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## Real and Complex Analysis

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**REAL AND COMPLEX ANALYSIS**

# Bergman Type Operators on Mixed Norm Spaces over the Ball in $\mathbb{C}^n$

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**Abstract**—The paper considers Bergman type operators introduced by Shields and Williams depending on a normal pair of weight functions. We prove that there exist values of parameter $\beta$ for which these operators are bounded on mixed norm spaces $L(p, q, \beta)$ on the unit ball in $\mathbb{C}^n$.

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## 1. INTRODUCTION AND NOTATION

We first set down some notation. Let $B = B_n := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \ |z| < 1\}$ be an open unit ball in $\mathbb{C}^n$, and let $S := \partial B$ be its boundary, that is, the unit sphere. The inner product in $\mathbb{C}^n$ we denote by $\langle z, w\rangle := z_1\overline{w}_1 + \dots + z_n\overline{w}_n$ for $z, w \in \mathbb{C}^n$. We set $z = r\zeta,\ w = \rho\eta \in B,\ 0 \le r, \rho < 1,\ \zeta, \eta \in S$, $r = |z| = \sqrt{\langle z, z\rangle}$. The set of all holomorphic functions in the ball $B$ we denote by $H(B)$. For a function $f(z) = f(r\zeta)$, defined in the ball $B$, by $M_p(f; r)$ we denote its integral means of order $p$ over the sphere $|z| = r$, defined by

$$M_p(f; r) = \big\|f(r\cdot)\big\|_{L^p(S;d\sigma)}, \quad 0 \le r < 1,\ 0 < p \le \infty,$$

where $d\sigma$ stands for the $(2n-1)$-dimensional spherical Lebesgue measure on the sphere $S$, normed so that $\sigma(S) = 1$. Note that the class of functions $f \in H(B)$ with "norm" $\|f\|_{H^p} = \sup\limits_{0<r<1} M_p(f; r)$ is the ordinary Hardy space $H^p(B)$ in the unit ball $B$.

We define the mixed norm space $L(p, q, \beta)$ $\big(0 < p, q \le \infty,\ \beta \in \mathbb{R}\big)$ to be the space of those measurable functions $f(z) = f(r\zeta)$ defined in the ball $B$, for which the following pre-norm is finite

$$\|f\|_{L(p,q,\beta)} = \|f\|_{p,q,\beta} := \begin{cases} \left(\displaystyle\int_0^1 (1-r)^{\beta q-1} M_p^q(f; r)\, dr\right)^{1/q}, & 0 < q < \infty, \\ \operatorname*{ess\,sup}\limits_{0<r<1}\ (1-r)^{\beta} M_p(f; r), & q = \infty. \end{cases}$$

The subspace of $L(p, q, \beta)$, consisting of holomorphic functions, we denote by $H(p, q, \beta)$, that is, $H(p, q, \beta) := H(B) \cap L(p, q, \beta),\ \beta > 0$.

Observe that if $1 \le p, q \le \infty$, then the spaces $L(p, q, \beta)$ and $H(p, q, \beta)$ become Banach spaces with the norm $\|\cdot\|_{p,q,\beta}$. For $p = q < \infty$ the spaces $H(p, p, \beta)$ coincide with the weighted Bergman spaces, while for $q = \infty$ these spaces often are referred as weighted Hardy spaces.

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The mixed norm spaces for holomorphic in the unit disk functions have been introduced by G. Hardy and J. Littlewood in [1], [2], and later were developed by T. Flett [3]. We refer the reader also the monographs [4], [5], devoted to the weighted Bergman spaces $H(p,p,\beta)$ in the unit disk.

A large number of papers are devoted to the study of the mixed norm spaces $L(p,q,\beta)$ or their subspaces, consisting of holomorphic, pluriharmonic or harmonic functions in the disk or ball in $\mathbb{C}^n$ or in $\mathbb{R}^n$. The spaces $H(p,q,\beta)$ for holomorphic functions in the unit disk $B\subset\mathbb{C}^n$ and the Bergman operators on these spaces have been extensively studied (see [6]–[10]), as for the case of holomorphic and $n$-harmonic functions in the polydisk from $\mathbb{C}^n$, we refer the reader, for example, the paper [11].

