

# Accelerating the Convergence of Trigonometric Interpolation

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## Abstract

We discuss the problem of recovering a function by its discrete Fourier coefficients and consider acceleration of convergence of the classical trigonometric interpolation.

*Key Words:* Trigonometric interpolation, rational interpolation, convergence acceleration

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**1. Introduction** It is well known that reconstruction of a smooth on  $[-1, 1]$  function by trigonometric interpolation

$$I_N(f) = \sum_{n=-N}^N \check{f}_n e^{i\pi n x}, \quad (1)$$

$$\check{f}_n = \frac{1}{2N+1} \sum_{k=-N}^N f(x_k) e^{-i\pi n x_k}, \quad x_k = \frac{2k}{2N+1} \quad (2)$$

is noneffective when the 2-periodic extension of the interpolated function is discontinuous. The oscillations caused by the Gibbs phenomenon are typically propagated into regions away from the singularities and degrade the quality of interpolation.

Here we consider the convergence acceleration of the classical trigonometric interpolation applying similar idea as in the Fourier-Pade approximation [1], [2].

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**2. Basic Formulae** Consider a finite sequence of complex numbers  $\theta := \{\theta_k\}_{k=-p}^p$  and denote

$$\begin{aligned}\Delta_{n,+}^0(c_n) &= c_n, \quad \Delta_{n,+}^1(c_n) = c_n + \theta_1 c_{n+1}, \\ \Delta_{n,+}^k(c_n) &= \Delta_{n,+}^{k-1}(c_n) + \theta_k \Delta_{n+1,+}^{k-1}(c_n), \\ \Delta_{n,-}^0(c_n) &= c_n, \quad \Delta_{n,-}^1(c_n) = c_n + \theta_{-1} c_{n-1}, \\ \Delta_{n,-}^k(c_n) &= \Delta_{n,-}^{k-1}(c_n) + \theta_{-k} \Delta_{n-1,-}^{k-1}(c_n).\end{aligned}$$

By  $R_N(f)$  denote the error of the classical trigonometric interpolation

$$R_N(f) = f(x) - I_N(f) \quad (3)$$

and by  $f_n$  denote the  $n$ -th Fourier coefficient of  $f$

$$f_n = \frac{1}{2} \int_{-1}^1 f(x) e^{-i\pi n x} dx.$$

It is easy to verify that Equations (1), (2) and (3) imply

$$R_N(f) = \sum_{n=-N}^N (f_n - \check{f}_n) + \sum_{n=N+1}^{\infty} f_n e^{i\pi n x} + \sum_{n=-\infty}^{-N-1} f_n e^{i\pi n x}$$

in view of the relation

$$\check{f}_n = \sum_{s=-\infty}^{\infty} f_{n+s(2N+1)}.$$

The following transformation is easy to check

$$\begin{aligned}R_N(f) &= \check{f}_N \theta_{-1} \frac{e^{-i\pi N x} - e^{i\pi(N+1)x}}{(1 + \theta_{-1} e^{i\pi x})(1 + \theta_1 e^{-i\pi x})} \\ &\quad + \check{f}_{-N} \theta_1 \frac{e^{i\pi N x} - e^{-i\pi(N+1)x}}{(1 + \theta_{-1} e^{i\pi x})(1 + \theta_1 e^{-i\pi x})} \\ &\quad + \frac{1}{(1 + \theta_{-1} e^{i\pi x})(1 + \theta_1 e^{-i\pi x})} \sum_{|n|=N+1}^{\infty} \Delta_{n,+}^1(\Delta_{n,-}^1(f_n)) e^{i\pi n x} \\ &\quad + \frac{1}{(1 + \theta_{-1} e^{i\pi x})(1 + \theta_1 e^{-i\pi x})} \sum_{n=-N}^N \Delta_{n,+}^1(\Delta_{n,-}^1(f_n - \check{f}_n)) e^{i\pi n x}.\end{aligned}$$

Reiteration of this transformation up to  $p$  times yields the following expansion of the error

$$\begin{aligned}
 R_N(f) &= (e^{-i\pi Nx} - e^{i\pi(N+1)x}) \sum_{k=1}^p \frac{\theta_{-k} \Delta_{N,+}^{k-1} (\Delta_{n,-}^{k-1}(\check{f}_n))}{\prod_{s=1}^k (1 + \theta_{-s} e^{i\pi x})(1 + \theta_s e^{-i\pi x})} \\
 &+ (e^{i\pi Nx} - e^{-i\pi(N+1)x}) \sum_{k=1}^p \frac{\theta_k \Delta_{-N,+}^{k-1} (\Delta_{n,-}^{k-1}(\check{f}_n))}{\prod_{s=1}^k (1 + \theta_{-s} e^{i\pi x})(1 + \theta_s e^{-i\pi x})} \\
 &+ \frac{1}{\prod_{s=1}^p (1 + \theta_{-s} e^{i\pi x})(1 + \theta_s e^{-i\pi x})} \sum_{|n|=N+1}^{\infty} \Delta_{n,+}^p (\Delta_{n,-}^p(f_n)) e^{i\pi n x} \\
 &+ \frac{1}{\prod_{s=1}^p (1 + \theta_{-s} e^{i\pi x})(1 + \theta_s e^{-i\pi x})} \sum_{n=-N}^N \Delta_{n,+}^p (\Delta_{n,-}^p(f_n - \check{f}_n)) e^{i\pi n x}.
 \end{aligned}$$

