ &= (p-1) \left( \frac{\partial u_0(y')}{\partial y_0} \right)^2 + \sum_{k=1}^{3} \left( \frac{\partial u_k(y')}{\partial y_k} \right)^2 + p \sum_{k=1}^{3} \left( \frac{\partial u_0(y')}{\partial y_k} \right)^2 \\ &\geq (p-1) \left( \frac{\partial u_0(y')}{\partial y_0} \right)^2 + \sum_{k=1}^{3} \left( \frac{\partial u_k(y')}{\partial y_k} \right)^2. \end{aligned}$$

Next, by the Cauchy-Schwarz inequality and (2.1),

$$E(y') \geq (p-1) \left( \frac{\partial u_0}{\partial y_0} \right)^2 + \frac{1}{3} \left( \sum_{k=1}^{3} \frac{\partial u_k}{\partial y_k} \right)^2 = \left( p - \frac{2}{3} \right) \left( \frac{\partial u_0(y')}{\partial y_0} \right)^2 \geq 0.$$

Thus, $\Delta s_1(x) = \Delta |f(x)|^p \geq 0$, and $s_1$ is subharmonic in $B$. For exponents $p < 2/3$, the assertion is no longer true. A relevant counterexample is, for instance, $f(x) = \frac{\mathbf{e}_0 - \overline{x}}{|\mathbf{e}_0 - x|^4}$.

(b) Denote by $Z$ and $Z_0$ the zero sets of the functions $f$ and $f_0$ respectively. Also, set $B_+ := B \setminus Z$ and $B_0 := B \setminus Z_0$, hence $Z \subset Z_0$ and $B_0 \subset B_+$.

Note that $s_2 \in C^2(B_0)$. First, we will prove that $\Delta s_2(x) \geq 0$ on the set $B_0$. For an estimation, we will use identity (2.2) for general vectors $f$. System

(2.1) implies

$$\begin{aligned}
\left(\frac{\partial u_0}{\partial x_0}\right)^2 &\le 3\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2+\left(\frac{\partial u_2}{\partial x_2}\right)^2+\left(\frac{\partial u_3}{\partial x_3}\right)^2\right],\\
\left(\frac{\partial u_0}{\partial x_1}\right)^2 &\le 3\left[\left(\frac{\partial u_1}{\partial x_0}\right)^2+\left(\frac{\partial u_2}{\partial x_3}\right)^2+\left(\frac{\partial u_3}{\partial x_2}\right)^2\right],\\
\left(\frac{\partial u_0}{\partial x_2}\right)^2 &\le 3\left[\left(\frac{\partial u_1}{\partial x_3}\right)^2+\left(\frac{\partial u_2}{\partial x_0}\right)^2+\left(\frac{\partial u_3}{\partial x_1}\right)^2\right],\\
\left(\frac{\partial u_0}{\partial x_3}\right)^2 &\le 3\left[\left(\frac{\partial u_1}{\partial x_2}\right)^2+\left(\frac{\partial u_2}{\partial x_1}\right)^2+\left(\frac{\partial u_3}{\partial x_0}\right)^2\right].
\end{aligned} \tag{2.4}$$

Summing all four inequalities (2.4), we obtain $|\nabla u_0|^2 \le 3|\nabla f_0|^2$, which can be rewritten in the equivalent forms

$$|\nabla u_0|^2 \le \frac{3}{4}|\nabla f|^2 \qquad \text{or} \qquad |\nabla f| \le 2|\nabla f_0|. \tag{2.5}$$

