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REAL AND COMPLEX ANALYSIS

Fractional Integration in Weighted Lebesgue Spaces

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Abstract—The action of the fractional integration operator in weighted Lebesgue classes and mixed-norm spaces is studied in the unit ball from $\mathbb{R}^n$. Some results of Hardy, Littlewood, and Flett are refined and generalized.

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1. INTRODUCTION AND DESIGNATIONS

The first results on action of fractional integration operators in Lebesgue classes go back to Hardy and Littlewood [1–4]. In particular, they showed how much the fractional integral $D^{-\alpha}f$ is "better" than the function $f \in L^p$ itself. Namely, the fractional integration operator $D^{-\alpha}$ $(0 < \alpha < 1)$ boundedly acts from the space L^p $(1 < p < 1/\alpha)$ to the space L^q $\left(q = \frac{p}{1-\alpha p}\right)$. In addition, Hardy and Littlewood [1–4] established various weighted analogs, including those for holomorphic functions. Various generalizations may be found in works of Flett [5, 6] and Karapetyants and Rubin [7]. In the current work we aim at clarifying the action of fractional integration operators in mixed-norm weighted Lebesgue spaces $L(p, q, \alpha)$.

Suppose $B = B_n$ is an open unit ball in $\mathbb{R}^n (n \ge 2)$ and $S = \partial B$ is its boundary, the unit sphere. The p-order integral means of the function $f(x) = f(r\zeta)$ on the sphere $|x| = r$ are denoted as

$$M_p(f; r) := ||f(r\cdot)||_{L^p(S;d\sigma)}, \quad 0 \le r < 1, \quad 0 < p \le \infty,$$

where $d\sigma$ is the $(n-1)$-dimensional Lebesgue surface measure on S normalized by $\sigma(S) = 1$.

By definition, the mixed-norm space $L(p, q, \alpha)\big(0 < p, q \le \infty, \alpha \in \mathbb{R}\big)$ is a set of functions f measurable in the unit ball B for which the following norm (quasi-norm) is finite

$$||f||_{L(p,q,\alpha)} := \begin{cases} \left(\int\limits_0^1 (1-r)^{\alpha q - 1} M_p^q(f; r)\, dr\right)^{1/q}, & 0 < q < \infty, \\ \operatorname*{ess\,sup}\limits_{0 \le r < 1} (1-r)^{\alpha} M_p(f; r), & q = \infty. \end{cases}$$

Sometimes, we will invoke the more general version of the mixed norm with $q < \infty$ and $\gamma \in \mathbb{R}$

$$||f||_{L(p,q,\alpha;r^\gamma dr)} := \left[\int\limits_0^1 (1-r)^{\alpha q - 1} M_p^q(f; r) \quad r^\gamma dr\right]^{1/q} = ||r^{\gamma/q} f||_{L(p,q,\alpha)}.$$

We represent the points in $\mathbb{R}^n$ in the form $x = r\zeta$, where $\zeta \in S$, $|x| = r$. The symbols $C(\alpha, \beta, \dots)$, C_α, etc., denote different positive constants that depend only on the mentioned parameters. Suppose that

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$\mathbb{N}$ only denotes the set of positive integer numbers and $[\alpha]$ is the larger integer number not exceeding $\alpha \in \mathbb{R}$, that is, the integer part of the number α. For real expressions A and B, the symbol $A \approx B$ means the two-sided inequality $c_1|A| \le |B| \le c_2|A|$ with some insignificant positive constants c_1 and c_2 independent of the participating variables.

Definition 1.1. *(Fractional integral and Riemann–Liouville derivative)*

For a function $f(r)$ of one variable $r \in [0,1)$, we define

$$D^{-\alpha} f(r) := \frac{1}{\Gamma(\alpha)} \int_0^r (r-t)^{\alpha-1} f(t)dt = \frac{r^\alpha}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tr)dt,$$

$$D^m f(r) := \left(\frac{d}{dr}\right)^m f(r), \quad D^\alpha f(r) := D^{-(m-\alpha)} D^m f(r), \quad D^0 f := f,$$

where $\alpha > 0$, $m \in \mathbb{N}$, and $m-1 < \alpha \le m$.

Definition 1.2. *(Fractional integral and Riemann–Liouville derivative on $\mathbb{R}^n$, $n \ge 2$) For a given function $f(x)$ in the unit ball B, we define*

$$\mathcal{D}^{-\alpha} f(x) \equiv \mathcal{D}_n^{-\alpha} f(x) := r^{-(\alpha+n/2-1)} D^{-\alpha}\left\{r^{n/2-1} f(x)\right\} = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tx) t^{n/2-1} dt,$$

$$\mathcal{D}^{\alpha} f(x) \equiv \mathcal{D}_n^{\alpha} f(x) := r^{-(n/2-1)} D^{\alpha}\left\{r^{\alpha+n/2-1} f(x)\right\}, \quad r = |x|, \quad \alpha > 0.$$

The latter version of the radial fractional derivative on $\mathbb{R}^n$ was introduced in [8] and allows simplifying its application in weighted spaces. We also define a somewhat more general fractional integral on $\mathbb{R}^n$.