Throughout the paper the symbols $C(\alpha,\beta,\dots)$, $c_\alpha$, etc. will denote positive constants, depending only on the indicated indices $\alpha,\beta,\dots$, the values of which can vary from line to line. Also, by $dV$ we will denote the Lebesgue measure on $B$, normed so that $V(B)=1$. In polar coordinates we have $dV(z)=2n\,r^{2n-1}drd\sigma(\zeta)$.

A. Shields and D. Williams were the first who suggested, instead of the standard exponential weight functions, to use more general normal weight functions (see [12]). In fact, these are those weight functions that possess exponential minorants and majorants with positive exponents.

**Definition 1.1** (Normal weight function, [12]). *A positive continuous function $\varphi(r)$, $0\le r<1$, is said to be normal if there exist constants $0<a<b$ and $0\le r_0<1$ to satisfy*

$$\frac{\varphi(r)}{(1-r)^a}\searrow 0 \qquad \textit{and} \qquad \frac{\varphi(r)}{(1-r)^b}\nearrow +\infty \quad \textit{as} \quad r\to 1^-, \quad r_0\le r<1. \tag{1.1}$$

Here and in what follows, monotonicity of functions always will mean in wide (non-strong) sense. Notice that the exponents $a$ and $b$ in the definition of a normal function $\varphi$ are not determined uniquely.

Typical examples of normal functions are functions of the form:

$$\varphi_{c,d}(r)=(1-r)^c\left(\log\frac{e}{1-r}\right)^d, \quad c>0,\ d\in\mathbb{R}.$$

Notice that for $c=0$, that is, the function $\varphi_{0,d}=\left(\log\frac{e}{1-r}\right)^d$ is not a normal function.

**Definition 1.2** (Normal pair, [12]). *A pair of functions $\{\varphi,\psi\}$ is said to be a normal pair, if the function $\varphi$ is normal, and there exists a number $\alpha>b-1$ (called the index of the pair), such that*

$$\varphi(r)\,\psi(r)=(1-r^2)^\alpha, \qquad 0\le r<1. \tag{1.2}$$

Note that due to the condition $\alpha>b-1$ the function $\psi$ is integrable over the interval $(0,1)$. In [12], it was proved that for a given normal function $\varphi$ always can be found its normal pair, and under more stronger condition $\alpha>b$, the function $\psi$ itself is normal with indices $\alpha-b$ and $\alpha-a$.

We enlarge the domain of definition of such radial weight functions to ball $B$, by setting $\varphi(z):=\varphi(|z|)=\varphi(r),\ \psi(z):=\psi(|z|)=\psi(r)$.

Using the notion of normal weight functions, A. Shields and D. Williams [12], have generalized Bergman operators for the unit disk $\mathbb{D}=B_1$, which for the ball $B$ are defined in the papers by A. Petrosyan [13], [14] as follows:

$$Q_{\varphi,\psi}(f)(z):=\int_B\frac{\psi(z)\,\varphi(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}\,f(w)\,dV(w), \quad z\in B, \tag{1.3}$$

$$\widetilde{Q}_{\varphi,\psi}(f)(z):=\int_B\frac{\psi(z)\,\varphi(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}\,f(w)\,dV(w), \quad z\in B. \tag{1.4}$$

In the special case where $\varphi(r)=(1-r^2)^\alpha,\ \psi\equiv 1$ the operators (1.3), (1.4), called Bergman type operators, reduce to the classical Bergman projectors (see [4]–[10]). In the case $\varphi(r)=(1-r^2)^c,\ \psi(r)=(1-r^2)^d,\ c+d=\alpha$, the Bergman type operators (1.3), (1.4), are also well known (see [7]–[11]).

In the present paper we prove that there exist values of the parameter $\beta$, for which the operators (1.3), (1.4) are bounded on the mixed norm spaces $L(p,q,\beta)$ in the ball $B$.

The main result of the paper is the following theorem.

