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On the Fractional Integro-Differentiation Operator in $\mathbb{R}^n$

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Abstract—In the paper, a new form of fractional derivative is introduced for functions defined in the unit ball in $\mathbb{R}^n$. As an application, an integral representation for harmonic functions with finite mixed-norm in the ball is obtained.

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1. INTRODUCTION

The concept of fractional integration and differentiation, originating in the XIX century, at the beginning was connected with the theory of integral and differential equations. The fractional integrals and derivatives were in the focus of Abel, Liouville, Riemann, Hadamard, Weyl, Hardy, Littlewood, M. Riesz and others (see the monograph by Samko, Kilbas and Marichev [1], and references therein). Later the fractional integro-differentiation was applied in function theory. In particular, it was used as an effective tool in the study of classical functional spaces (see [2] − [9]).

One of the disadvantages of fractional integrals and derivatives is their variety and often incompatibility with each other, including in various applications. A large number of choices of various types of fractional integrals and derivatives becomes an advantage when it is necessary to adjust them to various applications.

In the present paper we develop and apply the classical Riemann-Liouville fractional integro-differentiation in the problems of the theory of harmonic spaces in $\mathbb{R}^n$ $(n \geq 2)$.

We first introduce necessary notation. Let $B = B_n$ be the open unit ball of $\mathbb{R}^n$, and let $S = \partial B$ be its boundary, that is, the unit sphere. The integral means of order p of a harmonic function $u(x) = u(r\zeta)$ on the sphere $|x| = r$ we denote by

$$M_p(u;r) = \big\|u(r\cdot)\big\|_{L^p(S;\,d\sigma)}, \qquad 0 \leq r < 1,\ 0 < p \leq \infty,$$

where $d\sigma$ is the $(n-1)$-dimensional surface Lebesgue measure on S normed by condition $\sigma(S) = 1$. The set of all real harmonic functions in the ball B we denote by $h(B)$.

Define the space $h(p,q,\alpha)\big(0 < p, q \leq \infty, \alpha \in \mathbb{R}\big)$ with mixed norm to be the space of those harmonic functions $u \in h(B)$, for which the following quasi-norm is finite

$$\|u\|_{h(p,q,\alpha)} = \begin{cases} \left(\displaystyle\int_0^1 (1-r)^{\alpha q - 1} M_p^q(u;r)\,dr\right)^{1/q}, & 0 < q < \infty, \\ \displaystyle\sup_{0<r<1} (1-r)^{\alpha} M_p(u;r), & q = \infty. \end{cases}$$

Note that for $p = q < \infty$ the spaces $h(p,q,\alpha)$ coincide with weighted Bergman spaces, while for $q = \infty$ these spaces are called weighted Hardy spaces. A number of papers are devoted to the study

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of spaces $h(p, q, \alpha)$ with mixed norm or to their analogs, consisting of holomorphic, pluriharmonic or harmonic functions in the disk or in the ball of $\mathbb{C}^n$ or $\mathbb{R}^n$.

The mixed-norm spaces for holomorphic in the unit disk functions were introduced by Hardy and Littlewood (see [2], [3]). Then, Flett (see [6], [7]) considerably developed the theory. Later were obtained multivariate generalizations of the results by Hardy, Littlewood and Flett in the domains from $\mathbb{C}^n$ and $\mathbb{R}^n$. The spaces $h(p, p, \alpha)$ and $h(p, q, \alpha)$ in the unit ball of $\mathbb{R}^n$ were studied in the papers [10]–[21], while the spaces $h(p, q, \alpha)$, consisting of n-harmonic functions in the polydisk of $\mathbb{C}^n$, were studied in the papers [22] – [24].

Our basic supplement of fractional operators into the theory of harmonic spaces is the following integral representation in the mixed-norm spaces $h(p, q, \alpha)$.

Theorem 1.1. *Let $0 < p, q \le \infty$, $\alpha > 0$, and let $u \in h(p, q, \alpha)$ be an arbitrary function. Assume that the following condition is satisfied: either $0 < p, q \le \infty$ and $\beta > \max\{\alpha + (n-1)(1/p - 1), \alpha\}$, or $1 \le p \le \infty$, $0 < q \le 1$ and $\beta \ge \alpha$. Then*

$$u(x) = \frac{2}{n\,\Gamma(\beta)} \int_B (1 - |y|^2)^{\beta - 1} P_\beta(x, y)\, u(y)\, dV(y), \qquad x \in B, \tag{1.1}$$

where $P_\beta(x, y)$ is the Poisson-Bergman kernel, defined in Section 5.