Definition 1.3. *For $\alpha > 0$, $\gamma \in \mathbb{R}$, we define*

$$\widetilde{\mathcal{D}}_{n,\gamma}^{-\alpha} f(x) := r^{-(\alpha+\gamma+n/2-1)} D^{-\alpha}\left\{r^{\gamma+n/2-1} f(x)\right\} = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tx) t^{\gamma+n/2-1} dt. \tag{1.1}$$

We see that $\widetilde{\mathcal{D}}_{n,\gamma}^{-\alpha}$ differs from $\mathcal{D}^{-\alpha}$ only by an insignificant multiplier t^γ in the integrand. We may set $\gamma + n/2 > 0$. It is clear that $\widetilde{\mathcal{D}}_{n,0}^{-\alpha} = \mathcal{D}^{-\alpha}$ and $\widetilde{\mathcal{D}}_{n,\gamma}^{-\alpha}|f(x)| \le \mathcal{D}^{-\alpha}|f(x)|$ if $\gamma \ge 0$. From the general formula

$$\mathcal{D}^m f(x) = r^{-(n/2-1)} \frac{\partial^m}{\partial r^m}\left[r^{m-1+n/2} f(x)\right], \quad m \in \mathbb{N},$$

it is fruitful to express the first- and second-order fractional derivatives explicitly

$$\mathcal{D}^1 f(x) = \frac{n}{2} f + r\frac{\partial f}{\partial r},$$

$$\mathcal{D}^2 f(x) = \left(1+\frac{n}{2}\right)\frac{n}{2} f(x) + 2\left(1+\frac{n}{2}\right) r\frac{\partial f}{\partial r} + r^2 \frac{\partial^2 f}{\partial r^2}. \tag{1.2}$$

It is well-known how perfect is the action of fractional integro-differentiation in weighted Bergman spaces in the unit disc $\mathbb{D} = B_2$, that is, in weighted classes of holomorphic (or harmonic) are integrable functions. In particular,

$$||\mathcal{D}_2^{-\beta} f||_{L(p,p,\alpha-\beta)} \le C(p,\alpha)||f||_{L(p,p,\alpha)}, \quad \alpha > \beta > 0, \quad 0 < p < \infty, \tag{1.3}$$

for all holomorphic functions in the disc (see, e.g., [3, Theorem 8] and [6, Theorem 6]). The inverse inequality is also true. Therefore, it may seem somewhat strange that inequality (1.2) at $p = 1$ is violated for general measurable functions. Indeed, let us choose the function $h_0(x) = h_0(r) := r^{-1}\left(\log\frac{2}{r}\right)^{-2}$, $0 < r < 1$, compute, and estimate the norms

$$||h_0||_{L(1,1,\alpha)} = \int_0^1 (1-r)^{\alpha-1} h_0(r)dr = \int_0^1 \frac{(1-r)^{\alpha-1}}{r\left(\log\frac{2}{r}\right)^2} dr < +\infty, \quad \text{i.e.} \quad h_0 \in L(1,1,\alpha),$$

$$||\mathcal{D}_2^{-\beta}h_0||_{L(1,1,\alpha-\beta)} \geq C\int_0^1 \frac{(1-r)^{\alpha-\beta-1}}{r\log\frac{4}{r}}dr = +\infty, \quad \text{i.e.} \quad \mathcal{D}_2^{-\beta}h_0 \notin L(1,1,\alpha-\beta).$$

The same counterexample h_0 is applicable when we consider the mixed-norm spaces $L(p,1,\alpha)$, $1 \leq p \leq \infty$, instead of $L(p,p,\alpha)$ in (1.2).

This work is aimed at deriving the inequalities of type (1.2) with fractional integration on mixed-norm Lebesgue spaces $L(p,q,\alpha)$. Different fractional operators, weighted functions, and indices may probably lead to different results.

2. FLETT INEQUALITIES AND THEIR COROLLARIES

We provide some Hardy-type inequalities obtained by Flett [5, pp. 490–491], [6, p. 758].

Lemma A. ([5, 6]). *For $1 \leq q < \infty$, $\alpha > \beta > 0$, $\lambda < 1 - \frac{1}{q}$, and a measurable function $h(r) \geq 0$, the following inequality is valid*

$$\int_0^1 (1-r)^{(\alpha-\beta)q-1} r^{q(\lambda-\beta)} \left(D^{-\beta}h(r)\right)^q dr \leq C\int_0^1 (1-r)^{\alpha q-1} r^{q\lambda} h^q(r)dr, \tag{2.1}$$

where $C = C(q,\alpha,\beta,\lambda)$, together with the simpler inequality

$$\int_0^1 (1-r)^{(\alpha-\beta)q-1} \left(D^{-\beta}h(r)\right)^q dr \leq C(q,\alpha,\beta)\int_0^1 (1-r)^{\alpha q-1} h^q(r)dr. \tag{2.2}$$

Remark 2.1. *Inequalities (2.1) and (2.2) may be considered the generalizations of the well-known Hardy inequalities (see, for instance, [4]) to the case of fractional integrals of arbitrary order. At $\beta = 1$ inequality (2.2) reduces to the one of the classical Hardy inequalities. Nevertheless, inequalities (2.1) and (2.2) surely may be refined and updated in several directions. For instance, for more general weighted functions (the so called normal functions), the inequalities of the type (2.1), (2.2) were obtained in [9, 10].*

On the other side, inequality (2.2) at $q = 1$ may be generalized to the identity. In addition, we are interested in the possibility of replacing the fractional integration operator $D^{-\beta}$ by another one. These and some other refinements are contained in the following lemma.

Lemma 2.1. (**i**) *At $q = 1$, $\alpha > \beta > 0$, $\gamma \in \mathbb{R}$, $n \geq 2$, the identities are true*

$$\int_0^1 (1-r)^{\alpha-\beta-1} D^{-\beta}h(r)dr = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 (1-r)^{\alpha-1}h(r)dr, \tag{2.3}$$

$$\int_0^1 (1-r)^{\alpha-\beta-1} r^{\beta+\gamma+n/2-1} \widetilde{\mathcal{D}}_{n,\gamma}^{-\beta}h(r)dr = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 (1-r)^{\alpha-1} r^{\gamma+n/2-1}h(r)dr, \tag{2.4}$$