Since $1 < p \le 2$, dropping the nonpositive term in formula (2.2), we immediately obtain

$$\Delta|f|^p \le p|f|^{p-2}|\nabla f|^2. \tag{2.6}$$

On the other hand, formula (2.2) written for the function $f_0$ immediately implies

$$\Delta|f_0|^p \ge p(p-1)|f_0|^{p-2}|\nabla f_0|^2. \tag{2.7}$$

It follows from (2.7), (2.6), (2.5) and $|f_0|^{p-2} \ge |f|^{p-2}$ that

$$\begin{aligned}
\Delta s_2 = A\Delta|f_0|^p - \Delta|f|^p &\ge Ap(p-1)|f_0|^{p-2}|\nabla f_0|^2 - p|f|^{p-2}|\nabla f|^2\\
&\ge p|f|^{p-2}\Big[A(p-1)|\nabla f_0|^2 - |\nabla f|^2\Big]\\
&\ge p|f|^{p-2}|\nabla f|^2\left[\frac{A}{4}(p-1)-1\right] \ge 0.
\end{aligned}$$

Thus, we have proved that $\Delta s_2(x) \ge 0$ on the set $B_0$. Now we have to extend the subharmonicity of $s_2$ from the set $B_0$ to the whole ball $B$.

In the case $p = 2$, obviously the function $s_2 = A|f_0|^2 - |f|^2$ $(A \ge 4)$ is twice differentiable on $B$. Hence the above estimates for the Laplacian $\Delta s_2$ are valid on the whole ball $B$. Therefore, $s_2$ is subharmonic on $B$, so we can write the mean value inequality in particular at the points $x \in Z_0$ for sufficiently small $\varepsilon > 0$

$$-|f(x)|^2 \le \frac{1}{\varepsilon^3}\int_{|y-x|=\varepsilon}\Big[4|f_0(y)|^2 - |f(y)|^2\Big]\,d\sigma(y), \qquad x \in Z_0. \tag{2.8}$$

In the case $1 < p < 2$, we have $A^{2/p} \ge \left(\frac{4}{p-1}\right)^{2/p} \ge \frac{4}{p-1} \ge 4$, and hence

$$\Big(|f|^p - A|f_0|^p\Big)^{2/p} + 4|f_0|^2 \le \Big(|f|^p - A|f_0|^p\Big)^{2/p} + (A|f_0|^p)^{2/p} \le |f|^2. \tag{2.9}$$

Consider a point $x \in Z_0 \setminus Z$, that is, $f_0(x) = 0$ but $f(x) \neq 0$. Then $|f(x)| > |f_0(x)| = 0$, and by continuity, $s_2(y) = A|f_0(y)|^p - |f(y)|^p \leq 0, \quad |y-x| \leq \varepsilon_0$, for an $\varepsilon_0 > 0$ small enough. Then by Hölder's inequality and (2.9), (2.8)

$$\begin{aligned}
\frac{1}{\varepsilon_0^3}\int_{|y-x|=\varepsilon_0}\Big(-s_2(y)\Big)\,d\sigma(y) &= \frac{1}{\varepsilon_0^3}\int_{|y-x|=\varepsilon_0}\Big(|f(y)|^p - A|f_0(y)|^p\Big)\,d\sigma(y)\\
&\leq \left[\frac{1}{\varepsilon_0^3}\int_{|y-x|=\varepsilon_0}\Big(|f(y)|^p - A|f_0(y)|^p\Big)^{2/p}\,d\sigma(y)\right]^{p/2}\\
&\leq \left[\frac{1}{\varepsilon_0^3}\int_{|y-x|=\varepsilon_0}\Big(|f(y)|^2 - 4|f_0(y)|^2\Big)\,d\sigma(y)\right]^{p/2}\\
&\leq \Big(|f(x)|^2\Big)^{p/2} = |f(x)|^p - A|f_0(x)|^p = -s_2(x).
\end{aligned}$$

Thus, the mean value inequality holds for $s_2$ on $B_+$, so the function $s_2$ is subharmonic on the open set $B_+$.

Coifman and Weiss [10, p.81] proved for more general vectors $f$ that the zero set $Z$ is a polar set. The continuous function $s_2$ subharmonic on the open set $B_+ = B \setminus Z$, must be subharmonic on the whole ball $B$, by a result of Brelot, see, for example, [15, Sec.5.5.2].