**Theorem 1.1.** *Let $1 \le p, q \le \infty$, $\beta \in \mathbb{R}$, and let $\{\varphi, \psi\}$ be a normal pair of functions with indices $a$ and $b$ $(0 < a < b)$ and with the index of pair $\alpha$ $(\alpha > b - 1)$ in the sense of Definitions 1.1–1.2.*

*If $b - \alpha < \beta < 1 + a$, then the operators $Q_{\varphi,\psi}$ and $\widetilde{Q}_{\varphi,\psi}$, defined by (1.3), (1.4), act boundedly from the space $L(p, q, \beta)$ to itself, that is,*

$$Q_{\varphi,\psi} : L(p, q, \beta) \longrightarrow L(p, q, \beta), \tag{1.5}$$

$$\widetilde{Q}_{\varphi,\psi} : L(p, q, \beta) \longrightarrow L(p, q, \beta). \tag{1.6}$$

**Remark 1.1.** In the special case where $1 \le p = q = 1/\beta < \infty$, that is, for the non-weighted class $L(p, p, 1/p)$, Theorem 1.1 was proved in [13], [14], by a different method, using the so-called Schur test (see ([4]–[7])), which is not applicable for our case. More special cases of the Bergman operators with exponential weights have been studied in [5]–[10].

**Remark 1.2.** In fact, in Theorem 1.1 we extend the result from [13], [14] in three directions by considering: 1) all values of parameter $p$: $1 \le p \le \infty$, 2) weighted spaces, 3) more general mixed norm spaces $L(p, q, \beta)$, instead of Bergman spaces. Also, instead of the not applicable for our case Schur test, we apply generalized versions of Hardy inequalities.

## 2. HARDY INEQUALITIES AND SOME OTHER INTEGRAL INEQUALITIES

The following classical Hardy inequalities are well-known (see [3], [15]):

$$\int_0^1 x^{-\beta-1} \left( \int_0^x h(t)\, dt \right)^p dx \le C(p, \beta) \int_0^1 x^{p-\beta-1}\, h^p(x)\, dx, \tag{2.1}$$

$$\int_0^1 (1-r)^{\beta-1} \left( \int_0^r h(t)\, dt \right)^p dr \le C(p, \beta) \int_0^1 (1-r)^{p+\beta-1}\, h^p(r)\, dr, \tag{2.2}$$

$$\int_0^1 (1-r)^{-\beta-1} \left( \int_r^1 h(t)\, dt \right)^p dr \le C(p, \beta) \int_0^1 (1-r)^{p-\beta-1}\, h^p(r)\, dr, \tag{2.3}$$

where $1 \le p < \infty$, $\beta > 0$ and $h(r) \ge 0$.

Note that the inequality (2.3) can be deduced from (2.1) by linear change of the integration variables.

In our subsequent proofs we also need some generalizations of the inequalities (2.2) and (2.3).

**Lemma 2.1.** *Let $1 \le p < \infty$, $\gamma > 0$, and $h(r) \ge 0$. Let for a positive continuous function $\varphi(r)$, $0 \le r < 1$ some constants $b \in \mathbb{R}$, $\gamma - pb > 0$ and $0 \le r_0 < 1$ can be found to satisfy*

$$\frac{\varphi(r)}{(1-r)^b} \nearrow \quad \text{for} \quad r_0 \le r < 1. \tag{2.4}$$

*Then*

$$\int_0^1 \frac{(1-r)^{\gamma-1}}{\varphi^p(r)} \left( \int_0^r h(t)\, dt \right)^p dr \le C(p, \gamma, b, r_0) \int_0^1 \frac{(1-r)^{p+\gamma-1}}{\varphi^p(r)}\, h^p(r)\, dr. \tag{2.5}$$

**Proof.** We apply the Hardy inequality (2.2) to the function $\frac{(1-r)^b}{\varphi(r)} h(r)$ and with index $\beta = \gamma - pb > 0$, to obtain

$$\int_0^1 (1-r)^{\gamma-pb-1} \left( \int_0^r \frac{(1-t)^b}{\varphi(t)}\, h(t)\, dt \right)^p dr \le C \int_0^1 (1-r)^{p+\gamma-pb-1} \left( \frac{(1-r)^b}{\varphi(r)}\, h(r) \right)^p dr,$$

where the constant $C$ depends only on $p, \gamma, b$.

Taking into account (2.4), and that the function $\frac{(1-r)^b}{\varphi(r)}$ monotone decreases on the interval $(r_0, 1)$ and is continuous on $[0, 1)$, we obtain

$$\int_0^1 (1-r)^{\gamma-pb-1} \frac{(1-r)^{pb}}{\varphi^p(r)} \left( \int_0^r h(t)\, dt \right)^p dr \le C(p, \gamma, b, r_0) \int_0^1 \frac{(1-r)^{p+\gamma-1}}{\varphi^p(r)}\, h^p(r)\, dr,$$

implying (2.5).