Remark 1.1. *In the special case where $p = q \ge 1$ and $\beta = \alpha$, or even in more narrow classes, a similar integral representation was obtained in [10], [11], [13], [14], [18], and for more general weight functions, in [17], [19]. For holomorphic functions in the unit ball of $\mathbb{C}^n$, Theorem 1.1 for $1 \le p, q \le \infty$ can be found in [25], [26], while for n-harmonic functions in the polydisk it was proved in [22].*

2. FRACTIONAL INTEGRO-DIFFERENTIATION IN THE UNIT DISK OF A PLANE

In this section we give the definition of the well-known Riemann-Liouville fractional integro-differentiation operator D^α and one of its modification denoted by $\mathcal{D}^\alpha$. In what follows, we will assume that for positive order α the operators D^α and $\mathcal{D}^\alpha$ represent fractional differentiation of order $\alpha > 0$, while for negative order α the operators D^α and $\mathcal{D}^\alpha$ will stand for fractional integration of order $|\alpha| > 0$. In the case $\alpha = 0$ we assume that D^0 and $\mathcal{D}^0$ are identity operators, that is, $D^0 f = f$ and $\mathcal{D}^0 f = f$.

Definition 2.1 (Riemann-Liouville fractional integro-differentiation on $\mathbb{R}^2$). *Given a function $f(r)$ of a single variable $r \in [0, 1)$, define*

$$D_2^{-\alpha} f(r) := \frac{1}{\Gamma(\alpha)} \int_0^r (r - t)^{\alpha - 1} f(t)\, dt = \frac{r^\alpha}{\Gamma(\alpha)} \int_0^1 (1 - t)^{\alpha - 1} f(tr)\, dt, \tag{2.1}$$

$$D_2^m f(r) := \left(\frac{d}{dr}\right)^m f(r), \qquad D_2^\alpha f(r) := D_2^{-(m-\alpha)} D_2^m f(r), \tag{2.2}$$

$$\mathcal{D}_2^{-\alpha} f(r) := r^{-\alpha} D_2^{-\alpha} f(r) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - t)^{\alpha - 1} f(tr)\, dt, \tag{2.3}$$

$$\mathcal{D}_2^\alpha f(r) := D_2^\alpha \Big\{ r^\alpha f(r) \Big\}, \tag{2.4}$$

where $0 \le r < 1,\ \alpha > 0,\ m \in \mathbb{Z},\ m \ge 0,\ m - 1 < \alpha \le m.$

The lower index 2 in the operators indicates that below the operators will be applied with respect to functions that are defined on the plane, in particular, on the unit disk $\mathbb{D}$. In subsequent sections we also will consider the operators $\mathcal{D}^\alpha$ with respect to functions that are defined in $\mathbb{R}^n$. The consideration of the modification $\mathcal{D}_2^\alpha$ is stipulated by the fact that the operator D_2^α does not preserve holomorphic or harmonic functions, while the operator $\mathcal{D}_2^\alpha$ does, that is, $\mathcal{D}_2^\alpha : H(\mathbb{D}) \longrightarrow H(\mathbb{D})$ and $\mathcal{D}_2^\alpha : h(\mathbb{D}) \longrightarrow h(\mathbb{D})$. It is worth to note that the first order differentiation operator $\mathcal{D}_2^1$ does not coincide with ordinary partial or complex derivative, namely, $\mathcal{D}_2^1 f = \frac{\partial}{\partial r}(rf)$, which is called radial derivative of function f. Sometimes we write $D^\alpha_{2,r}$ and $\mathcal{D}^\alpha_{2,r}$, to indicate the variable r, with respect to which differentiation or integration is carried out.

The operators D^α_2 and $\mathcal{D}^\alpha_2$ are well adapted in applications to power functions. The next lemma can be proved by direct calculations.

Lemma 2.1. *For $\alpha > 0$, $k > -1$, $m \in \mathbb{Z}$, $m \geq 0$, $D^\alpha \equiv D^\alpha_2$, $\mathcal{D}^\alpha \equiv \mathcal{D}^\alpha_2$ the following identities hold:*

$$D^{-\alpha}\{r^k\} = \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}\, r^{k+\alpha}, \tag{2.5}$$

$$D^{m}\{r^k\} = \frac{\Gamma(k+1)}{\Gamma(k-m+1)}\, r^{k-m}, \qquad k-m \neq -1,-2,\dots, \tag{2.6}$$

$$D^{\alpha}\{r^{k+\alpha}\} = \frac{\Gamma(\alpha+k+1)}{\Gamma(k+1)}\, r^{k}, \tag{2.7}$$

$$\mathcal{D}^{-\alpha}\{r^k\} = \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}\, r^{k}, \tag{2.8}$$

$$\mathcal{D}^{\alpha}\{r^k\} = \frac{\Gamma(\alpha+k+1)}{\Gamma(k+1)}\, r^{k}. \tag{2.9}$$

Note that except Definition 2.1, there are other definitions of integro-differentiation by means of fractional integrals via expansions of power series (see, e.g., [1], [3], [5] – [7]). These expansions are convenient for extension of fractional integro-differentiation to functions defined in $\mathbb{R}^n$.