(c) To prove the assertion (c), first note that $f_\lambda(x) \neq 0$ for the points $x \in B_+$. Therefore, $s_3 \in C^2(B_+)$. Now prove that $\Delta s_3(x) \geq 0$ on the set $B_+$. We need the identities $\Delta(\varphi\psi) = \varphi\Delta\psi + \psi\Delta\varphi + 2\nabla\varphi \cdot \nabla\psi$, and $\nabla|f|^p = p|f|^{p-2}\left(f \cdot \frac{\partial f}{\partial x_0}, f \cdot \frac{\partial f}{\partial x_1}, f \cdot \frac{\partial f}{\partial x_2}, f \cdot \frac{\partial f}{\partial x_3}\right)$. These identities together with formula (2.2) imply

$$\begin{aligned}
\Delta s_3 &= A|f_0|^2\Delta|f_\lambda|^{p-2} + A|f_\lambda|^{p-2}\Delta|f_0|^2 + 2A\nabla|f_0|^2 \cdot \nabla|f_\lambda|^{p-2} - \Delta|f_\lambda|^p\\
&= 2A|f_\lambda|^{p-2}|\nabla f_0|^2 + A(p-2)|f_0|^2|f_\lambda|^{p-4}\left[(p-4)\sum_{j=0}^{3}\left(\frac{f_\lambda}{|f_\lambda|}\cdot\frac{\partial f_\lambda}{\partial x_j}\right)^2 + \right.\\
&\quad \left. + |\nabla f_\lambda|^2\right] + 4A(p-2)|f_\lambda|^{p-4}\sum_{j=0}^{3}\left(f_0 \cdot \frac{\partial f_0}{\partial x_j}\right)\left(f_\lambda \cdot \frac{\partial f_\lambda}{\partial x_j}\right) - \Delta|f_\lambda|^p.
\end{aligned}$$

Multiply both sides by $|f_\lambda|^{2-p}$ and then estimate it from below, by dropping the two positive terms with $|\nabla f_\lambda|^2$ and $p$ in the brackets, and using the trivial inequalities $|f_0| \leq |f_\lambda| \leq |f|$ and $|\lambda|\left|\frac{\partial f}{\partial x_j}\right| \leq \left|\frac{\partial f_\lambda}{\partial x_j}\right| \leq \left|\frac{\partial f}{\partial x_j}\right|$. It leads to

$$
\begin{aligned}
|f_\lambda|^{2-p}\Delta s_3 &\geq 2A|\nabla f_0|^2 - 4A(p-2)|f_0|^2|f_\lambda|^{-2}\sum_{j=0}^{3}\left(\frac{f_\lambda}{|f_\lambda|}\cdot\frac{\partial f_\lambda}{\partial x_j}\right)^2 + \\
&\quad + 4A(p-2)|f_\lambda|^{-2}\sum_{j=0}^{3}\left(f_0\cdot\frac{\partial f_0}{\partial x_j}\right)\left(f_\lambda\cdot\frac{\partial f_\lambda}{\partial x_j}\right) - |f_\lambda|^{2-p}\Delta|f_\lambda|^p \\
&\geq 2A|\nabla f_0|^2 - |f_\lambda|^{2-p}\Delta|f_\lambda|^p + \\
&\quad + 4A(p-2)|f_\lambda|^{-2}\sum_{j=0}^{3}\left(f_\lambda\cdot\frac{\partial f_\lambda}{\partial x_j}\right)\left[\left(f_0\cdot\frac{\partial f_0}{\partial x_j}\right) - \left(f_\lambda\cdot\frac{\partial f_\lambda}{\partial x_j}\right)\right].
\end{aligned}
$$