**Remark 2.1.** Another, similar to (2.5), Hardy type inequality involving normal weight functions, can be found in [10].

We also need one more version of the inequality (2.5).

**Lemma 2.2.** *Let* $1 \le p < \infty$, $\gamma > 0$, *and* $h(r) \ge 0$. *Let for a positive continuous function* $\varphi(r)$, $0 \le r < 1$, *some constants* $a \in \mathbb{R}$, $\gamma - pa < 0$, *and* $0 \le r_0 < 1$ *can be found to satisfy*

$$\frac{\varphi(r)}{(1-r)^a} \searrow \quad for \quad r_0 \le r < 1. \tag{2.6}$$

*Then*

$$\int_0^1 \frac{(1-r)^{\gamma-1}}{\varphi^p(r)} \left(\int_r^1 h(t)\,dt\right)^p dr \le C(p,\gamma,a,r_0) \int_0^1 \frac{(1-r)^{p+\gamma-1}}{\varphi^p(r)}\, h^p(r)\,dr. \tag{2.7}$$

**Proof.** We apply the Hardy inequality (2.3) to the function $\frac{(1-r)^a}{\varphi(r)}\, h(r)$ and with index $-\beta = \gamma - pa < 0$, to obtain

$$\int_0^1 (1-r)^{\gamma-pa-1} \left(\int_r^1 \frac{(1-t)^a}{\varphi(t)}\, h(t)\,dt\right)^p dr \le C \int_0^1 (1-r)^{p+\gamma-pa-1} \left(\frac{(1-r)^a}{\varphi(r)}\, h(r)\right)^p dr,$$

where the constant $C$ depends only on $p, \gamma, a$. Taking into account (2.6), and that the function $\frac{(1-r)^a}{\varphi(r)}$ monotone increases on the interval $(r_0, 1)$ and is continuous on $[0,1)$, we obtain

$$\int_0^1 (1-r)^{\gamma-pa-1} \frac{(1-r)^{pa}}{\varphi^p(r)} \left(\int_r^1 h(t)\,dt\right)^p dr \le C(p,\gamma,a,r_0) \int_0^1 \frac{(1-r)^{p+\gamma-1}}{\varphi^p(r)}\, h^p(r)\,dr,$$

implying (2.7).

**Lemma 2.3** ([6], [7]). *For* $\alpha > 0$ *the following estimate holds:*

$$\int_S \frac{d\sigma(\xi)}{|1-\langle z,\xi\rangle|^{n+\alpha}} \le \frac{C(\alpha,n)}{(1-|z|)^\alpha}, \qquad z \in B.$$

**Lemma 2.4** ([12]). *For* $m > \beta > 0$ *the following inequality holds:*

$$\int_0^1 \frac{(1-\rho)^{\beta-1}}{(1-r\rho)^m}\, d\rho \le \frac{C(\beta,m)}{(1-r)^{m-\beta}}, \qquad 0 \le r < 1.$$

The next lemma contains an estimate similar to that of obtained in [9], [12]–[14].

**Lemma 2.5.** *Let* $\varphi$ *be a normal function with indices* $a$ *and* $b$ $(0 < a < b)$ *and with the index of pair* $\alpha$ $(\alpha > b - 1)$ *in the sense of Definitions 1.1–1.2.*

*If* $b - \alpha < \beta < 1 + a$, *then*

$$\int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\, d\rho \le C(\alpha,\beta,a,b,r_0)\, \frac{\varphi(r)}{(1-r)^{\alpha+\beta}}, \qquad 0 \le r < 1. \tag{2.8}$$

**Proof.** Observe first that the condition $\beta < 1 + a$ ensures convergence of the integral in (2.8). It is enough to prove the inequality (2.8) for $r$, close to 1. We take $r$, $r_0 < r < 1$, and split the integral (2.8) into three parts:

$$J := \int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\, d\rho = \left(\int_0^{r_0} + \int_{r_0}^r + \int_r^1\right) \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\, d\rho =: J_1 + J_2 + J_3.$$

