ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2019, Vol. 40, No. 8, pp. 1025–1033. © Pleiades Publishing, Ltd., 2019.

Poisson–Bergman Type Operators on Lipschitz and Mixed Norm Spaces in the Real Ball

K. L. Avetisyan*

(Submitted by S. A. Grigoryan)

Faculty of Mathematics and Mechanics, Yerevan State University, Alex Manoogian st. 1, Yerevan, 0025, Armenia

Received January 15, 2019; revised February 14, 2019; accepted March 1, 2019

Abstract—Boundedness of some Poisson–Bergman type operators is stated over the unit ball in $\mathbb{R}^n$. Forelli–Rudin type theorems are proved and bounded harmonic projections are found on Lipschitz and mixed norm spaces.

DOI: 10.1134/S1995080219080043

Keywords and phrases: *unit ball in $\mathbb{R}^n$, harmonic function, Bergman operator, projection, Lipschitz space, mixed norm space.*

1. INTRODUCTION

In this paper, we develop and improve our recent results in harmonic function spaces and operators on them, see [1, 2, 4, 5, 21]. Some results of the paper were announced in [3].

Let $B = B_n$ be the open unit ball in $\mathbb{R}^n (n \geq 2)$ and $S = \partial B$ be its boundary, the unit sphere. The integral means of order p of a function $f(x) = f(r\zeta)$ on the sphere $|x| = r$ are denoted by

$$M_p(f; r) := ||f(r\cdot)||_{L^p(S;d\sigma)}, \quad 0 \leq r < 1, \quad 0 < p \leq \infty,$$

where $d\sigma$ is the $(n-1)$-dimensional area-surface Lebesgue measure on S normalized so that $\sigma(S) = 1$. The set of all (real) harmonic functions in the unit ball B is denoted by $h(B)$. Let dV be the Lebesgue volume measure on B normalized so that $V(B) = 1$. In the polar coordinates, we have $dV(x) = nr^{n-1} dr d\sigma(\zeta)$.

By definition, the mixed norm space $L(p, q, \alpha)\big(0 < p, q \leq \infty, \alpha \in \mathbb{R}\big)$ is the set of those functions f measurable in the unit ball B, for which the quasi-norm

$$||f||_{L(p,q,\alpha)} := \begin{cases} \left(\int\limits_0^1 (1-r)^{\alpha q - 1} M_p^q(f; r) dr\right)^{1/q}, & 0 < q < \infty, \\ \text{ess sup}_{0<r<1} \quad (1-r)^{\alpha} M_p(f; r), & q = \infty, \end{cases}$$

is finite. For the subspace of $L(p, q, \alpha)$ consisting of harmonic functions let $h(p, q, \alpha) := h(B) \cap L(p, q, \alpha)$.

The mixed norm spaces $h(p, q, \alpha)$ and their analogues consisting of holomorphic, pluriharmonic or harmonic functions in the disc, the ball in $\mathbb{C}^n$ or $\mathbb{R}^n$ have been extensively discussed in the past three decades. The mixed norm spaces of holomorphic functions in the unit disc were introduced by Hardy and Littlewood [11, 12], and developed later by Flett [10]. For $p = q < \infty$ the spaces $h(p, q, \alpha)$ coincide with weighted Bergman spaces, see [9, 13], while for $q = \infty$ these spaces are referred to as weighted Hardy spaces. The spaces $h(p, p, \alpha), h(p, q, \alpha)$ on the unit ball in $\mathbb{R}^n$ were studied in [6, 7, 16, 14, 19,

*E-mail: avetkaren@ysu.am

18, 4, 5], while the space $h(p, q, \alpha)$ consisting of n-harmonic functions on a polydisc in $\mathbb{C}^n$ were studied in [1, 2].

In the recent paper [5], we established a reproducing integral formula of Poisson–Bergman type for functions in $h(p, q, \alpha)$ with a wide range of parameters.

Theorem A. *Let $\alpha > 0$ and $u \in h(p, q, \alpha)$ be an arbitrary function. If either $0 < p, q \le \infty, \beta > \max\{\alpha + (n-1)(1/p - 1), \alpha\}$, or $1 \le p \le \infty, 0 < q \le 1, \beta \ge \alpha$, then*

$$u(x) = \frac{2}{n\Gamma(\beta)} \int_B (1 - |y|^2)^{\beta-1} P_\beta(x, y) u(y) dV(y), \quad x \in B, \tag{1}$$

where P_β is the Poisson–Bergman type kernel defined in Section 2 below.

Integral formula (1) induces a linear integral operator of Bergman type

$$T_\beta(u)(x) := \frac{2}{n\Gamma(\beta)} \int_B (1 - |y|^2)^{\beta-1} P_\beta(x, y) u(y) dV(y), \quad x \in B.$$

In fact, Theorem A asserts that operator T_β is the identity map on $h(p, q, \alpha)$, that is, $T_\beta(u) = u, \forall u \in h(p, q, \alpha)$ for suitable parameters.