3. RIEMANN-LIOUVILLE FRACTIONAL INTEGRATION FOR HARMONIC AND NONHARMONIC FUNCTIONS IN $\mathbb{R}^n$

We first define fractional integro-differentiation for harmonic in $\mathbb{R}^n$ functions. Assume that a harmonic function $f(x) \in h(B)$ is expanded into a series of homogeneous spherical harmonics Y_{kj}:

$$f(x) = \sum_{k=0}^{\infty}\sum_{j=1}^{d_k} a_{kj}\, r^k\, Y_{kj}(\zeta) \equiv \sum_{k,j} a_{kj}\, r^k\, Y_{kj}(\zeta), \qquad x = r\zeta \in B, \tag{3.1}$$

where $d_k = \frac{(2k+n-2)(n+k-3)!}{(n-2)!\,k!}$ is the dimension of the space $\mathcal{H}_k(S)$ of spherical harmonics of order k. The theory of spaces $\mathcal{H}_k(S)$ can be found, for instance, in monographs [27] and [16].

The concept of fractional integro-differentiation is well-known for harmonic functions (3.1) in $\mathbb{R}^n$, represented in the form of a series by spherical harmonics (see [1], [8], [10]–[20]).

Definition 3.1. *Define*

$$\mathcal{D}^{-\alpha}_{n(ser)} f(x) := \sum_{k=0}^{\infty}\sum_{j=1}^{d_k} \frac{\Gamma(k+n/2)}{\Gamma(\alpha+k+n/2)}\, a_{kj}\, r^k\, Y_{kj}(\zeta), \qquad \alpha \geq 0, \tag{3.2}$$

$$\mathcal{D}^{\alpha}_{n(ser)} f(x) := \sum_{k=0}^{\infty}\sum_{j=1}^{d_k} \frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\, a_{kj}\, r^k\, Y_{kj}(\zeta), \qquad \alpha \geq 0. \tag{3.3}$$

Another, more general way to define the concept of fractional integration of order $\alpha > 0$ is to use Riemann-Liouville fractional integrals, similar to the case $n = 2$.

Definition 3.2. *Define*

$$\begin{aligned} D^{-\alpha}_n f(x) &:= \frac{1}{\Gamma(\alpha)} \int_0^r (r-t)^{\alpha-1} f(t\zeta)\, t^{n/2-1}\, dt \\ &= \frac{r^{\alpha+n/2-1}}{\Gamma(\alpha)} \int_0^1 (1-\xi)^{\alpha-1} f(\xi x)\, \xi^{n/2-1}\, d\xi = D^{-\alpha}_2\Big\{r^{n/2-1} f(x)\Big\}. \end{aligned} \tag{3.4}$$

This is the direct extension of the operator $D_2^{-\alpha}$ (see (2.1)) to the n-dimensional case. As in the case $n = 2$, we define its modification $\mathcal{D}_n^{-\alpha}$ as follows (cf. [12], [8]).

Definition 3.3. *Define*

$$\mathcal{D}_{n(int)}^{-\alpha} f(x) := r^{-(\alpha+n/2-1)} D_n^{-\alpha} f(x)$$
$$= r^{-(\alpha+n/2-1)} D_2^{-\alpha}\Big\{r^{n/2-1} f(x)\Big\} = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\xi)^{\alpha-1} f(\xi x)\,\xi^{n/2-1}\,d\xi. \tag{3.5}$$

The advantage of the modification $\mathcal{D}_{n(int)}^{-\alpha}$ over the operator $D_n^{-\alpha}$ is that the operator $\mathcal{D}_{n(int)}^{-\alpha}$ preserves harmonic functions, that is, $\mathcal{D}_{n(int)}^{-\alpha} : h(B) \longrightarrow h(B)$. The same property possess the operators $\mathcal{D}_{n(ser)}^{\pm\alpha}$.

Since both operators $\mathcal{D}_{n(ser)}^{-\alpha}$ and $\mathcal{D}_{n(int)}^{-\alpha}$ represent the antiderivative of order $\alpha > 0$ of a function f, a natural question arise concerning the relationship of these operators. In fact, these operators coincide for harmonic functions. This result is stated in the next lemma.