Next, since $f_0\cdot\frac{\partial f_0}{\partial x_j} - f_\lambda\cdot\frac{\partial f_\lambda}{\partial x_j} = -\lambda^2 u_0\frac{\partial u_0}{\partial x_j}$ for each $j$, we continue the estimation

$$
|f_\lambda|^{2-p}\Delta s_3 \geq 2A|\nabla f_0|^2 - |f_\lambda|^{2-p}\Delta|f_\lambda|^p - 4A|\lambda|(p-2)\sum_{j=0}^{3}\left|\frac{\partial f}{\partial x_j}\right|^2,
$$

where we have used the Cauchy-Schwarz inequality

$$
|f_\lambda|^{-2}\left|u_0\frac{\partial u_0}{\partial x_j}\left(f_\lambda\cdot\frac{\partial f_\lambda}{\partial x_j}\right)\right| \leq \frac{1}{|\lambda|}\frac{|\lambda u_0|}{|f_\lambda|}\left|\frac{\partial u_0}{\partial x_j}\right|\left|\frac{\partial f_\lambda}{\partial x_j}\right| \leq \frac{1}{|\lambda|}\left|\frac{\partial f}{\partial x_j}\right|^2.
$$

Consequently

$$
\begin{aligned}
|f_\lambda|^{2-p}\Delta s_3 &\geq 2A|\nabla f_0|^2 - |f_\lambda|^{2-p}\Delta|f_\lambda|^p - 4A|\lambda|(p-2)|\nabla f|^2 \\
&= A\Big[|\nabla f_0|^2 - 4|\lambda|(p-2)|\nabla f|^2\Big] + \Big[A|\nabla f_0|^2 - |f_\lambda|^{2-p}\Delta|f_\lambda|^p\Big].
\end{aligned}
$$

It suffices to show that the expression in each bracket is nonnegative. By (2.5),

$$
|\nabla f_0|^2 - 4|\lambda|(p-2)|\nabla f|^2 \geq \frac{1}{4}\Big(1 - 16|\lambda|(p-2)\Big)|\nabla f|^2 \geq 0,
$$

because $|\lambda| \leq \frac{1}{16(p-2)}$. By identity (2.2) for the Laplacian, the Cauchy-Schwarz inequality and (2.5), we get

$$
\begin{aligned}
&A|\nabla f_0|^2 - |f_\lambda|^{2-p}\Delta|f_\lambda|^p \\
&\qquad \geq A|\nabla f_0|^2 - p(p-2)|f_\lambda|^{-2}\sum_{j=0}^{3}|f_\lambda|^2\left|\frac{\partial f_\lambda}{\partial x_j}\right|^2 + p|\nabla f_\lambda|^2 \\
&\qquad = A|\nabla f_0|^2 - p(p-2)|\nabla f_\lambda|^2 + p|\nabla f_\lambda|^2 \\
&\qquad \geq \left[\frac{A}{4} - p(p-1)\right]|\nabla f_\lambda|^2 \geq 0, \qquad \text{since} \quad A \geq 4p(p-1).
\end{aligned}
$$

Thus, we have proved that $\Delta s_3(x) \geq 0$ on the open set $B_+$. The same argument of Brelot ([10, p.81], [15, Sec.5.5.2]) used in the proof of (b) works here, and we conclude that $s_3$ is subharmonic on whole $B$. □

*Remark* 2.3. In part (c) of Lemma 2.1, we slightly correct the range of the parameter $\lambda$ in comparison with Theorem (c) [10, p.78].

## 3. Proof of Theorem 1.1

We now turn to the proof of a monogenic version of the M. Riesz theorem, the main result of the paper.

*Proof of Theorem* 1.1. *Case* $1 < p \le 2$. By Lemma 2.1, the function $s_2 = \frac{4}{p-1}|f_0|^p - |f|^p$ is subharmonic in $B$. Then compute $s_2(0) = \frac{5-p}{p-1}\,|f(0)|^p \ge 0$. By the sub-mean-value property of the subharmonic function $s_2$,

$$0 \le s_2(0) \le \int_S s_2(r\zeta)\,d\sigma(\zeta) = \int_S \Big(\frac{4}{p-1}|f_0(r\zeta)|^p - |f(r\zeta)|^p\Big)\,d\sigma(\zeta),$$

$$\int_S |f(r\zeta)|^p\,d\sigma(\zeta) \le \frac{4}{p-1}\int_S |f_0(r\zeta)|^p\,d\sigma(\zeta),$$

$$M_p(f;r) \le \left(\frac{4}{p-1}\right)^{1/p} M_p(f_0;r), \qquad 0 \le r < 1.$$