Along with the operator T_β, define more general operators of Bergman type,

$$T_{\beta,\lambda}(f)(x) := \frac{2(1 - |x|^2)^\lambda}{n\Gamma(\beta + \lambda)} \int_B (1 - |y|^2)^{\beta-1} P_{\beta+\lambda}(x, y) f(y) dV(y),$$

$$S_{\beta,\lambda}(f)(x) := \frac{2(1 - |x|^2)^\lambda}{n\Gamma(\beta + \lambda)} \int_B (1 - |y|^2)^{\beta-1} |P_{\beta+\lambda}(x, y)| f(y) dV(y).$$

Note that $T_{\beta,0} \equiv T_\beta$. We introduce some more operators of Poisson–Bergman type with the use of ordinary Poisson kernel $P_0(x, y)$,

$$\Phi_{\beta,\delta}(f)(x) := \frac{2}{n\Gamma(\beta)} \int_B (1 - |y|^2)^{\beta-1} P_0(x, y) \mathcal{D}^\delta f(y) dV(y), \quad \beta, \delta > 0,$$

where $\mathcal{D}^\delta$ is an operator of fractional differentiation defined in Section 2.

Our purpose is to find out the action of operators $T_{\beta,\lambda}, S_{\beta,\lambda}, \Phi_{\beta,\delta}$ on Lipschitz and much wider mixed norm spaces.

Theorem 1. *For any $\alpha + \beta > \delta > \alpha > 0$, the operator $\Phi_{\beta,\delta}$ continuously maps the Lipschitz space Λ_α onto harmonic Lipschitz space $h\Lambda_{\alpha+\beta-\delta}$, i.e. $\Phi_{\beta,\delta} : \Lambda_\alpha \xrightarrow{onto} h\Lambda_{\alpha+\beta-\delta}$, with the norm inequality*

$$||\Phi_{\beta,\delta}(f)||_{h\Lambda_{\alpha+\beta-\delta}} \le C(\alpha, \beta, \delta, n) ||f||_{\Lambda_\alpha}.$$

In particular, for $\beta = \delta$, the operator $\Phi_{\beta,\beta}$ is a continuous projection of the Lipschitz space Λ_α onto its harmonic subspace, $\Phi_{\beta,\beta} : \Lambda_\alpha \xrightarrow{\text{onto}} h\Lambda_\alpha$.

The Bergman projection T_β also preserves classical Lipschitz spaces Lip α, as stated in Theorem 2 below.

Theorem 2. *For $0 < \alpha < 1$, $\beta > 0$, the operator T_β continuously projects the Lipschitz space Lip α onto its harmonic subspace hLip α, T_β : Lip $\alpha \xrightarrow{onto}$ hLip α, that is,*

$$||T_\beta f||_{\mathrm{hLip}\,\alpha} \le C(\alpha, \beta, n) ||f||_{\mathrm{Lip}\,\alpha}.$$

Remark 1. *Theorems 1 and 2 assert that Poisson–Bergman operators $\Phi_{\beta,\beta}$ and T_β act as bounded harmonic projections in Lipschitz spaces. Preservation of Lipschitz spaces under the Bergman projection was established earlier in [15] (for unweighted Bergman projection) and [8] (for integer β).*

Having established Theorem A, it is natural to ask whether the operators $T_{\beta,\lambda}, S_{\beta,\lambda}$ are bounded in mixed norm spaces.

Theorem 3. *If $1 \le p, q \le \infty$, $\beta > \alpha > -\lambda$, then the operators $T_{\beta,\lambda}$, $S_{\beta,\lambda}$ continuously map the space $L(p,q,\alpha)$ into itself: $T_{\beta,\lambda}, S_{\beta,\lambda} : L(p,q,\alpha) \xrightarrow{into} L(p,q,\alpha)$, that is,*

$$||T_{\beta,\lambda} f||_{L(p,q,\alpha)} \le C||f||_{L(p,q,\alpha)}, \tag{2}$$

$$||S_{\beta,\lambda} f||_{L(p,q,\alpha)} \le C||f||_{L(p,q,\alpha)}, \tag{3}$$

where $C = C(p,q,\alpha,\beta,\lambda,n)$ is a positive constant depending only on the parameters indicated. Moreover, the operator $T_{\beta,0}$ continuously projects $L(p,q,\alpha)$ onto $h(p,q,\alpha)$:

$$T_{\beta,0} : L(p,q,\alpha) \xrightarrow{\text{onto}} h(p,q,\alpha). \tag{4}$$

Remark 2. *Theorem 3 is an analogue of the well-known Forelli–Rudin theorem in Bergman spaces, see, e.g., [9, 13]. Various generalizations for holomorphic and harmonic functions can also be found in [9, 13, 7, 14, 18, 1].*

2. PRELIMINARIES

Throughout the paper, we assume $x = r\zeta$, $y = \rho\eta$, $0 \le r, \rho < 1$, $\zeta, \eta \in S$, and the letters $C(\alpha, \beta, \dots)$, C_α etc. stand for different positive constants depending only on the parameters indicated.