Observe that the integral $J_1$ is bounded by some constant $C(\alpha,\beta,r_0)$. To estimate the integrals $J_2$ and $J_3$ we use the normality condition (1.1) and Lemma 2.4, to obtain

$$J_2 = \int_{r_0}^r \frac{\varphi(\rho)}{(1-\rho)^b}\, \frac{(1-\rho)^b}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\, d\rho \le \frac{\varphi(r)}{(1-r)^b} \int_{r_0}^r \frac{(1-\rho)^{b-\beta}}{(1-r\rho)^{1+\alpha}}\, d\rho$$

$$\leq C(\alpha,\beta,b)\,\frac{\varphi(r)}{(1-r)^b}\,\frac{1}{(1-r)^{\alpha+\beta-b}}=C(\alpha,\beta,b)\,\frac{\varphi(r)}{(1-r)^{\alpha+\beta}}.$$

Similarly, taking into account that $\beta > b-\alpha > a-\alpha$, we get

$$J_3=\int_r^1\frac{\varphi(\rho)}{(1-\rho)^a}\,\frac{(1-\rho)^a}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\,d\rho\leq\frac{\varphi(r)}{(1-r)^a}\int_r^1\frac{(1-\rho)^{a-\beta}}{(1-r\rho)^{1+\alpha}}\,d\rho$$
$$\leq C(\alpha,\beta,a)\,\frac{\varphi(r)}{(1-r)^a}\,\frac{1}{(1-r)^{\alpha+\beta-a}}=C(\alpha,\beta,a)\,\frac{\varphi(r)}{(1-r)^{\alpha+\beta}},$$

and the result follows.

## 3. BOUNDEDNESS OF BERGMAN TYPE OPERATORS ON MIXED NORM SPACES

**Lemma 3.1.** *Let $1\leq p\leq\infty$, $\alpha>-1$, and let $\{\varphi,\psi\}$ be a pair of positive weight functions. Then the estimate holds:*

$$M_p\big(\widetilde{Q}_{\varphi,\psi}(f);r\big)\leq C(p,n,\alpha)\,\psi(r)\int_0^1\frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}}\,M_p(f;\rho)\,d\rho,\quad 0\leq r<1. \tag{3.1}$$

**Proof.** We first write the integral representation of $\widetilde{Q}_{\varphi,\psi}(f)(z)$ in term of polar coordinates to obtain

$$\big|\widetilde{Q}_{\varphi,\psi}(f)(z)\big|\leq\psi(z)\int_B\frac{\varphi(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}\,|f(w)|\,dV(w)$$
$$=2n\,\psi(z)\int_0^1\left[\int_S\frac{|f(\rho\eta)|}{|1-\langle z,\rho\eta\rangle|^{n+1+\alpha}}\,d\sigma(\eta)\right]\varphi(\rho)\,\rho^{2n-1}\,d\rho,$$

which we can write in the form:

$$\big|\widetilde{Q}_{\varphi,\psi}(f)(r\zeta)\big|\leq 2n\,\psi(r)\int_0^1\left[\int_S\frac{|f(\rho\eta)|}{|1-\langle r\zeta,\rho\eta\rangle|^{n+1+\alpha}}\,d\sigma(\eta)\right]\varphi(\rho)\,\rho^{2n-1}\,d\rho \tag{3.2}$$
$$=2n\,\psi(r)\int_0^1 g(r,\rho,\zeta)\,\varphi(\rho)\,\rho^{2n-1}\,d\rho,$$

where

$$g(r,\rho,\zeta)=\int_S\frac{|f(\rho\eta)|}{|1-\langle r\zeta,\rho\eta\rangle|^{n+1+\alpha}}\,d\sigma(\eta).$$

If $p=\infty$, then in view of (3.2) and Lemma 2.3, we obtain

$$M_\infty\big(\widetilde{Q}_{\varphi,\psi}(f);r\big)\leq 2n\,\psi(r)\int_0^1 M_\infty(f;\rho)\sup_{\zeta\in S}\left[\int_S\frac{d\sigma(\eta)}{|1-\langle r\zeta,\rho\eta\rangle|^{n+1+\alpha}}\right]\varphi(\rho)\,\rho^{2n-1}\,d\rho$$
$$\leq C(n,\alpha)\,\psi(r)\int_0^1\frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}}\,M_\infty(f;\rho)\,d\rho.$$