(**ii**) *If $1 \leq q < \infty$, $\alpha > \beta > 0$, $\lambda < 1 - 1/q$, $\gamma \in \mathbb{R}$, $n \geq 2$, $h(r) \geq 0$, then*

$$\int_0^1 (1-r)^{(\alpha-\beta)q-1} r^{q(\lambda+\gamma+n/2-1)} \left(\widetilde{\mathcal{D}}_{n,\gamma}^{-\beta}h(r)\right)^q dr$$
$$\leq C(q,\alpha,\beta,\lambda)\int_0^1 (1-r)^{\alpha q-1} r^{q(\lambda+\gamma+n/2-1)} h^q(r)dr, \tag{2.5}$$

$$\int_0^1 (1-r)^{(\alpha-\beta)q-1} r^{q(\beta+\gamma+n/2-1)} \left(\widetilde{\mathcal{D}}_{n,\gamma}^{-\beta}h(r)\right)^q dr$$

$$\leq C(q,\alpha,\beta)\int\limits_0^1(1-r)^{\alpha q-1}r^{q(\gamma+n/2-1)}h^q(r)dr. \tag{2.6}$$

In particular, if $1<q<\infty$ *or* $1\leq q<\infty$, $n\geq 3$, *then*

$$\int\limits_0^1(1-r)^{(\alpha-\beta)q-1}\left(\mathcal{D}^{-\beta}h(r)\right)^q dr\leq C(q,\alpha,\beta)\int\limits_0^1(1-r)^{\alpha q-1}h^q(r)dr. \tag{2.7}$$

Proof. **(i)** The Fubini theorem and the replacement of variables $r-t=\xi(1-t)$, $1-r=(1-t)(1-\xi)$ lead to

$$\int\limits_0^1(1-r)^{\alpha-\beta-1}D^{-\beta}h(r)dr=\int\limits_0^1(1-r)^{\alpha-\beta-1}\left[\frac{1}{\Gamma(\beta)}\int\limits_0^r(r-t)^{\beta-1}h(t)dt\right]dr$$

$$=\frac{1}{\Gamma(\beta)}\int\limits_0^1\left[\int\limits_t^1(r-t)^{\beta-1}(1-r)^{\alpha-\beta-1}dr\right]h(t)dt$$

$$=\frac{1}{\Gamma(\beta)}\int\limits_0^1(1-t)^{\alpha-1}\left[\int\limits_0^1\xi^{\beta-1}(1-\xi)^{\alpha-\beta-1}d\xi\right]h(t)dt=\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int\limits_0^1(1-t)^{\alpha-1}h(t)dt.$$

Identity (2.3) is proved. Then, we replace $h(r)$ with $r^{\gamma+n/2-1}h(r)$ in (2.3) and obtain identity (2.4),

$$\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int\limits_0^1(1-r)^{\alpha-1}r^{\gamma+n/2-1}h(r)dr$$

$$=\int\limits_0^1(1-r)^{\alpha-\beta-1}r^{\beta+\gamma+n/2-1}r^{-(\beta+\gamma+n/2-1)}D^{-\beta}\{r^{\gamma+n/2-1}h(r)\}dr$$

$$=\int\limits_0^1(1-r)^{\alpha-\beta-1}r^{\beta+\gamma+n/2-1}\widetilde{\mathcal{D}}_{n,\gamma}^{-\beta}h(r)dr.$$

(ii) To prove (2.5) and (2.6), it is sufficient to replace $h(r)$ with $r^{\gamma+n/2-1}h(r)$ in (2.1) and (2.2), respectively.

In the special case $1<q<\infty$ or $1\leq q<\infty$, $n\geq 3$, we can choose $\gamma=0$ and $\lambda=-(n/2-1)\leq 0$ and obtain (2.7). □

Remark 2.2. *We may admit a simplifying denotation* $s:=q(\lambda+\gamma+n/2-1)$ *and rewrite (2.5) in the equivalent form*

$$\int\limits_0^1(1-r)^{(\alpha-\beta)q-1}r^s\left(\widetilde{\mathcal{D}}_{n,\gamma}^{-\beta}h(r)\right)^q dr\leq C\int\limits_0^1(1-r)^{\alpha q-1}r^sh^q(r)dr \tag{2.8}$$

only if $s<q(\gamma+n/2)-1$ *or* $\gamma>\frac{s+1}{q}-\frac{n}{2}$. *Here, the constant* $C=C(q,\alpha,\beta,\gamma,s,n)$ *depends only on the mentioned parameters.*

3. SEMIGROUP PROPERTIES OF OPERATORS $\mathcal{D}^{\pm\alpha}$ AND $\widetilde{\mathcal{D}}_{n,\gamma}^{-\alpha}$

In this section we derive several auxiliary semigroup or commutative formulas for fractional operators $\mathcal{D}^{\pm\alpha}$ and $\widetilde{\mathcal{D}}_{n,\gamma}^{-\alpha}$. We already established some identities of this type in [11] at $n=2$.

Lemma 3.1. *For $m \in \mathbb{N}$ and a sufficiently smooth function $f(x)$ in the unit ball B, the following semigroup formulas take place*:

$$\mathcal{D}^{-\alpha-\beta} f = r^{-\beta}\mathcal{D}^{-\alpha}\{r^\beta \mathcal{D}^{-\beta} f\} = \widetilde{\mathcal{D}}^{-\alpha}_{n,\beta}\mathcal{D}^{-\beta} f, \quad \alpha, \beta > 0, \tag{3.1}$$

$$\mathcal{D}^{-\alpha}\mathcal{D}^{\beta} f = r^{-\beta}\mathcal{D}^{-(\alpha-\beta)}\{r^\beta f\} = \widetilde{\mathcal{D}}^{-(\alpha-\beta)}_{n,\beta} f, \quad \alpha > \beta > 0, \tag{3.2}$$

$$r^{-\beta}\mathcal{D}^{-\alpha}\{r^\beta f\} = \widetilde{\mathcal{D}}^{-\alpha}_{n,\beta} f, \quad \alpha > 0, \quad \beta \in \mathbb{R}, \tag{3.3}$$

$$\mathcal{D}^{-\alpha}\mathcal{D}^{\beta} f = r^{-\alpha}\mathcal{D}^{\beta-\alpha}\{r^\alpha f\}, \quad \beta > \alpha > 0, \tag{3.4}$$

$$\mathcal{D}^{\delta} f = r^{m-\delta}\mathcal{D}^{-(m-\delta)}\mathcal{D}^{m}\{r^{-(m-\delta)} f\}, \quad 0 < \delta \le m, \tag{3.5}$$

$$\mathcal{D}^{\delta} f = \widetilde{\mathcal{D}}^{-(1-\delta)}_{n,\delta-1}\Big[(\delta-1) f + \mathcal{D}^1 f\Big], \quad 0 < \delta \le 1, \tag{3.6}$$

$$\mathcal{D}^{-\alpha}\mathcal{D}^{m} f = \mathcal{D}^{m}\mathcal{D}^{-\alpha} f, \quad \alpha > 0. \tag{3.7}$$

Proof. We successively prove all seven identities.