Definition 1. *(Riemann–Liouville fractional integral and derivative for $\mathbb{R}^2$) For a function f of one variable $r \in [0,1)$, let*

$$D^{-\alpha} f(r) := \frac{1}{\Gamma(\alpha)} \int_0^r (r-t)^{\alpha-1} f(t)dt = \frac{r^\alpha}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tr)dt,$$

$$D^m f(r) := \left(\frac{d}{dr}\right)^m f(r), \quad D^\alpha f(r) := D^{-(m-\alpha)} D^m f(r),$$

where $0 \le r < 1$, $\alpha > 0$, $m \in \mathbb{Z}$, $m \ge 0$, $m-1 < \alpha \le m$.

Definition 2. *(Fractional integral and derivative for $\mathbb{R}^n, n \ge 2$) Given a function f in the unit ball B, let*

$$\mathcal{D}_n^{-\alpha} f(x) := r^{-(\alpha+n/2-1)} D^{-\alpha}\Big\{r^{n/2-1} f(x)\Big\} = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tx) t^{n/2-1} dt,$$

$$\mathcal{D}^\alpha f(x) \equiv \mathcal{D}_n^\alpha f(x) := r^{-(n/2-1)} D^\alpha \Big\{r^{\alpha+n/2-1} f(x)\Big\}, \quad r = |x|.$$

This version of the fractional derivative in $\mathbb{R}^n$ is introduced in [5] and makes it possible to apply it to the extended Poisson kernel in B

$$P(x,y) \equiv P_0(x,y) := \frac{1-|x|^2|y|^2}{(1-2x\cdot y+|x|^2|y|^2)^{n/2}}, \quad x \in B, y \in \overline{B},$$

in order to obtain the Poisson–Bergman type kernel P_β in B mentioned in Theorem A. Here $x \cdot y$ means the Euclidean inner product.

Definition 3. *(Poisson–Bergman type kernel in B)*

$$P_\alpha(x,y) := \mathcal{D}_n^\alpha P(x,y), \quad x, y \in B, \quad \alpha \ge 0.$$

Similar and equivalent kernels mostly in series or integral forms are defined in [6, 9, 7, 16, 14, 18].

Note that Poisson–Bergman type kernel $P_\alpha(x,y)$ is a harmonic function with respect to both x and y, and $P_\alpha(x,y) = P_\alpha(y,x)$. Let us mention important estimates of the Poisson–Bergman type kernel (see, e.g., [16, 14, 19])

$$|P_\alpha(x,y)| \le \frac{C(\alpha,n)}{|\eta-\rho x|^{\alpha+n-1}} = \frac{C(\alpha,n)}{|\zeta-ry|^{\alpha+n-1}}, \quad \alpha \ge 0, \tag{5}$$

$$|\mathcal{D}_n^1 P_\alpha(x,y)| \le \frac{C(\alpha,n)}{|\eta-\rho x|^{\alpha+n}}, \quad x=r\zeta, \quad y=\rho\eta, \quad \alpha\ge 0. \tag{6}$$

Lemma 1. *For any $\beta>\alpha>0$, there hold the inequalities*

$$\int_S \frac{d\sigma(\xi)}{|\xi-x|^{\alpha+n-1}} \le C(\alpha,n)\frac{1}{(1-|x|)^\alpha}, \quad x\in B,$$

$$\int_0^1 \frac{(1-t)^{\alpha-1}}{(1-rt)^\beta}dt \le C(\alpha,\beta)\frac{1}{(1-r)^{\beta-\alpha}}, \quad 0\le r<1,$$

$$\int_B \frac{(1-|y|)^{\alpha-1}}{|\zeta-ry|^{\beta+n-1}}dV(y) \le C(\alpha,\beta,n)\frac{1}{(1-|x|)^{\beta-\alpha}}, \quad x=r\zeta\in B.$$

The estimates of Lemma 1 are well known and can be found, for example, in [14, 16, 19].

We also need well-known Hardy's inequalities, see, e.g., [20, 10].

Lemma 2. *(Hardy's inequalities) If $1\le q<\infty$, $\beta>0$, $h(r)\ge 0$, then*

$$\int_0^1 x^{-\beta-1}\left(\int_0^x h(t)dt\right)^q dx \le C\int_0^1 x^{q-\beta-1}h^q(x)dx, \tag{7}$$

$$\int_0^1 (1-r)^{\beta-1}\left(\int_0^r h(t)dt\right)^q dr \le C\int_0^1 (1-r)^{q+\beta-1}h^q(r)dr, \tag{8}$$

$$\int_0^1 (1-r)^{-\beta-1}\left(\int_r^1 h(t)dt\right)^q dr \le C\int_0^1 (1-r)^{q-\beta-1}h^q(r)dr, \tag{9}$$

where the constants $C=C(q,\beta)>0$ depend only on q and β.

Inequality (9) follows from (7) by a linear change of integration variables.

3. BOUNDED PROJECTIONS ON LIPSCHITZ SPACES

It would be of interest to find out the images of classical Lipschitz spaces under the Bergman operators. It turns out that both operators $\Phi_{\beta,\beta}$ and T_β preserve the Lipschitz classes in B, and so become bounded projections.