If $p=1$, then integrating (3.2) and using Fubini theorem and Lemma 2.3, we obtain the desired inequality (3.1). If $1<p<\infty$, then applying Hölder inequality and Lemma 2.3, we get

$$g(r,\rho,\zeta)\leq\left(\int_S\frac{|f(\rho\eta)|^p\,d\sigma(\eta)}{|1-\langle r\zeta,\rho\eta\rangle|^{n+1+\alpha}}\right)^{1/p}\left(\int_S\frac{d\sigma(\eta)}{|1-\langle r\zeta,\rho\eta\rangle|^{n+1+\alpha}}\right)^{1/p'}$$
$$\leq\frac{C(p,n,\alpha)}{(1-r\rho)^{(1+\alpha)/p'}}\left(\int_S\frac{|f(\rho\eta)|^p\,d\sigma(\eta)}{|1-\langle r\zeta,\rho\eta\rangle|^{n+1+\alpha}}\right)^{1/p},$$

where $p'$ is the conjugate index: $1/p+1/p'=1$.

Next, integrating over the sphere $S$ with respect to variable $\zeta$, and using Lemma 2.3, we can write

$$\big\|g(r,\rho,\cdot)\big\|^p_{L^p(S;d\sigma)}\leq\frac{C(p,n,\alpha)}{(1-r\rho)^{(1+\alpha)p/p'}}\int_S\left(\int_S\frac{d\sigma(\zeta)}{|1-\langle r\zeta,\rho\eta\rangle|^{n+1+\alpha}}\right)|f(\rho\eta)|^p\,d\sigma(\eta)$$

$$\leq \frac{C(p,n,\alpha)}{(1-r\rho)^{(1+\alpha)p/p'}(1-r\rho)^{1+\alpha}} \int_S |f(\rho\eta)|^p \, d\sigma(\eta) = \frac{C(p,n,\alpha)}{(1-r\rho)^{(1+\alpha)p}} M_p^p(f;\rho),$$

implying that

$$\big\|g(r,\rho,\cdot)\big\|_{L^p(S;d\sigma)} \leq \frac{C(p,n,\alpha)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho). \tag{3.3}$$

Finally, in the inequality (3.2), we apply Minkowski inequality and then the estimate (3.3), to obtain

$$\begin{aligned} M_p\big(\widetilde{Q}_{\varphi,\psi}(f);r\big) &\leq 2n\,\psi(r) \int_0^1 \big\|g(r,\rho,\cdot)\big\|_{L^p(S;d\sigma)} \varphi(\rho)\,\rho^{2n-1}\,d\rho \\ &\leq C(p,n,\alpha)\,\psi(r) \int_0^1 \tfrac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho)\,d\rho. \end{aligned}$$

This completes the proof of Lemma 3.1.

**Proof of Theorem 1.1.** Taking into account that $\big|Q_{\varphi,\psi}(f)(z)\big| \leq \widetilde{Q}_{\varphi,\psi}(|f|)(z)$, it is enough to prove only the boundedness of the operator $\widetilde{Q}_{\varphi,\psi}(|f|)$, that is, the inequality (1.6).

Consider first the case $1 \leq q < \infty$. By Lemma 3.1 we have

$$M_p\big(\widetilde{Q}_{\varphi,\psi}(f);r\big) \leq C(p,n,\alpha)\,\psi(r) \int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho)\,d\rho, \quad 0 \leq r < 1. \tag{3.4}$$

Then, we integrate by the radial variable to obtain the mixed norm

$$\begin{aligned} \big\|\widetilde{Q}_{\varphi,\psi}(f)\big\|^q_{L(p,q,\beta)} &= \int_0^1 (1-r)^{\beta q-1} M_p^q\big(\widetilde{Q}_{\varphi,\psi}(f);r\big)\,dr \\ &\leq C \int_0^1 (1-r)^{\beta q-1}\,\psi^q(r) \left[\int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho)\,d\rho\right]^q dr. \end{aligned}$$