(3.1) According to Definitions 1.1–1.3 of the operators $D^{\pm\alpha}$, $\mathcal{D}^{\pm\alpha}$, $\widetilde{\mathcal{D}}^{\pm\alpha}$, we obtain

$$\mathcal{D}^{-\alpha-\beta} f = r^{-(\alpha+\beta+n/2-1)} D^{-\alpha-\beta}\{r^{n/2-1} f(x)\}$$
$$= r^{-(\alpha+\beta+n/2-1)} D^{-\alpha}\Big[r^{\beta+n/2-1} r^{-(\beta+n/2-1)} D^{-\beta}\{r^{n/2-1} f(x)\}\Big]$$
$$= r^{-\beta}\mathcal{D}^{-\alpha}\Big[r^\beta \mathcal{D}^{-\beta} f(x)\Big] = r^{-\beta} r^\beta \frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}\mathcal{D}^{-\beta} f(tx) t^{\beta+n/2-1} dt = \widetilde{\mathcal{D}}^{-\alpha}_{n,\beta}\mathcal{D}^{-\beta} f(x).$$

(3.2) Using the inversion formulas for $\mathcal{D}^{\pm\alpha}$, (1.1), and (3.1), we obtain for $\alpha > \beta > 0$

$$\mathcal{D}^{-\alpha}\mathcal{D}^{\beta} f = \mathcal{D}^{-(\alpha-\beta)-\beta}\mathcal{D}^{\beta} f = r^{-\beta}\mathcal{D}^{-(\alpha-\beta)}\{r^\beta \mathcal{D}^{-\beta}\mathcal{D}^{\beta} f\} = r^{-\beta}\mathcal{D}^{-(\alpha-\beta)}\{r^\beta f\} = \widetilde{\mathcal{D}}^{-(\alpha-\beta)}_{n,\beta} f.$$

(3.3) Similarly,

$$r^{-\beta}\mathcal{D}^{-\alpha}\{r^\beta f(x)\} = r^{-\beta}\frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}(tr)^\beta f(tx) t^{n/2-1} dt = \widetilde{\mathcal{D}}^{-\alpha}_{n,\beta} f(x).$$

(3.4) We apply the inversion formulas and Definitions 1.1–1.3 with $\beta > \alpha > 0$ and obtain

$$\mathcal{D}^{-\alpha}\mathcal{D}^{\beta} f = r^{-(\alpha+n/2-1)} D^{-\alpha}\Big\{r^{n/2-1}\mathcal{D}^\beta f\Big\}$$
$$= r^{-(\alpha+n/2-1)} D^{\beta-\alpha} D^{-\beta}\Big\{r^{n/2-1}\mathcal{D}^\beta f\Big\}$$
$$= r^{-(\alpha+n/2-1)} D^{\beta-\alpha} r^{\beta+n/2-1}\mathcal{D}^{-\beta}\Big\{r^{-(n/2-1)} r^{n/2-1}\mathcal{D}^\beta f\Big\}$$
$$= r^{-(\alpha+n/2-1)} D^{\beta-\alpha}\Big\{r^{\beta+n/2-1} f\Big\} = r^{-\alpha}\mathcal{D}^{\beta-\alpha}\{r^\alpha f\},$$

which proves identity (3.4).

(3.5) To prove identity (3.5), it is sufficient to replace in (3.4) β by m, α by $m-\delta$, and the function $r^{m-\delta} f(x)$ by $f(x)$.

(3.6) We transform

$$\mathcal{D}^1\{r^{-(1-\delta)} f\} = \frac{n}{2} r^{-(1-\delta)} f + r D^1\{r^{-(1-\delta)} f\}$$
$$= \frac{n}{2} r^{-(1-\delta)} f + r\Big[-(1-\delta) r^{-(2-\delta)} f + r^{-(1-\delta)} D^1 f\Big]$$
$$= r^{-(1-\delta)}\Big[\Big(\delta + \frac{n}{2} - 1\Big) f + r D^1 f\Big] = r^{-(1-\delta)}\Big[(\delta-1) f + \mathcal{D}^1 f\Big].$$

Hence, using the proved identities (3.5) and (3.3), we arrive at

$$\mathcal{D}^{\delta} f = r^{1-\delta} \mathcal{D}^{-(1-\delta)} \mathcal{D}^1 \big\{ r^{-(1-\delta)} f \big\} = r^{1-\delta} \mathcal{D}^{-(1-\delta)} r^{-(1-\delta)} \Big[(\delta-1) f + \mathcal{D}^1 f \Big]$$
$$= \widetilde{\mathcal{D}}^{-(1-\delta)}_{n,\delta-1} \Big[(\delta-1) f + \mathcal{D}^1 f \Big],$$

which is the required result.

(3.7) Firstly, we prove a simpler commutative formula

$$r^m D^m \mathcal{D}^{-\alpha} f(x) = \mathcal{D}^{-\alpha} \big\{ r^m D^m f(x) \big\}, \quad x = r\zeta. \tag{3.8}$$

For this purpose, we expand it by using the obvious formula $\frac{\partial^m}{\partial r^m} f(tr\zeta) = t^m \frac{\partial^m}{\partial (tr)^m} f(tr\zeta)$,

$$r^m D^m \mathcal{D}^{-\alpha} f(x) = r^m \frac{\partial^m}{\partial r^m} \left[\frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tx) t^{n/2-1} dt \right]$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} \left[(tr)^m \frac{\partial^m}{\partial (tr)^m} f(tx) \right] t^{n/2-1} dt = \mathcal{D}^{-\alpha} \big\{ r^m D^m f(x) \big\},$$

which coincides with (3.8).