Definition 4. *A function f given in the unit ball B is said to belong to Lipschitz space Lip $\alpha(0<\alpha<1)$ if $|f(x)-f(y)|\le C(\alpha,n)|x-y|^\alpha, \quad x,y\in B$. The Lipschitz space Lip α is equipped with a seminorm*

$$||f||_{\mathrm{Lip}\,\alpha} := \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\alpha},$$

and Lip α becomes a Banach space with the norm $|f(0)|+||f||_{Lip\,\alpha}$. Let hLip α be the subspace of Lip α consisting of harmonic functions, $\mathrm{hLip}\,\alpha := h(B)\cap \mathrm{Lip}\,\alpha$.

Define now slightly different Lipschitz spaces for all $\alpha>0$.

Definition 5. *A function f smooth enough in the unit ball B is said to belong to Lipschitz space $\Lambda_\alpha(\alpha>0)$ if*

$$(1-|x|)^{\gamma-\alpha}|\mathcal{D}^\gamma f(x)|\le C(\alpha,\gamma,n), \quad x\in B,$$

for some $\gamma>\alpha$. Note that for different values of $\gamma>\alpha$, equivalent conditions appear. The fractional derivative $\mathcal{D}^\gamma$ may be replaced by a higher order gradient. A norm in Λ_α can be defined as follows:

$$||f||_{\Lambda_\alpha} := ||\mathcal{D}^\gamma f||_{L(\infty,\infty,\gamma-\alpha)}.$$

Denote by $h\Lambda_\alpha$ the subspace of Λ_α consisting of harmonic functions, $h\Lambda_\alpha := h(B) \cap \Lambda_\alpha$.

The following lemma is an analogue of the classical Hardy–Littlewood theorem [11]. The proof runs along essentially the same lines as in the classical theorem.

Lemma 3. *For $0 < \alpha < 1$, the harmonic Lipschitz spaces defined above coincide, $hLip\ \alpha = h\Lambda_\alpha$, and corresponding norms are equivalent:*

$$|u(0)| + ||u||_{\mathrm{Lip}\ \alpha} \approx |u(0)| + \sup_{x \in B} (1 - |x|)^{1-\alpha} |\nabla u(x)| \approx ||u||_{\Lambda_\alpha}.$$

The notation $A \approx B$ for some $A, B > 0$ denotes the two-sided estimate $c_1 A \leq B \leq c_2 A$ with some positive constants c_1 and c_2 independent of the variable involved.

We are now in a position to prove projection theorems for Lipschitz spaces.

Proof of Theorem 1. Differentiate $\Phi_{\beta,\delta}(f)$ with the operator $\mathcal{D}^\gamma$ for $\gamma > \alpha + \beta - \delta > 0$, to get

$$\mathcal{D}^\gamma \Phi_{\beta,\delta}(f)(x) = \frac{2}{n\Gamma(\beta)} \int_B (1 - |y|^2)^{\beta-1} P_\gamma(x, y) \mathcal{D}^\delta f(y) dV(y).$$

In view of the kernel estimate (5),

$$\begin{aligned} \left|(\mathcal{D}^\gamma \Phi_{\beta,\delta} f)(x)\right| &\leq C(\beta, n) \int_B (1 - |y|^2)^{\beta-1} \left|P_\gamma(x, y)\right| \left|\mathcal{D}^\delta f(y)\right| dV(y) \\ &\leq C(\beta, \gamma, n) \int_B \frac{(1 - \rho^2)^{\beta-1}}{|\rho x - \eta|^{\gamma+n-1}} \left|\mathcal{D}^\delta f(\rho\eta)\right| \rho^{n-1} d\rho d\sigma(\eta). \end{aligned}$$

Replace here x by Qx, where Q is an arbitrary orthogonal linear transformation $Q : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, that is, $|Qx| = |x|$ for all $x \in \mathbb{R}^n$. Recall that the measure σ is invariant under rotations, meaning $\sigma\big(Q(G)\big) = \sigma(G)$ for every Borel set $G \subset S$ and every orthogonal transformation Q. Applying also the change $\eta \mapsto Q\eta$, we find that

$$\begin{aligned} \left|(\mathcal{D}^\gamma \Phi_{\beta,\delta} f)(Qx)\right| &\leq C \int_B \frac{(1 - \rho^2)^{\beta-1}}{|\rho Qx - Q\eta|^{\gamma+n-1}} \left|\mathcal{D}^\delta f(\rho Q\eta)\right| \rho^{n-1} d\rho d\sigma(\eta) \\ &= C \int_0^1 \int_S \frac{(1 - \rho^2)^{\beta-1}}{|\rho x - \eta|^{\gamma+n-1}} \left|\mathcal{D}^\delta f(\rho Q\eta)\right| \rho^{n-1} d\rho d\sigma(\eta). \end{aligned}$$

Consequently, by Lemma 1,

$$M_\infty(\mathcal{D}^\gamma \Phi_{\beta,\delta} f; r) \leq C \int_0^1 \int_S \frac{(1 - \rho^2)^{\beta-1}}{|\rho x - \eta|^{\gamma+n-1}} M_\infty(\mathcal{D}^\delta f; \rho) d\rho d\sigma(\eta) \leq C \int_0^1 \frac{(1 - \rho^2)^{\beta-1}}{(1 - r\rho)^\gamma} M_\infty(\mathcal{D}^\delta f; \rho) d\rho.$$