Lemma 3.1. *Both definitions* (3.2) *and* (3.5) *for the antiderivative* $\mathcal{D}_n^{-\alpha}$ $(\alpha > 0)$ *coincide for harmonic functions in* B*, that is,*

$$\mathcal{D}_{n(ser)}^{-\alpha} f(x) = \mathcal{D}_{n(int)}^{-\alpha} f(x) =: \mathcal{D}_n^{-\alpha} f(x), \quad x \in B, \quad f \in h(B).$$

Proof. Let a harmonic function $f \in h(B)$ be expanded into a series of spherical harmonics by formula (3.1). Then according to definitions (3.5) and (3.3) we can write

$$\begin{aligned}
\mathcal{D}_{n(int)}^{-\alpha} f(x) &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1} f(tx)\, t^{n/2-1}\,dt \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}\left(\sum_{k,j} a_{kj}\, t^k r^k\, Y_{kj}(\zeta)\right) t^{n/2-1}\,dt \\
&= \frac{1}{\Gamma(\alpha)}\sum_{k,j} a_{kj}\, r^k\, Y_{kj}(\zeta)\left(\int_0^1 (1-t)^{\alpha-1}\, t^{k+n/2-1}\,dt\right) \\
&= \sum_{k,j}\frac{\Gamma(k+n/2)}{\Gamma(\alpha+k+n/2)}\, a_{kj}\, r^k\, Y_{kj}(\zeta) = \mathcal{D}_{n(ser)}^{-\alpha} f(x),
\end{aligned}$$

and the result follows.

In particular, the action of the operator $\mathcal{D}_n^{-\alpha}$ on monomials of type r^γ $(\alpha > 0,\ \gamma > -n/2)$ yields the following useful formula

$$\mathcal{D}_n^{-\alpha}\{r^\gamma\} = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}(tr)^\gamma\, t^{n/2-1}\,dt = \frac{\Gamma(\gamma+n/2)}{\Gamma(\alpha+\gamma+n/2)}\, r^\gamma.$$

4. DEFINITION OF THE MODIFICATION OF RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE IN $\mathbb{R}^n$

While the Definition 3.3 of the operator of fractional integration is quite natural and convenient, there arise some problems when we try to define the corresponding operator of fractional differentiation. Note that we already have defined the operator of fractional differentiation $\mathcal{D}_{n(ser)}^{\alpha}$ $(\alpha > 0)$ for harmonic functions via expansion into a series by spherical harmonics (3.2). Such a definition is quite natural and implies invertibility of the operator, given by the following formula, which is pretty useful in applications:

$$\mathcal{D}_{n(ser)}^{\alpha}\mathcal{D}_n^{-\alpha} f(x) = \mathcal{D}_n^{-\alpha}\mathcal{D}_{n(ser)}^{\alpha} f(x) = f(x), \qquad f \in h(B).$$

Nevertheless, we would like to overcome using harmonic functions and their expansions into a series by spherical harmonics, and to define the fractional derivatives as inverses to fractional antiderivatives for more general functions. Thus, we are going to solve the following problem: **find an explicit and convenient form for fractional derivative $\mathcal{D}_n^\alpha$ as an inverse operator to the fractional integral** $\mathcal{D}_n^{-\alpha}$. The natural semigroup property $\mathcal{D}^\alpha\mathcal{D}^\beta f = \mathcal{D}^{\alpha+\beta} f$ is desirable but not necessary.

A form for fractional derivative $\mathcal{D}_n^\alpha$ implicitly is contained in papers [10], [13] − [16], [19], [20]. We represent it in terms of operator D_2^α (see (2.2)).

Definition 4.1 (Fractional derivative of order $\alpha > 0$). *Define*

$$\mathcal{D}^\alpha_{n(int)} f(x) := D^\alpha_{2,t}\Big[t^{\alpha+n/2-1} f(tx)\Big]_{t=1}, \qquad \alpha > 0, \tag{4.1}$$

where the derivative $D^\alpha_{2,t}$ is taken by variable t.

Nevertheless, due to the presence of an additional variable t such a form for fractional derivative creates some inconvenience in applications. Our definition of the new form of fractional derivative is more convenient and adapted to subsequent applications.

Definition 4.2 (Fractional derivative of order $\alpha > 0$). *Define*

$$\mathfrak{D}^\alpha_n f(x) := r^{-(n/2-1)} D^\alpha_2\Big[r^{\alpha+n/2-1} f(x)\Big], \tag{4.2}$$

where the derivative $D_2^\alpha = D^\alpha_{2,r}$ (see (2.2)), as usual, is taken by the variable $r = |x|$.

It is clear that for $n = 2$, the operator $\mathfrak{D}_2^\alpha$ reduces to $\mathcal{D}_2^\alpha$ (see (2.4)).

Thus, at this point, in $\mathbb{R}^n$ we have three definitions (3.3), (4.1) and (4.2) for fractional derivative. In fact, all these definitions are equivalent for suitable functions.