Similarly to formulas (1.1) for $m = 1, 2$, for arbitrary $m \in \mathbb{N}$ there exist constants $c_k = c_k(n) > 0$ $(k = 0, 1, 2, \ldots, m-1)$ dependent only on n such that

$$\mathcal{D}^m f(x) = r^{-(n/2-1)} D^m \big\{ r^{m+n/2-1} f(x) \big\} = \sum_{k=0}^{n} c_k r^k D^k f(x). \tag{3.9}$$

Then, by using (3.8) and (3.9), we obtain

$$\mathcal{D}^m \mathcal{D}^{-\alpha} f(x) = \sum_{k=0}^{n} c_k r^k D^k \mathcal{D}^{-\alpha} f(x) = \mathcal{D}^{-\alpha} \left\{ \sum_{k=0}^{n} c_k r^k D^k f(x) \right\} = \mathcal{D}^{-\alpha} \mathcal{D}^m f(x).$$

Lemma 3.1 is completely proved. □

4. FRACTIONAL INTEGRATION IN MIXED-NORM LEBESGUE SPACES IN THE BALL

Because we are interested in questions of fractional integration on functions integrable with weight in the ball, we need accurate estimates of some reference integrals. The integral in the below lemma is well-known, in particular, from the theory of Gauss hypergeometric function, and we need the two-sided estimate of this integral. Similar estimates may be found in [12, Lemma 3.1], [13, Lemmas 6.1 and 6.3], and [14]. Here, we provide another direct and self-contained proof of inequalities based only on integral estimates without invoking the theory of special functions.

Lemma 4.1. *For $\alpha, \lambda > 0$ and $\beta \in \mathbb{R}$ the following two-sided estimates take place*

$$\int_0^1 \frac{t^{\lambda-1}(1-t)^{\alpha-1}}{(1-rt)^{\beta}} dt \approx \begin{cases} \frac{1}{(1-r)^{\beta-\alpha}}, & \beta > \alpha, \\ 1, & \beta < \alpha, \\ \log \frac{2}{1-r}, & \beta = \alpha, \end{cases} \qquad 0 \le r < 1, \tag{4.1}$$

where participating (but not explicitly mentioned) constants depend only on α, β, and λ.

Proof. All inequalities in (4.1) are trivial if $0 \le r \le \frac{1}{2}$. Therefore, it is enough to prove (4.1) only for $\frac{1}{2} \le r < 1$ or $r \to 1^-$. Firstly, supposing $\beta > \alpha > 0$, we majorize the integral (4.1)

$$\int_0^1 \frac{t^{\lambda-1}(1-t)^{\alpha-1}}{(1-rt)^{\beta}} dt = \int_0^r \frac{t^{\lambda-1}(1-t)^{\alpha-1}}{(1-rt)^{\beta}} dt + \int_r^1 \frac{t^{\lambda-1}(1-t)^{\alpha-1}}{(1-rt)^{\beta}} dt$$

$$\le \int_0^r \frac{t^{\lambda-1}(1-t)^{\alpha-1}}{(1-t)^{\beta}}dt + \frac{1}{(1-r)^{\beta}}\int_r^1 t^{\lambda-1}(1-t)^{\alpha-1}dt$$
$$= \int_0^r \frac{t^{\lambda-1}}{(1-t)^{1+\beta-\alpha}}dt + \frac{1}{(1-r)^{\beta}}\int_r^1 t^{\lambda-1}(1-t)^{\alpha-1}dt.$$

Taking into account two asymptotic relations with $r \to 1^-$

$$\int_0^r \frac{t^{\lambda-1}}{(1-t)^{1+\beta-\alpha}}dt \sim \frac{1}{(\beta-\alpha)(1-r)^{\beta-\alpha}}, \quad \int_r^1 t^{\lambda-1}(1-t)^{\alpha-1}dt \sim \frac{1}{\alpha}(1-r)^{\alpha},$$

we conclude that

$$\int_0^1 \frac{t^{\lambda-1}(1-t)^{\alpha-1}}{(1-rt)^{\beta}}dt \le C(\alpha,\beta,\lambda)\frac{1}{(1-r)^{\beta-\alpha}}, \quad \frac{1}{2} \le r < 1.$$

Inversely,

$$\int_0^1 \frac{t^{\lambda-1}(1-t)^{\alpha-1}}{(1-rt)^{\beta}}dt \ge \int_r^1 \frac{t^{\lambda-1}(1-t)^{\alpha-1}}{(1-rt)^{\beta}}dt \ge \frac{C_\lambda}{(1-r^2)^{\beta}}\int_r^1 (1-t)^{\alpha-1}dt$$
$$= \frac{C_\lambda}{\alpha(1-r^2)^{\beta}}(1-r)^{\alpha} \ge \frac{C_\lambda}{\alpha 2^{\beta}}\frac{1}{(1-r)^{\beta-\alpha}}.$$

The case $\beta > \alpha > 0$ is proved. We now proceed to the case $\beta < \alpha$, suppose $0 < \beta < \alpha$, and estimate

$$\int_0^1 \frac{t^{\lambda-1}(1-t)^{\alpha-1}}{(1-rt)^{\beta}}dt \le \int_0^1 \frac{t^{\lambda-1}(1-t)^{\alpha-1}}{(1-t)^{\beta}}dt = B(\lambda, \alpha-\beta),$$
$$\int_0^1 \frac{t^{\lambda-1}(1-t)^{\alpha-1}}{(1-rt)^{\beta}}dt \ge \int_0^1 t^{\lambda-1}(1-t)^{\alpha-1}dt = B(\lambda,\alpha),$$

where $B(\cdot,\cdot)$ is the Euler beta function. In the case $\beta < 0$ the proof is similar.