Note that the last integral converges since $f \in \Lambda_\alpha$,

$$(1 - r)^{\delta-\alpha} M_\infty(\mathcal{D}^\delta f; r) \leq ||\mathcal{D}^\delta f||_{L(\infty,\infty,\delta-\alpha)} = ||f||_{\Lambda_\alpha},$$

and

$$\int_0^1 (1 - \rho)^{\beta-1} M_\infty(\mathcal{D}^\delta f; \rho) d\rho \leq ||f||_{\Lambda_\alpha} \int_0^1 (1 - \rho)^{\alpha+\beta-\delta-1} d\rho < +\infty.$$

Hence, in view of the condition $\gamma > \alpha + \beta - \delta > 0$ and Lemma 1,

$$M_\infty\big(\mathcal{D}^\gamma \Phi_{\beta,\delta}(f); r\big) \leq C \int_0^1 \frac{(1 - \rho^2)^{\beta-1}}{(1 - r\rho)^\gamma} \frac{||\mathcal{D}^\delta f||_{L(\infty,\infty,\delta-\alpha)}}{(1 - \rho)^{\delta-\alpha}} d\rho$$

$$\leq C||\mathcal{D}^\delta f||_{L(\infty,\infty,\delta-\alpha)} \int_0^1 \frac{(1-\rho)^{\beta-\delta+\alpha-1}}{(1-r\rho)^\gamma} d\rho \leq C||\mathcal{D}^\delta f||_{L(\infty,\infty,\delta-\alpha)} \frac{1}{(1-r)^{\gamma-\beta+\delta-\alpha}},$$

where $C = C(\alpha, \beta, \gamma, \delta, n)$. Therefore,

$$(1-r)^{\gamma-\beta+\delta-\alpha} M_\infty\big(\mathcal{D}^\gamma \Phi_{\beta,\delta}(f); r\big) \leq C||\mathcal{D}^\delta f||_{L(\infty,\infty,\delta-\alpha)}, \quad 0 \leq r < 1,$$

that is, $||\Phi_{\beta,\delta}(f)||_{h\Lambda_{\alpha+\beta-\delta}} \leq C||f||_{\Lambda_\alpha}$.

For the surjectivity of $\Phi_{\beta,\delta} : \Lambda_\alpha \longrightarrow h\Lambda_{\alpha+\beta-\delta}$, we take arbitrary harmonic function $g \in h\Lambda_{\alpha+\beta-\delta}$, so $\mathcal{D}^\beta g \in h(\infty, \infty, \delta - \alpha)$. According to the continuous inclusions (ii),(vi) in [4, Thm 1.1],

$$\mathcal{D}^\beta g \in h(\infty, \infty, \delta - \alpha) \subset h(1, \infty, \delta - \alpha) \subset h(1, 1, \beta), \quad \beta > \delta - \alpha > 0.$$

By Theorem A, $\mathcal{D}^\beta g(x) = T_{\beta,0}(\mathcal{D}^\beta g)(x)$, and then taking the integral operator $\mathcal{D}^{-\beta}$, we obtain

$$g(x) = \frac{2}{n\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1} P_0(x,y) \mathcal{D}^\beta g(y) dV(y)$$

$$= \frac{2}{n\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1} P_0(x,y) \mathcal{D}^\delta \Big[\mathcal{D}^{-\delta}\mathcal{D}^\beta g(y)\Big] dV(y) = \Phi_{\beta,\delta}\big(\mathcal{D}^{-\delta}\mathcal{D}^\beta g\big)(x) =: \Phi_{\beta,\delta}(\psi)(x),$$

where we have used the inversion formula $\mathcal{D}^\delta \mathcal{D}^{-\delta} F = F$, see [5, Lemma 4.1]. Now it remains to show only that the function $\psi := \mathcal{D}^{-\delta}\mathcal{D}^\beta g$ is in $h\Lambda_\alpha$. For an integer $m \in \mathbb{N}, m > \delta > \alpha > 0$, we apply Hardy–Littlewood type theorems on (fractional) integro-differentiation in harmonic spaces $h(\infty, \infty, \alpha)$ (see [17]). Also, by using the commutation formula $\mathcal{D}^m \mathcal{D}^{-\delta} F = \mathcal{D}^{-\delta} \mathcal{D}^m F$ (see [2, Lemma 4] for $\mathbb{R}^2$), we consecutively obtain the following implications

$$\begin{aligned} g \in h\Lambda_{\alpha+\beta-\delta} \quad &\Longrightarrow \quad \mathcal{D}^\beta g \in h(\infty, \infty, \delta - \alpha) \quad \Longrightarrow \quad \mathcal{D}^m \mathcal{D}^\beta g \in h(\infty, \infty, \delta - \alpha + m) \\ &\Longrightarrow \quad \mathcal{D}^{-\delta}\mathcal{D}^m \mathcal{D}^\beta g = \mathcal{D}^m \mathcal{D}^{-\delta} \mathcal{D}^\beta g \in h(\infty, \infty, m - \alpha) \\ &\Longrightarrow \quad \mathcal{D}^m \psi = \mathcal{D}^m\big(\mathcal{D}^{-\delta}\mathcal{D}^\beta g\big) \in h(\infty, \infty, m - \alpha) \quad \Longrightarrow \quad \psi \in h\Lambda_\alpha. \end{aligned}$$