Theorem 4.1. *The three definitions of fractional derivative, given by* (3.3), (4.1) *and* (4.2) *coincide for harmonic functions $f \in h(B)$, that is,*

$$\mathcal{D}^\alpha_{n(ser)} f(x) = \mathcal{D}^\alpha_{n(int)} f(x) = \mathfrak{D}^\alpha_n f(x), \qquad \alpha > 0. \tag{4.3}$$

For nonharmonic but sufficiently smooth functions f the definitions (4.1) *and* (4.2) *coincide, that is,*

$$\mathcal{D}^\alpha_{n(int)} f(x) = \mathfrak{D}^\alpha_n f(x), \qquad \alpha > 0. \tag{4.4}$$

Proof. We first apply the operators to the monomials of type r^k $(k \geq 0)$. According to (4.1), (2.7) and (3.3), we have

$$\begin{aligned}\mathcal{D}^\alpha_{n(int)}\{r^k\} &= D^\alpha_{2,t}\Big[t^{\alpha+n/2-1}(tr)^k\Big]_{t=1} = r^k D^\alpha_{2,t}\Big[t^{\alpha+k+n/2-1}\Big]_{t=1} \\ &= r^k \frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\Big[t^{k+n/2-1}\Big]_{t=1} = \frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\, r^k.\end{aligned}$$

Next, in view of (4.2), (2.7) and (3.3), we obtain

$$\begin{aligned}\mathfrak{D}^\alpha_n\{r^k\} &= r^{-(n/2-1)} D^\alpha_2\Big[r^{\alpha+n/2-1}\, r^k\Big] = r^{-(n/2-1)} D^\alpha_2\Big[r^{\alpha+k+n/2-1}\Big] \\ &= r^{-(n/2-1)}\frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\Big[r^{k+n/2-1}\Big] = \frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\, r^k.\end{aligned}$$

Therefore

$$\mathcal{D}^\alpha_{n(int)}\{r^k\} = \mathfrak{D}^\alpha_n\{r^k\} = \mathcal{D}^\alpha_{n(ser)}\{r^k\} = \frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\, r^k. \tag{4.5}$$

Now let a harmonic in B function f be expanded into a series (3.1). In view of (4.5) and linearity of operators, we can write

$$\mathcal{D}^\alpha_{n(int)} f(x) = \mathcal{D}^\alpha_{n(int)}\Bigg[\sum_{k,j} a_{kj}\, r^k\, Y_{kj}(\zeta)\Bigg] = \sum_{k,j} a_{kj}\, \Big[\mathcal{D}^\alpha_{n(int)}\{r^k\}\Big]\, Y_{kj}(\zeta)$$

$$= \sum_{k,j} a_{kj} \left[\frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\, r^k \right] Y_{kj}(\zeta) = \mathcal{D}^{\alpha}_{n(ser)} f(x),$$

$$\mathfrak{D}^{\alpha}_n f(x) = \mathfrak{D}^{\alpha}_n \left[\sum_{k,j} a_{kj}\, r^k\, Y_{kj}(\zeta) \right] = \sum_{k,j} a_{kj} \left[\mathfrak{D}^{\alpha}_n \{r^k\} \right] Y_{kj}(\zeta)$$

$$= \sum_{k,j} a_{kj} \left[\frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\, r^k \right] Y_{kj}(\zeta) = \mathcal{D}^{\alpha}_{n(ser)} f(x),$$

which proves the desired equalities in (4.3).

For nonharmonic functions f we have to prove the equality (4.4), that is, $\mathcal{D}^{\alpha}_{n(int)} f(x) = \mathfrak{D}^{\alpha}_n f(x)$, or equivalently,

$$D^{\alpha}_{2,t}\Big[t^{\alpha+n/2-1} f(tx)\Big]_{t=1} = r^{-(n/2-1)} D^{\alpha}_{2,r}\Big[r^{\alpha+n/2-1} f(x)\Big], \quad x = r\zeta \in B.$$

To this end, observe first that for integers $m \geq 1$, we have

$$D^{m}_{2,t}\Big[t^{\alpha+\beta} f(tx)\Big]_{t=1} = r^{-\beta} D^{m}_{2,r}\Big[r^{\alpha+\beta} f(x)\Big], \qquad \alpha > 0,\ \beta \geq 0, \tag{4.6}$$

which can be verified by direct differentiation. This implies the equality (4.4) for integers $\alpha = m \geq 1$, that is, we have $\mathcal{D}^{\alpha}_{n(int)} f(x) = \mathfrak{D}^{\alpha}_n f(x)$. For non-integers $\alpha > 0$ we denote $m \in \mathbb{Z},\ m \geq 1,\ m-1 < \alpha < m$, that is, $[\alpha] = m-1$. Then, in view of (4.6), the equality (4.4) is equivalent to the following

$$D^{-(m-\alpha)}_{2,t} D^{m}_{2,t}\Big[t^{\alpha+n/2-1} f(tx)\Big]_{t=1} = r^{-(n/2-1)} D^{-(m-\alpha)}_{2,r} D^{m}_{2,r}\Big[r^{\alpha+n/2-1} f(x)\Big].$$

This completes the proof of (4.4). Theorem 4.1 is proved.