*Case* $2 < p < \infty$. By Lemma 2.1, the function $s_3 = A|f_0|^2|f_\lambda|^{p-2} - |f_\lambda|^p$ is subharmonic in $B$ for any $A \ge 4p(p-1)$, $|\lambda| \le \min\left\{1, \frac{1}{16(p-2)}\right\}$. Since $f(0) = f_0(0) = f_\lambda(0)$, we have $s_3(0) = (A-1)|f(0)|^p \ge 0$. By the sub-mean-value property of the subharmonic function $s_3$,

$$0 \le s_3(0) \le \int_S s_3(r\zeta)\,d\sigma(\zeta) = \int_S \Big(A|f_0(r\zeta)|^2|f_\lambda(r\zeta)|^{p-2} - |f_\lambda(r\zeta)|^p\Big)\,d\sigma(\zeta).$$

Hence, by Hölder's inequality with the indices $p/2$ and $p/(p-2)$,

$$\begin{aligned}\int_S |f_\lambda(r\zeta)|^p\,d\sigma(\zeta) &\le A\int_S |f_0(r\zeta)|^2|f_\lambda(r\zeta)|^{p-2}\,d\sigma(\zeta)\\ &\le A\left(\int_S |f_0(r\zeta)|^p\,d\sigma(\zeta)\right)^{2/p}\left(\int_S |f_\lambda(r\zeta)|^p\,d\sigma(\zeta)\right)^{\frac{p-2}{p}}.\end{aligned}$$

Since $1 - \frac{p-2}{p} = \frac{2}{p}$,

$$\int_S |f_\lambda(r\zeta)|^p\,d\sigma(\zeta) \le A^{p/2}\int_S |f_0(r\zeta)|^p\,d\sigma(\zeta). \tag{3.1}$$

Now estimate the left-hand side integral in (3.1) from below. Making use of the identity

$$f_\lambda(x) = (1-\lambda)f_0(x) + \lambda f(x), \qquad x \in B,$$

and the inequality $(a+b)^{2/p} \ge a^{2/p} + b^{2/p}$, we get

$$\begin{aligned}A^{p/2}\int_S |f_0|^p\,d\sigma \ge \int_S |f_\lambda|^p\,d\sigma &= \int_S |(1-\lambda)f_0 + \lambda f|^p\,d\sigma\\ &= \int_S \Big((1-\lambda)^2|f_0|^2 + \lambda^2|f|^p\Big)^{p/2}\,d\sigma\\ &\ge (1-\lambda)^p\int_S |f_0|^p\,d\sigma + |\lambda|^p\int_S |f|^p\,d\sigma.\end{aligned}$$

Thus,

$$\int_S |f(r\zeta)|^p \, d\sigma(\zeta) \le \frac{A^{p/2} - (1-\lambda)^p}{|\lambda|^p} \int_S |f_0(r\zeta)|^p \, d\sigma(\zeta), \qquad 0 \le r < 1,$$

or

$$M_p^p(f;r) \le C_p^p M_p^p(f_0;r), \qquad 0 \le r < 1.$$

This completes the proof of the main theorem. □

As a simple application, we obtain a result about the increasing character of the integral means as well as a Hadamard three spheres theorem for monogenic functions (cf. [15, Th. 2.12]).

**Corollary 3.1.** *Let $f$ be a monogenic function on the ball $|x| < R \le \infty$. Then $M_p(f;r)$ is an increasing function of $r \in (0,R)$ as long as $p \ge 2/3$.*

**Corollary 3.2.** *Let $f$ be a monogenic function on the spherical shell $\{x \in \mathbb{R}^4 : 0 \le r_1 < |x| = r < r_2 \le \infty\}$. Then $M_p\big(|f|^{2/3}; r\big)$ is a convex function of $r^{-2}$ on $(r_1, r_2)$ as long as $p \ge 1$.*

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