Finally, in the third case $\beta = \alpha$ we note that $\frac{1-t}{1-rt} < 1$ and replace the variable in the integral

$$\int_0^1 \frac{t^{\lambda-1}(1-t)^{\alpha-1}}{(1-rt)^{\alpha}}dt = \int_0^1 \frac{t^{\lambda-1}}{1-rt}\left(\frac{1-t}{1-rt}\right)^{\alpha}dt \le \int_0^1 \frac{t^{\lambda-1}}{1-rt}dt$$
$$= \frac{1}{r^{\alpha}}\int_0^r \frac{\eta^{\alpha-1}}{1-\eta}d\eta \sim \log\frac{1}{1-r} \quad \text{with} \quad r \to 1^-.$$

This proves the upper estimate. To prove the lower estimate, we use the monotonic decrease of the function $\frac{1-t}{1-rt}$ in t,

$$\int_0^1 \frac{t^{\lambda-1}(1-t)^{\alpha-1}}{(1-rt)^{\alpha}}dt = \int_0^1 \frac{t^{\lambda-1}}{1-rt}\left(\frac{1-t}{1-rt}\right)^{\alpha}dt \ge \int_0^r \frac{t^{\lambda-1}}{1-rt}\left(\frac{1-t}{1-rt}\right)^{\alpha}dt$$
$$\ge \int_0^r \frac{t^{\lambda-1}}{1-rt}\left(\frac{1-r}{1-r^2}\right)^{\alpha}dt \ge \frac{1}{2^{\alpha}}\int_0^r \frac{t^{\lambda-1}}{1-rt}dt \sim \frac{1}{2^{\alpha}}\log\frac{1}{1-r}$$

with $r \to 1^-$. This proves Lemma 4.1. □

In the following, we need accurate estimates of integrals similar to (4.1). In the next lemma we refine the estimates of some integrals considered by Flett [5, Eqs. (15.3) and (15.4)].

Lemma 4.2. *For $a \in \mathbb{R}$, b, and $c > 0$, the following two-sided estimates are true*

$$J_1 = J_1(r) := \int_0^r \frac{(r-t)^{b-1}t^{c-1}}{(1-t)^a} dt \approx \begin{cases} \frac{r^{b+c-1}}{(1-r)^{a-b}}, & a > b, \\ r^{b+c-1}, & a < b, \\ r^{b+c-1}\log\frac{2}{1-r}, & a = b, \end{cases} \qquad (0 < r < 1), \tag{4.2}$$

$$J_2 = J_2(\rho) := \int_\rho^1 \frac{(t-\rho)^{b-1}(1-t)^{c-1}}{t^a} dt \approx \begin{cases} \frac{(1-\rho)^{b+c-1}}{\rho^{a-b}}, & a > b, \\ (1-\rho)^{b+c-1}, & a < b, \\ (1-\rho)^{b+c-1}\log\frac{2}{\rho}, & a = b, \end{cases} \qquad (0 < \rho < 1), \tag{4.3}$$

where implicitly participating constants depend only on a, b, and c.

Proof. The first integral J_1 is immediately estimated by replacement of the variable $t = r\eta$,

$$J_1 = \int_0^r \frac{(r-t)^{b-1}t^{c-1}}{(1-t)^a} dt = r^{b+c-1}\int_0^1 \frac{(1-\eta)^{b-1}\eta^{c-1}}{(1-r\eta)^a} d\eta,$$

with further application of Lemma 4.1. The second integral J_2 reduces to J_1 by replacement of variables $1 - t = \eta(1-\rho)$ and $r = 1 - \rho$. □

On the basis of the proved Lemmas 4.1 and 4.2, we provide accurate estimates of integrals (2.1) and (2.2) at $q = 1$ and therefore refine Lemma A of Flett.

Lemma 4.3. (**i**) *For $\lambda < 0 < \beta < \alpha$, the two-sided estimate is true*

$$\int_0^1 (1-r)^{\alpha-\beta-1}r^{\lambda-\beta}D^{-\beta}h(r)dr \approx \int_0^1 (1-r)^{\alpha-1}r^{\lambda}h(r)dr. \tag{4.4}$$

(**ii**) *In the limit case $\lambda = 0 < \beta < \alpha$, we have*

$$\int_0^1 (1-r)^{\alpha-\beta-1}\mathcal{D}_2^{-\beta}h(r)dr \approx \int_0^1 (1-r)^{\alpha-1}h(r)\log\frac{2}{r}dr. \tag{4.5}$$

Proof. We prove only the second estimate of two similar estimates (4.4) and (4.5). By the Fubini theorem and (4.3) for the operator $\mathcal{D}_2^{-\beta} = r^{-\beta}D^{-\beta}$, at $\lambda = 0$ we derive

$$\int_0^1 (1-r)^{\alpha-\beta-1}r^{-\beta}D^{-\beta}h(r)dr = \int_0^1 (1-r)^{\alpha-\beta-1}r^{-\beta}\left[\frac{1}{\Gamma(\beta)}\int_0^r (r-t)^{\beta-1}h(t)dt\right]dr$$

$$= \frac{1}{\Gamma(\beta)}\int_0^1\left[\int_t^1 \frac{(r-t)^{\beta-1}(1-r)^{\alpha-\beta-1}}{r^\beta}dr\right]h(t)dt \approx \int_0^1 (1-t)^{\alpha-1}\log\frac{2}{t}h(t)dt,$$

which is the required result. □

We proceed to the formulation and proof of the main result of this paper.

Theorem 4.1. *For $1 \le p, q \le \infty$ the following five statements are true; in all of them, we assume that the norm in the right-hand sides of inequalities (4.6)–(4.11) is finite.*

(**i**) *For all $\alpha > \beta > 0$ the inequality is true*

$$||\mathcal{D}^{-\beta}f||_{L(p,q,\alpha-\beta)} \le C(q,\alpha,\beta)||f||_{L(p,q,\alpha)}, \tag{4.6}$$

except for the case $q = 1$, $n = 2$, when inequality (4.6) is violated.