Thus, for all types of operators of fractional differentiation we will use the same notation $\mathcal{D}^{\alpha}_n$:

$$\mathcal{D}^{\alpha}_{n(ser)} f(x) = \mathcal{D}^{\alpha}_{n(int)} f(x) = \mathfrak{D}^{\alpha}_n f(x) =: \mathcal{D}^{\alpha}_n f(x), \qquad \alpha > 0. \tag{4.7}$$

Lemma 4.1. *For sufficiently smooth functions f in the ball B, and in particular, for harmonic functions $f \in h(B)$ the following inversion formula holds:*

$$\mathcal{D}^{\alpha}_n \mathcal{D}^{-\alpha}_n f(x) = \mathcal{D}^{-\alpha}_n \mathcal{D}^{\alpha}_n f(x) = f(x), \qquad x \in B, \quad \alpha > 0. \tag{4.8}$$

Proof. For harmonic functions the formula (4.8) immediately follows from the expansions of such functions into series of spherical harmonics and Definitions (3.2) and (3.3).

For more general functions, that are sufficiently smooth, we use Definitions (3.4), (3.5) and (4.2) to prove (4.8). In view of these definitions and the identity $D^{\alpha}_2 D^{-\alpha}_2 g(x) = g(x)$, for integrable functions g, we obtain

$$\mathcal{D}^{\alpha}_n \mathcal{D}^{-\alpha}_n f(x) = \mathcal{D}^{\alpha}_n \Big[r^{-(\alpha+n/2-1)} D^{-\alpha}_2 \big\{ r^{n/2-1} f(x) \big\} \Big]$$
$$= r^{-(n/2-1)} D^{\alpha}_2 \Big[r^{\alpha+n/2-1}\, r^{-(\alpha+n/2-1)} D^{-\alpha}_2 \big\{ r^{n/2-1} f(x) \big\} \Big]$$
$$= r^{-(n/2-1)} D^{\alpha}_2 D^{-\alpha}_2 \big\{ r^{n/2-1} f(x) \big\} = r^{-(n/2-1)} r^{n/2-1} f(x) = f(x).$$

Similarly we prove that

$$\mathcal{D}^{-\alpha}_n \mathcal{D}^{\alpha}_n f(x) = \mathcal{D}^{-\alpha}_n \Big[r^{-(n/2-1)} D^{\alpha}_2 \big\{ r^{\alpha+n/2-1} f(x) \big\} \Big]$$
$$= r^{-(\alpha+n/2-1)} D^{-\alpha}_2 \Big[r^{n/2-1}\, r^{-(n/2-1)} D^{\alpha}_2 \big\{ r^{\alpha+n/2-1} f(x) \big\} \Big]$$
$$= r^{-(\alpha+n/2-1)} D^{-\alpha}_2 D^{\alpha}_2 \big\{ r^{\alpha+n/2-1} f(x) \big\} = r^{-(\alpha+n/2-1)} r^{\alpha+n/2-1} f(x) = f(x).$$

Here we have used the identity $D^{-\alpha}_2 D^{\alpha}_2 \big\{ r^{\beta} f(x) \big\} = r^{\beta} f(x), \quad \beta \geq \alpha$. Lemma 4.1 is proved.

5. AN INTEGRAL REPRESENTATION OF HARMONIC FUNCTIONS WITH MIXED NORM

In this section we apply fractional integro-differentiation operators to study weighted classes of harmonic functions. In what follows we will use standard notation: $x = r\zeta,\ y = \rho\eta,\ 0 \le r, \rho < 1,\ \zeta, \eta \in S$, dV is a normalized n-dimensional volume measure on the ball B, $dV(x) = n\,r^{n-1}drd\sigma(\zeta),\ V(B) = 1$. Each harmonic function $u(x) \in h(B)$ can be expanded into a series of homogeneous spherical harmonics

$$u(x) = \sum_{k=0}^{\infty}\sum_{j=1}^{d_k} a_{kj}\, r^k\, Y_{kj}(\zeta) \equiv \sum_{k,j} a_{kj}\, r^k\, Y_{kj}(\zeta), \qquad x = r\zeta \in B, \tag{5.1}$$

Now we recall the definition of the extended Poisson kernel in the ball B (see [16, Ch.6]).

Definition 5.1. *The extended Poisson kernel in the ball B is defined to be*

$$P(x,y) \equiv P_0(x,y) := \sum_{k=0}^{\infty} Z_k(x,y) = \frac{1-|x|^2|y|^2}{(1-2x\cdot y+|x|^2|y|^2)^{n/2}}, \qquad x \in B,\ y \in \overline{B},$$

where Z_k stand for zonal harmonics (see [27], [16]).