Note that the surjectivity of $\Phi_{\beta,\beta}$ immediately follows from the fact that $\Phi_{\beta,\beta}$ is the identity operator on $h\Lambda_\alpha$. This completes the proof of Theorem 1. □

Proof of Theorem 2. Let f be an arbitrary Lipschitz function in the space $\mathrm{Lip}\,\alpha$ not necessarily harmonic. Because of Theorem A

$$1 = \frac{2}{n\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1} P_\beta(x,y) dV(y), \quad x \in B.$$

It follows that for any fixed point $z \in B$, $0 < |z - x| < 1 - |x|$,

$$T_\beta(f)(x) = \frac{2}{n\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1} P_\beta(x,y) \Big(f(y) - f(z)\Big) dV(y) + f(z).$$

Further differentiation with $\mathcal{D}_n^1$ and estimation by using of (6) lead to

$$\big|\mathcal{D}_n^1 T_\beta(f)(x)\big| \leq \frac{2}{n\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1} \big|\mathcal{D}_n^1 P_\beta(x,y)\big| |f(y) - f(z)| dV(y)$$

$$\leq C(\alpha, \beta, n) \int_B (1-|y|^2)^{\beta-1} ||f||_{\mathrm{Lip}\,\alpha} \frac{|y-z|^\alpha}{|\rho x - \eta|^{\beta+n}} dV(y)$$

$$\leq C(\alpha, \beta, n) ||f||_{\mathrm{Lip}\,\alpha} \int_B \frac{(1-|y|)^{\beta-1} |y-z|^\alpha}{|\rho x - \eta|^{\beta+n}} dV(y).$$

The further application of the triangle inequality

$$|y-z|^\alpha \le |y-x|^\alpha + |x-z|^\alpha \quad (0<\alpha<1)$$

and another obvious inequality $|y-x| < |\rho x-\eta|$, $x, y \in B$, together with Lemma 1, finally yields

$$\left|\mathcal{D}_n^1 T_\beta(f)(x)\right| \le C\|f\|_{\mathrm{Lip}\,\alpha}\int\limits_B \frac{(1-|y|)^{\beta-1}|y-x|^\alpha}{|\rho x-\eta|^{\beta+n}}dV(y) + C\|f\|_{\mathrm{Lip}\,\alpha}|x-z|^\alpha \int\limits_B \frac{(1-|y|)^{\beta-1}}{|\rho x-\eta|^{\beta+n}}dV(y)$$

$$\le C\|f\|_{\mathrm{Lip}\,\alpha}\int\limits_B \frac{(1-|y|)^{\beta-1}}{|\rho x-\eta|^{\beta+n-\alpha}}dV(y) + C\|f\|_{\mathrm{Lip}\,\alpha}(1-|x|)^\alpha \int\limits_B \frac{(1-|y|)^{\beta-1}}{|\rho x-\eta|^{\beta+n}}dV(y)$$

$$\le C\|f\|_{\mathrm{Lip}\,\alpha}\frac{1}{(1-|x|)^{1-\alpha}} + C\|f\|_{\mathrm{Lip}\,\alpha}\frac{(1-|x|)^\alpha}{1-|x|} = C(\alpha,\beta,n)\|f\|_{\mathrm{Lip}\,\alpha}\frac{1}{(1-|x|)^{1-\alpha}}.$$

This, together with Lemma 3, completes the proof of Theorem 2. □

4. BOUNDED OPERATORS ON MIXED NORM SPACES

Proof of Theorem 3. Let f be an arbitrary function in the mixed norm space $L(p,q,\alpha)$. Since $\left|T_{\beta,\lambda}(f)\right| \le S_{\beta,\lambda}(|f|)$, it suffices to prove only the inequality (3). An application of the estimate (5) gives

$$\left|(S_{\beta,\lambda}f)(x)\right| \le C(1-|x|^2)^\lambda \int\limits_B (1-|y|^2)^{\beta-1}|P_{\beta+\lambda}(x,y)||f(y)|dV(y)$$

$$\le C(1-|x|^2)^\lambda \int\limits_B \frac{(1-\rho^2)^{\beta-1}}{|\rho x-\eta|^{\beta+\lambda+n-1}}|f(y)|\rho^{n-1}d\rho d\sigma(\eta).$$