Next, we use the condition (1.2) of Definition 1.2, and split the integral into two parts:

$$\begin{aligned} \big\|\widetilde{Q}_{\varphi,\psi}(f)\big\|^q_{L(p,q,\beta)} &\leq C \int_0^1 \frac{(1-r)^{\alpha q+\beta q-1}}{\varphi^q(r)} \left[\int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho)\,d\rho\right]^q dr \\ &\leq C \int_0^1 \frac{(1-r)^{\alpha q+\beta q-1}}{\varphi^q(r)} \left[\left(\int_0^r + \int_r^1\right) \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho)\,d\rho\right]^q dr =: I_1 + I_2. \end{aligned} \tag{3.5}$$

Now we estimate the integrals $I_1$ and $I_2$, using the Hardy type inequalities established in Lemmas 2.1 and 2.2. To estimate the integral $I_1$, observe that since the condition $\alpha q + \beta q - bq > 0$ is equivalent to $\beta > b - \alpha$, we can apply the inequality (2.5), to obtain

$$\begin{aligned} I_1 &= C \int_0^1 \frac{(1-r)^{\alpha q+\beta q-1}}{\varphi^q(r)} \left[\int_0^r \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho)\,d\rho\right]^q dr \\ &\leq C \int_0^1 \frac{(1-r)^{\alpha q+\beta q-1+q}}{\varphi^q(r)} \left[\frac{\varphi(r)}{(1-r^2)^{1+\alpha}} M_p(f;r)\right]^q dr \\ &\leq C \int_0^1 (1-r)^{\beta q-1} M_p^q(f;r)\,dr = C(n,p,q,\beta,\alpha,b,r_0)\,\|f\|^q_{L(p,q,\beta)}. \end{aligned} \tag{3.6}$$

As for estimation of the integral $I_2$, since the condition $\beta q - q - aq < 0$ is equivalent to $\beta < 1 + a$, we can apply the inequality (2.7), to obtain

$$\begin{aligned} I_2 &= C \int_0^1 \frac{(1-r)^{\alpha q+\beta q-1}}{\varphi^q(r)} \left[\int_r^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho)\,d\rho\right]^q dr \\ &\leq C \int_0^1 \frac{(1-r)^{\beta q-q-1}}{\varphi^q(r)} \left[\int_r^1 \varphi(\rho)\,M_p(f;\rho)\,d\rho\right]^q dr \\ &\leq C \int_0^1 \frac{(1-r)^{\beta q-q-1+q}}{\varphi^q(r)} \Big[\varphi(r)\,M_p(f;r)\Big]^q dr \end{aligned}$$

$$= C\int_0^1 (1-r)^{\beta q-1} M_p^q(f;r)\,dr = C(n,p,q,\beta,\alpha,a,r_0)\,\|f\|^q_{L(p,q,\beta)}. \qquad (3.7)$$

A combination of the inequalities (3.5) − (3.7) yields the desired inequality (in the case $1 \le q < \infty$):

$$\big\|\widetilde{Q}_{\varphi,\psi}(f)\big\|_{L(p,q,\beta)} \le C\,\|f\|_{L(p,q,\beta)},$$

where the constant $C = C(n,p,q,\beta,\alpha,a,b,r_0) > 0$ depends only on indicated parameters.

Now let $q = \infty$. Then, applying Lemma 2.5, from (3.4), we can deduce

$$M_p\big(\widetilde{Q}_{\varphi,\psi}(f);r\big) \le C(p,n,\alpha)\,\psi(r)\int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\,(1-\rho)^\beta\,M_p(f;\rho)\,d\rho$$

$$\le C\,\psi(r)\,\|f\|_{L(p,\infty,\beta)}\int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\,d\rho$$

$$\le C\,\psi(r)\,\|f\|_{L(p,\infty,\beta)}\frac{\varphi(r)}{(1-r)^{\alpha+\beta}} \le C(p,n,\alpha,\beta,a,b,r_0)\,\|f\|_{L(p,\infty,\beta)}\frac{1}{(1-r)^\beta},$$

implying that

$$\big\|\widetilde{Q}_{\varphi,\psi}(f)\big\|_{L(p,\infty,\beta)} \le C\,\|f\|_{L(p,\infty,\beta)},$$

where constant $C = C(p,n,\alpha,\beta,a,b,r_0) > 0$ depends only on indicated parameters. Theorem 1.1 is proved.

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