Observe that the extended Poisson kernel is harmonic in the ball B with respect to each variable x and y, and is also harmonic by x for $y \in \overline{B}$. It possesses the following properties: $P(x,y) = P(y,x),\ P(x,y) = P(ry,\zeta),\ x = r\zeta$, and can be expanded into a series of spherical harmonics

$$P(x,y) = \sum_{k=0}^{\infty} Z_k(x,y) = \sum_{k=0}^{\infty}\sum_{j=1}^{d_k} \overline{Y_{k_j}(y)}\, r^k\, Y_{k_j}(\zeta).$$

If $\eta \in S$, then $P(x,\eta)$ is the usual (non-extended) Poisson kernel in B.

Definition 5.2. *The Poisson-Bergman kernel in the ball B is defined to be*

$$P_\alpha(x,y) := \mathcal{D}_n^\alpha P(x,y) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k+n/2)}{\Gamma(k+n/2)}\, Z_k(x,y), \quad x,y \in B, \quad \alpha \ge 0. \tag{5.2}$$

Note that similar kernels were defined in the papers [8], [10] (for integer values of α), [11] − [15], [18] − [20]. Observe that for $\alpha = 0$, the kernel (5.2) reduces to the extended Poisson kernel in the ball B, that is, $P \equiv P_0$. The Poisson-Bergman kernel (5.2) together with the technique of fractional integro-differentiation allows to deduce the integral formula from Theorem 1.1 of Poisson-Bergman type for functions from $h(p,q,\alpha)$.

Proof of Theorem 1.1. Denote the integral on the right-hand side of formula (1.1) as operator:

$$(T_\alpha u)(x) := \frac{2}{n\,\Gamma(\alpha)} \int_B (1-|y|^2)^{\alpha-1}\, P_\alpha(x,y)\, u(y)\, dV(y), \qquad x \in B, \qquad \alpha > 0.$$

We first prove the theorem for functions $u(x) \in h(1,1,\beta)$. To this end, we assume first that $u(x) \in h(B) \cap C(\overline{B})$. Assuming that the function $u(x)$ is expanded into a series of spherical harmonics (5.1), we transform the integral $(T_\beta u)(x)$ to obtain

$$\begin{aligned}
(T_\beta u)(x) &= \frac{2}{n\,\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1}\, P_\beta(x,y)\, u(y)\, dV(y) \\
&= \frac{2}{n\,\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1} \left[\sum_{k=0}^{\infty} \frac{\Gamma(\beta+k+n/2)}{\Gamma(k+n/2)}\, Z_k(x,y)\right] \left(\sum_{m,j} a_{mj}\, Y_{mj}(y)\right) dV(y) \\
&= \frac{2}{n\,\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\beta+k+n/2)}{\Gamma(k+n/2)} \sum_{m=0}^{\infty}\sum_{j=1}^{d_m} a_{mj} \int_B (1-|y|^2)^{\beta-1}\, Z_k(x,y)\, Y_{mj}(y)\, dV(y).
\end{aligned}$$

Taking into account that $Z_k(x,y) = r^k \rho^k Z_k(\zeta,\eta)$ and $Z_k(\zeta,\eta) = \sum_{i=1}^{d_k} Y_{ki}(\zeta)\,\overline{Y_{ki}(\eta)}$, and assuming $k \geq 0$ to be fixed, we simplify the last integral separately, to get

$$\sum_{m=0}^{\infty}\sum_{j=1}^{d_m} a_{mj} \int_B (1-|y|^2)^{\beta-1}\, Z_k(x,y)\, \rho^m\, Y_{mj}(\eta)\, dV(y)$$

$$= r^k \sum_{m=0}^{\infty}\sum_{j=1}^{d_m} a_{mj} \int_B (1-|y|^2)^{\beta-1}\, \rho^k \rho^m\, Z_k(\zeta,\eta)\, Y_{mj}(\eta)\, dV(y)$$

$$= n\, r^k \sum_{m=0}^{\infty}\sum_{j=1}^{d_m} a_{mj} \int_0^1 (1-\rho^2)^{\beta-1}\, \rho^k \rho^m\ \rho^{n-1} \left(\int_S Z_k(\zeta,\eta)\, Y_{mj}(\eta)\, d\sigma(\eta)\right) d\rho$$

$$= n\, r^k \sum_{m=0}^{\infty}\sum_{j=1}^{d_m} a_{mj} \int_0^1 (1-\rho^2)^{\beta-1}\, \rho^{k+m+n-1} \sum_{i=1}^{d_k} Y_{ki}(\zeta) \left[\int_S \overline{Y_{ki}(\eta)}\, Y_{mj}(\eta)\, d\sigma(\eta)\right] d\rho$$

$$= n\, r^k \sum_{j=1}^{d_k} a_{kj}\, Y_{kj}(\zeta) \int_0^1 (1-\rho^2)^{\beta-1}\, \rho^{2k+n-1}\, d\rho = \frac{n}{2}\, r^k\, \frac{\Gamma(\beta)\,\Gamma(k+n/2)}{\Gamma(\beta+k+n/2)} \sum_{j=1}^{d_k} a_{kj}\, Y_{kj}(\zeta).$$