(**ii**) *For all* $1 \leq q < \infty$, $\alpha > \beta > 0$, *and* $\gamma < \frac{qn}{2} - 1$, *the inequality is true*

$$||\mathcal{D}^{-\beta} f||_{L(p,q,\alpha-\beta;r^\gamma dr)} \leq C(q,\alpha,\beta,\gamma)||f||_{L(p,q,\alpha;r^\gamma dr)}. \tag{4.7}$$

In particular, at $q = 1$, $n = 2$, *we have*

$$||\mathcal{D}_2^{-\beta} f||_{L(p,1,\alpha-\beta;r^\gamma dr)} \leq C(\alpha,\beta,\gamma)||f||_{L(p,1,\alpha;r^\gamma dr)}, \quad \gamma < 0 < \alpha < \beta. \tag{4.8}$$

(**iii**) *For any* $\alpha > \beta > 0$, $\ell > \frac{1}{q} - \frac{n}{2}$ *it is true, that*

$$||\widetilde{\mathcal{D}}_{n,\ell}^{-\beta} f||_{L(p,q,\alpha-\beta)} \leq C(q,\alpha,\beta,\ell)||f||_{L(p,q,\alpha)}. \tag{4.9}$$

(**iv**) *If* $0 < \alpha < \delta \leq k$, $j \leq \delta$ $(j,k \in \mathbb{N})$, *then*

$$||\mathcal{D}^j f||_{L(p,q,\delta-\alpha)} \leq C(q,\alpha,\delta,j,k)||\mathcal{D}^k f||_{L(p,q,k-\alpha)} \tag{4.10}$$

for all sufficiently smooth functions f *in* B.

(**v**) *If* $0 < \alpha < \delta \leq [\alpha] + 1 < \delta + \dfrac{n}{2} - \dfrac{1}{q}$, *then*

$$||\mathcal{D}^\delta f||_{L(p,q,\delta-\alpha)} \leq C(q,\alpha,\delta)||\mathcal{D}^{[\alpha]+1} f||_{L(p,q,[\alpha]+1-\alpha)} \tag{4.11}$$

for all sufficiently smooth functions f *in* B.

Proof. **(i) Case $q = \infty$.** We suppose that $f(x) \in L(p,\infty,\alpha)$ and by the definition have

$$(1-r)^\alpha M_p(f;r) \leq ||f||_{L(p,\infty,\alpha)}, \quad 0 \leq r < 1. \tag{4.12}$$

We apply the Minkowski inequality and (4.12) successively and obtain

$$M_p(\mathcal{D}^{-\beta} f;r) \leq \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1} M_p(f;tr)dt$$

$$\leq ||f||_{L(p,\infty,\alpha)} \frac{1}{\Gamma(\beta)} \int_0^1 \frac{(1-t)^{\beta-1}}{(1-rt)^\alpha} dt \leq C(\alpha,\beta) \frac{||f||_{L(p,\infty,\alpha)}}{(1-r)^{\alpha-\beta}}.$$

Consequently, $||\mathcal{D}^{-\beta} f||_{L(p,\infty,\alpha-\beta)} \leq C||f||_{L(p,\infty,\alpha)}$, which is the required result.

Case $1 \leq q < \infty$. For $1 < q < \infty$ or $1 \leq q < \infty$, $n \geq 3$, by the Minkowski inequality and (2.7) we obtain

$$||\mathcal{D}^{-\beta} f||^q_{L(p,q,\alpha-\beta)} = \int_0^1 (1-r)^{(\alpha-\beta)q-1} \left|\left| \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1} f(tr\zeta) t^{n/2-1} dt \right|\right|^q_{L^p(S;d\sigma)} dr$$

$$\leq \int_0^1 (1-r)^{(\alpha-\beta)q-1} \left(\frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1} ||f(tr\zeta)||_{L^p(S;d\sigma)} t^{n/2-1} dt \right)^q dr$$

$$= \int_0^1 (1-r)^{(\alpha-\beta)q-1} \left(\mathcal{D}^{-\beta}||f||_{L^p(S)}\right)^q dr \leq C(\alpha,\beta,q,n) \int_0^1 (1-r)^{\alpha q-1} ||f||^q_{L^p(S)} dr = C||f||^q_{L(p,q,\alpha)}.$$

Violation of (4.6) at $q = 1$, $n = 2$ follows from (4.5), or we convince ourselves of that by applying the counterexample h_0 from the Introduction.

(ii) Now, we apply the more general inequality (2.8) with $\gamma = 0$ and the Minkowski inequality and estimate similarly to the previous statement **(i)**. As a result, we get

$$||\mathcal{D}^{-\beta} f||^q_{L(p,q,\alpha-\beta;r^\gamma dr)} \leq C \int_0^1 (1-r)^{\alpha q-1} ||f||^q_{L^p(S)} \quad r^\gamma dr = C||f||^q_{L(p,q,\alpha;r^\gamma dr)}.$$

In particular, at $q=1$, $n=2$, we have $\gamma<0$.

(iii) Case $1\le q<\infty$. Again, by the Minkowski inequality and (2.8), by taking into account $\ell>\frac{1}{q}-\frac{n}{2}$, we obtain

$$||\widetilde{\mathcal{D}}^{-\beta}_{n,\ell}f||^q_{L(p,q,\alpha-\beta)}\le\int\limits_0^1(1-r)^{(\alpha-\beta)q-1}\left(\widetilde{\mathcal{D}}^{-\beta}_{n,\ell}||f||_{L^p(S)}\right)^q dr$$

$$\le C(\alpha,\beta,q,\ell,n)\int\limits_0^1(1-r)^{\alpha q-1}||f||^q_{L^p(S)}dr=C||f||^q_{L(p,q,\alpha)}.$$

In particular, at $q=1$, $n=2$, we obtain $||\widetilde{\mathcal{D}}^{-\beta}_{2,\ell}f||_{L(p,1,\alpha-\beta)}\le C||f||_{L(p,1,\alpha)}$, $\ell>0$.