Replace here x by Qx, where Q is an arbitrary orthogonal linear transformation $Q:\mathbb{R}^n \longrightarrow \mathbb{R}^n$, that is, $|Qx|=|x|$ for all $x\in\mathbb{R}^n$. After changing the variable $\eta\mapsto Q\eta$, we find that

$$\left|(S_{\beta,\lambda}f)(Qx)\right| \le C(\beta,\lambda,n)(1-|Qx|^2)^\lambda \int\limits_B \frac{(1-\rho^2)^{\beta-1}}{|\rho Qx-Q\eta|^{\beta+\lambda+n-1}}|f(\rho Q\eta)|\rho^{n-1}d\rho d\sigma(\eta)$$

$$= C(\beta,\lambda,n)(1-|x|^2)^\lambda \int\limits_0^1\int\limits_S \frac{(1-\rho^2)^{\beta-1}}{|\rho x-\eta|^{\beta+\lambda+n-1}}|f(\rho Q\eta)|\rho^{n-1}d\rho d\sigma(\eta).$$

Further, we use Minkowski's inequality in the continuous form, Lemma 1, and the identity $M_p(F;|z|) = \left(\int |F(Qz)|^p dQ\right)^{1/p}$, $z\in B$, where the integral is taken over the orthogonal group. Hence,

$$M_p(S_{\beta,\lambda}f;r) \le C(1-|x|^2)^\lambda \int\limits_0^1\int\limits_S \frac{(1-\rho^2)^{\beta-1}}{|\rho x-\eta|^{\beta+\lambda+n-1}}M_p(f;\rho)d\rho d\sigma(\eta)$$

$$\le C(\beta,\lambda,n)(1-r^2)^\lambda \int\limits_0^1 \frac{(1-\rho^2)^{\beta-1}}{(1-r\rho)^{\beta+\lambda}}M_p(f;\rho)d\rho. \tag{10}$$

Case $1 \le q < \infty$. First note that the last integral in (10) is convergent since $\beta>\alpha$ and, by Hölder's inequality ($1<q<\infty, 1/q+1/q'=1$),

$$\int\limits_0^1 (1-\rho^2)^{\beta-1}M_p(f;\rho)d\rho = \int\limits_0^1 (1-\rho^2)^{\beta-\alpha+\alpha}M_p(f;\rho)\frac{d\rho}{1-\rho}$$

$$\le C_\beta\left(\int\limits_0^1 (1-\rho)^{q'(\beta-\alpha)}\frac{d\rho}{1-\rho}\right)^{1/q'}\left(\int\limits_0^1 (1-\rho)^{\alpha q}M_p^q(f;\rho)\frac{d\rho}{1-\rho}\right)^{1/q}$$

$$= C(\alpha,\beta,q)||f||_{L(p,q,\alpha)} < +\infty.$$

Therefore, the evaluation of $M_p(S_{\beta,\lambda}f;r)$ can be continued as follows:

$$M_p(S_{\beta,\lambda}f;r) \le C(\beta,\lambda,n)(1-r^2)^{\lambda}\left(\int_0^r + \int_r^1\right)\frac{(1-\rho^2)^{\beta-1}}{(1-r\rho)^{\beta+\lambda}}M_p(f;\rho)d\rho$$

$$\le C(1-r^2)^{\lambda}\int_0^r \frac{M_p(f;\rho)}{(1-\rho)^{1+\lambda}}d\rho + C(1-r)^{-\beta}\int_r^1 (1-\rho^2)^{\beta-1}M_p(f;\rho)d\rho.$$

By the triangle inequality and next by Hardy's inequalities (8) and (9) in Lemma 2,

$$||S_{\beta,\lambda}f||_{L(p,q,\alpha)} = ||(1-r)^{\alpha}M_p(S_{\beta,\lambda}f;r)||_{L^q(dr/(1-r))}$$

$$\le C\left|\left|(1-r)^{\alpha+\lambda}\int_0^r M_p(f;\rho)\frac{d\rho}{(1-\rho)^{1+\lambda}}\right|\right|_{L^q(dr/(1-r))}$$

$$+ C\left|\left|(1-r)^{\alpha-\beta}\int_r^1 (1-\rho)^{\beta-1}M_p(f;\rho)d\rho\right|\right|_{L^q(dr/(1-r))}$$

$$\le C\left[\int_0^1 (1-r)^{(\alpha+\lambda)q-1}\left(\frac{1-r}{(1-r)^{1+\lambda}}M_p(f;r)\right)^q dr\right]^{1/q}$$

$$+ C\left[\int_0^1 (1-r)^{(\alpha-\beta)q-1}\Big((1-r)^{\beta}M_p(f;r)\Big)^q dr\right]^{1/q} \le C||f||_{L(p,q,\alpha)}.$$

Case $q=\infty$. Since $(1-r)^{\alpha}M_p(f;r) \le ||f||_{L(p,\infty,\alpha)}$ and $\beta > \alpha$, the integral in (10) again converges,

$$\int_0^1 (1-\rho^2)^{\beta-1}M_p(f;\rho)d\rho \le C_{\beta}||f||_{L(p,\infty,\alpha)}\int_0^1 (1-\rho)^{\beta-\alpha-1}d\rho < +\infty.$$