Here we have used the orthogonality in $L^2(S)$ of spherical harmonics of distinct orders ($k \neq m$). Next, a substitution yields

$$(T_\beta u)(x) = \frac{2}{n\,\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\beta+k+n/2)}{\Gamma(k+n/2)} \left(\frac{n}{2}\, r^k\, \frac{\Gamma(\beta)\,\Gamma(k+n/2)}{\Gamma(\beta+k+n/2)} \sum_{j=1}^{d_k} a_{kj}\, Y_{kj}(\zeta)\right)$$

$$= \sum_{k=0}^{\infty}\sum_{j=1}^{d_k} a_{kj}\, r^k\, Y_{kj}(\zeta) = u(x).$$

Thus, the desired formula $u(x) = (T_\beta u)(x)$ in (1.1) is obtained. For an arbitrary function $u \in h(1,1,\beta)$, we apply the above arguments to the dilated function $u_\delta(x) := u(\delta x),\ 0 < \delta < 1$, yielding to $u_\delta(x) = (T_\beta u_\delta)(x)$. Now it remains to pass to the limit as $\delta \to 1^-$ (see, e.g., [17], [20])

$$\|u - u_\delta\|_{h(1,1,\beta)} \longrightarrow 0, \qquad \delta \to 1^-, \tag{5.3}$$

and as a result, we obtain the desired formula (1.1) for arbitrary function $u \in h(1,1,\beta)$. Now we prove the theorem for a general function $u \in h(p,q,\alpha)$.

If $1 \leq p \leq \infty$, $0 < q \leq 1$ and $\beta \geq \alpha$, then according to the embeddings (see [21, Theorem 1 (i)-(iii)] we have $h(p,q,\alpha) \subset h(1,1,\beta)$.

If $1 \leq p \leq \infty$, $0 < q \leq \infty$ and $\beta > \alpha$, then according to the embeddings (see [21, Theorem 1 (ii), (vi)] we have $h(p,q,\alpha) \subset h(1,q,\alpha) \subset h(1,1,\beta)$.

If $0 < p < 1$, $0 < q \leq \infty$ and $\beta > \alpha + (n-1)(1/p-1)$, then according to the embeddings (see [21, Theorem 1 (iv), (vi)] we have $h(p,q,\alpha) \subset h\big(1,q,\alpha+(n-1)(1/p-1)\big) \subset h(1,1,\beta)$.

In all the above cases, we obtain $u(x) \in h(1,1,\beta)$, which reduces the proof to the already proved case. Theorem 1.1 is proved.

Remark 5.1. *It is easy to see that in the above proof we have verified only the validity of formula* (1.1)*, without constructive derivation of that formula. This method of proof in special cases were applied, for instance, in [11], [12], [8], [19], [20]. It would be of interest to find a constructive derivation of formula* (1.1)*. Our definition of fractional derivative* (4.2)*, Theorem 4.1 and Lemma 4.1 allow to give a constructive and short proof of Theorem 1.1.*

Second proof of Theorem 1.1. Let $u(x) \in h(1,1,\beta),\ \beta > 0$, be an arbitrary function. By the inversion formula (4.8) and Definition 4.2, we can write

$$u(x) = \mathcal{D}_n^{-\beta}\mathcal{D}_n^{\beta} u(x) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1} \mathcal{D}_n^{\beta} u(tx)\, t^{n/2-1}\, dt$$

$$= \frac{1}{\Gamma(\beta)} \int_0^1 (1-\rho^2)^{\beta-1} \mathcal{D}_n^\beta u(\rho^2 x)\, \rho^{2(n/2-1)}\, 2\rho\, d\rho$$
$$= \frac{2}{\Gamma(\beta)} \int_0^1 (1-\rho^2)^{\beta-1} \mathcal{D}_n^\beta \left[\int_S P(x,\rho\eta)\, u(\rho\eta)\, d\sigma(\eta) \right] \rho^{n-1}\, d\rho$$
$$= \frac{2}{n\,\Gamma(\beta)} \int_0^1 \int_S (1-\rho^2)^{\beta-1} \mathcal{D}_n^\beta P(x,\rho\eta)\, u(\rho\eta)\, n\, \rho^{n-1}\, d\rho\, d\sigma(\eta)$$
$$= \frac{2}{n\,\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1}\, P_\beta(x,y)\, u(y)\, dV(y), \qquad x \in B.$$

The convergence of the last integral inside the ball B follows from the following estimate of Poisson-Bergman kernel (see, e.g., [13], [14], [18])

$$|P_\beta(x,y)| \le C(\beta,n) \frac{1}{(1-|x||y|)^{\beta+n-1}}, \qquad x,y \in B.$$

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