Case $q=\infty$. By the definition we have $(1-r)^\alpha M_p(f;r)\le||f||_{L(p,\infty,\alpha)}$ for $0\le r<1$, assuming that $f(x)\in L(p,\infty,\alpha)$. For estimating, we apply Lemma 4.1,

$$M_p(\widetilde{\mathcal{D}}^{-\beta}_{n,\ell}f;r)\le\frac{1}{\Gamma(\beta)}\int\limits_0^1(1-t)^{\beta-1}M_p(f;tr)t^{\ell+n/2-1}dt$$

$$\le||f||_{L(p,\infty,\alpha)}\frac{1}{\Gamma(\beta)}\int\limits_0^1\frac{(1-t)^{\beta-1}}{(1-rt)^\alpha}t^{\ell+n/2-1}dt\le C(\alpha,\beta,\ell,n)\frac{||f||_{L(p,\infty,\alpha)}}{(1-r)^{\alpha-\beta}},\quad\alpha>\beta>0.$$

Thus, $||\widetilde{\mathcal{D}}^{-\beta}_{n,\ell}f||_{L(p,\infty,\alpha-\beta)}\le C(\alpha,\beta,\ell,n)||f||_{L(p,\infty,\alpha)}$, which is the required result.

(iv) For sufficiently smooth functions and any $0<\alpha<\delta$, $1\le j\le\delta\le k$ $(j,k\in\mathbb{N})$, we estimate, following the inversion formula $f=\mathcal{D}^{-k}\mathcal{D}^kf$ and formulas (3.7), (3.2), and (4.9),

$$||\mathcal{D}^jf||_{L(p,q,\delta-\alpha)}=||\mathcal{D}^j\mathcal{D}^{-k}\mathcal{D}^kf||_{L(p,q,\delta-\alpha)}=||\mathcal{D}^{-k}\mathcal{D}^j\mathcal{D}^kf||_{L(p,q,\delta-\alpha)}$$
$$=||\widetilde{\mathcal{D}}^{-(k-j)}_{n,j}\mathcal{D}^kf||_{L(p,q,\delta-\alpha)}\le C||\mathcal{D}^kf||_{L(p,q,\delta-\alpha+k-j)}\le C||\mathcal{D}^kf||_{L(p,q,k-\alpha)}.$$

(v) Case $1\le q<\infty$**.** Denote $m:=[\alpha]+1$ so that $0<\alpha<\delta\le m=[\alpha]+1<\delta+\dfrac{n}{2}-\dfrac{1}{q}$. We transform the mth derivative in Eq. (3.5) using decomposition (3.9) and the Leibniz rule of differentiation,

$$\mathcal{D}^m\big\{r^{-(m-\delta)}f(x)\big\}=r^{-(n/2-1)}D^m\Big[r^{\delta+n/2-1}f(x)\Big]=r^{-(n/2-1)}\sum_{j=0}^m\binom{m}{j}\Big[D^{m-j}r^{\delta+n/2-1}\Big]D^jf$$

$$=\sum_{j=0}^m\binom{m}{j}C(\delta,n,j)r^{\delta-m+j}D^jf=r^{-(m-\delta)}\left[\sum_{j=0}^{m-1}C(\delta,m,n,j)r^jD^jf+r^mD^mf\right],$$

or, in more accurate form in terms of $\mathcal{D}^k$,

$$r^{m-\delta}\mathcal{D}^m\big\{r^{-(m-\delta)}f(x)\big\}=\sum_{j=0}^{m-1}C_j(\delta,m)\mathcal{D}^jf+\mathcal{D}^mf.\qquad(4.13)$$

Now, we transform the derivative $\mathcal{D}^\delta f$ using (3.5), (4.13), and (3.3),

$$\mathcal{D}^\delta f(x)=r^{m-\delta}\mathcal{D}^{-(m-\delta)}\Big[\mathcal{D}^m\big\{r^{-(m-\delta)}f(x)\big\}\Big]$$

$$=r^{m-\delta}\mathcal{D}^{-(m-\delta)}r^{-(m-\delta)}\left[\sum_{j=0}^{m-1}C_j(\delta,m)\mathcal{D}^jf+\mathcal{D}^mf\right]$$

$$= \widetilde{\mathcal{D}}_{n,\delta-m}^{-(m-\delta)} \left[\sum_{j=0}^{m-1} C_j(\delta, m)\mathcal{D}^j f + \mathcal{D}^m f \right], \quad 0 < \delta \le m < \delta + 1. \tag{4.14}$$

We proceed to mixed norms which we estimate using (4.14), (4.9), (4.10), and the condition $\delta - m > \frac{1}{q} - \frac{n}{2}$,

$$||\mathcal{D}^\delta f||_{L(p,q,\delta-\alpha)} = \left|\left| \widetilde{\mathcal{D}}_{n,\delta-m}^{-(m-\delta)} \left[\sum_{j=0}^{m-1} C_j(\delta, m)\mathcal{D}^j f + \mathcal{D}^m f \right] \right|\right|_{L(p,q,\delta-\alpha)}$$

$$\le C \left|\left| \sum_{j=0}^{m-1} C_j(\delta, m)\mathcal{D}^j f + \mathcal{D}^m f \right|\right|_{L(p,q,m-\alpha)}$$

$$\le C \sum_{j=0}^{m-1} C_j(\delta, m)||\mathcal{D}^j f||_{L(p,q,m-\alpha)} + ||\mathcal{D}^m f||_{L(p,q,m-\alpha)} \le C||\mathcal{D}^m f||_{L(p,q,m-\alpha)},$$

where the latter constant $C = C(q, \alpha, \delta, m, n)$ depends on the mentioned parameters. The case $1 \le q < \infty$ of inequality (4.11) is proved.

Case $q = \infty$ is similar, so we omit the proof details. Theorem 4.1 is completely proved. □

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