Therefore, by (10) and Lemma 1,

$$M_p(S_{\beta,\lambda}f;r) \le C(\beta,\lambda,n)(1-r^2)^{\lambda}\int_0^1 \frac{(1-\rho^2)^{\beta-1}}{(1-r\rho)^{\beta+\lambda}}\frac{||f||_{L(p,\infty,\alpha)}}{(1-\rho)^{\alpha}}d\rho$$

$$\le C||f||_{L(p,\infty,\alpha)}(1-r^2)^{\lambda}\int_0^1 \frac{(1-\rho)^{\beta-\alpha-1}}{(1-r\rho)^{\beta+\lambda}}d\rho \le C(\alpha,\beta,\lambda,n)||f||_{L(p,\infty,\alpha)}(1-r^2)^{\lambda}\frac{1}{(1-r)^{\alpha+\lambda}}.$$

Thus, $(1-r)^{\alpha}M_p(S_{\beta,\lambda}f;r) \le C(\alpha,\beta,\lambda,n)||f||_{L(p,\infty,\alpha)}$, $0 \le r < 1$, so (2) and (3) are proved.

Since the operator $T_{\beta,0}$ ($\lambda=0$) is bounded on $L(p,q,\alpha)$ and, by Theorem A, is the identity map on $h(p,q,\alpha)$, the mapping (4), $T_{\beta,0} : L(p,q,\alpha) \longrightarrow h(p,q,\alpha)$ is a harmonic projection of $L(p,q,\alpha)$ onto $h(p,q,\alpha)$. The proof of Theorem 3 is complete. □

ACKNOWLEDGMENTS

Supported by the Mathematical Studies Center, Yerevan State University.

REFERENCES

1. K. Avetisyan, "Continuous inclusions and Bergman type operators in n-harmonic mixed norm spaces on the polydisc," J. Math. Anal. Appl. **291**, 727–740 (2004).
2. K. Avetisyan, "Weighted integrals and Bloch spaces of n-harmonic functions on the polydisc," Potential Anal. **29**, 49–63 (2008).
3. K. Avetisyan, "On harmonic projection operators on mixed norm spaces over the real ball," in *Proceedings of the International Conference Dedicated to 100th Anniversary of M. Djrbashian, Oct. 22–24, 2018, Yerevan, Armenia* (NAS RA, Yerevan, 2018), p. 13.
4. K. Avetisyan and Y. Tonoyan, "Continuous embeddings in harmonic mixed norm spaces on the unit ball in $\mathbb{R}^n$," J. Contemp. Math. Anal. **47**, 209–220 (2012).
5. K. Avetisyan and Y. Tonoyan, "On the fractional integro-differentiation operator in $\mathbb{R}^n$," J. Contemp. Math. Anal. **50**, 236-245 (2015).
6. R. Coifman and R. Rochberg, "Representation theorems for holomorphic and harmonic functions in L^p," Asterisque **77**, 11–66 (1980).
7. A. E. Djrbashian, "Integral representations and continuous projectors in some spaces of harmonic functions," Mat. Sb. **121**, 259–271 (1983).
8. A. E. Djrbashian, "Integral representations for Riesz systems in the unit ball and some applications," Proc. Am. Math. Soc. **117**, 395–403 (1993).
9. A. E. Djrbashian and F. A. Shamoian, *Topics in the Theory of A^p_α Spaces*, Vol. 105 of *Teubner–Texte zur Mathematics* (Teubner, Leipzig, 1988).
10. T. M. Flett, "The dual of an inequality of Hardy and Littlewood and some related inequalities," J. Math. Anal. Appl. **38**, 746–765 (1972).
11. G. H. Hardy and J. E. Littlewood, "Some properties of fractional integrals (II)," Math. Z. **34**, 403–439 (1932).
12. G. H. Hardy and J. E. Littlewood, "Theorems concerning mean values of analytic or harmonic functions," Quart. J. Math. **12**, 221–256 (1941).
13. H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces* (Springer, New York, Berlin, Heidelberg, 2000).
14. M. Jevtić and M. Pavlović, "Harmonic Bergman functions on the unit ball in $\mathbb{R}^n$," Acta Math. Hung. **85**, 81–96 (1999).
15. E. Ligocka, "The Hölder continuity of the Bergman projection and proper holomorphic mappings," Studia Math. **80**, 89–107 (1984).
16. J. Miao, "Reproducing kernels for harmonic Bergman spaces of the unit ball," Monatsh. Math. **125**, 25–35 (1998).
17. M. Pavlović, "A proof of the Hardy–Littlewood theorem on fractional integration and a generalization," Publ. Inst. Math. (Belgrade) **59** (73), 31–38 (1996).
18. A. I. Petrosyan, "On weighted harmonic Bergman spaces," Demonstratio Math. **41**, 73–83 (2008).
19. G. Ren, "Harmonic Bergman spaces with small exponents in the unit ball," Collect. Math. **53**, 83–98 (2002).
20. E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces* (Princeton Univ. Press, Princeton, NJ, 1971).
21. Y. Tonoyan, "On integral operators of Bergman type on the unit ball of $\mathbb{R}^n$," Proc. Yerevan State Univ., Phys. Math. Sci. **2**, 23–30 (2015).