This leads to the following interpolation formula by rational functions that contains some correction terms compared with the classical interpolation

$$\begin{aligned}
 I_{p,N}(f) &= \sum_{n=-N}^N \check{f}_n e^{i\pi n x} \\
 &- (e^{-i\pi Nx} - e^{i\pi(N+1)x}) \sum_{k=1}^p \frac{\theta_{-k} \Delta_{N,+}^{k-1} (\Delta_{n,-}^{k-1}(\check{f}_n))}{\prod_{s=1}^k (1 + \theta_{-s} e^{i\pi x})(1 + \theta_s e^{-i\pi x})} \\
 &- (e^{i\pi Nx} - e^{-i\pi(N+1)x}) \sum_{k=1}^p \frac{\theta_k \Delta_{-N,+}^{k-1} (\Delta_{n,-}^{k-1}(\check{f}_n))}{\prod_{s=1}^k (1 + \theta_{-s} e^{i\pi x})(1 + \theta_s e^{-i\pi x})}
 \end{aligned}$$

with the error

$$\begin{aligned}
 R_{p,N}(f) &= \frac{1}{\prod_{s=1}^p (1 + \theta_{-s} e^{i\pi x})(1 + \theta_s e^{-i\pi x})} \sum_{|n|=N+1}^{\infty} \Delta_{n,+}^p (\Delta_{n,-}^p(f_n)) e^{i\pi n x} \\
 &+ \frac{1}{\prod_{s=1}^p (1 + \theta_{-s} e^{i\pi x})(1 + \theta_s e^{-i\pi x})} \sum_{n=-N}^N \Delta_{n,+}^p (\Delta_{n,-}^p(f_n - \check{f}_n)) e^{i\pi n x}.
 \end{aligned}$$

Explicit form of the error prompts the way of calculation of the sequence  $\theta$ . Trying to minimize the error and following the idea of Fourier-Pade approximation we consider the following system for getting the values of  $\theta_k$

$$\Delta_{n,+}^p (\Delta_{n,-}^p(\check{f}_n)) = 0, n = -N, \dots, -N+p-1; n = N-p+1, \dots, N.$$

**3. Numerical Illustrations** Consider the following simple function

$$f(x) = (1 - x^2)^2 \sin(x - 0.5). \tag{4}$$

Figure 1 shows the plots of the absolute errors on the interval  $[-0.5, 0.5]$  away from the singularities of the function (4) while interpolating it by  $I_{p,N}(f)$  with different values of  $p$ . We see the accelerated convergence of the rational interpolation compared with the classical one.

Figures 2 and 3 compare classical interpolation and rational interpolation at the point  $x = 1$ .

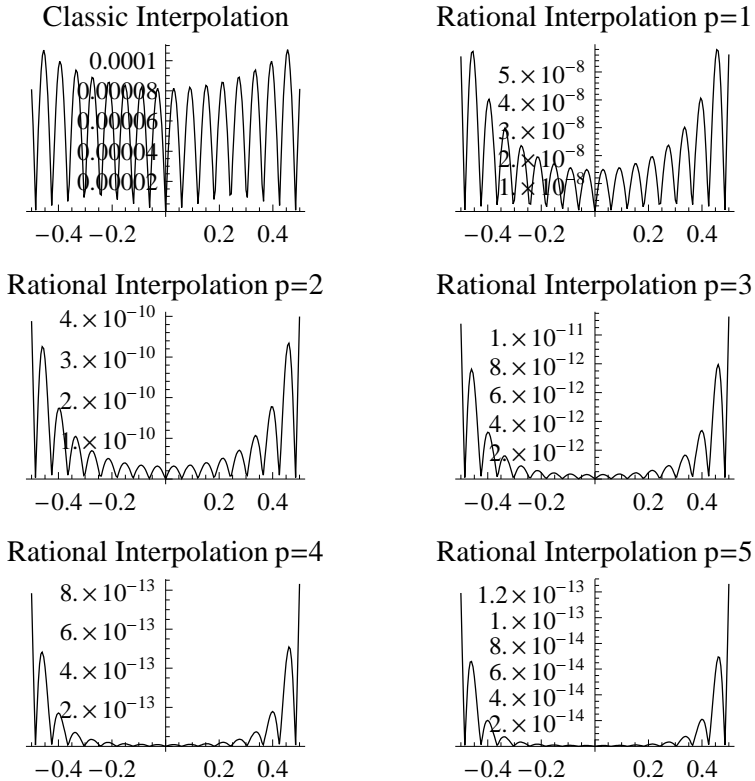


Figure 1: Plots of the absolute errors while interpolating (4) by classical interpolation and rational interpolations when  $p = 1, \dots, 5$  for  $N = 16$  in the interval  $[-0.5, 0.5]$ .

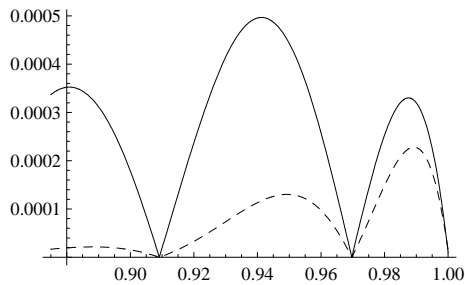


Figure 2: Plots of the absolute errors while interpolating (4) by the classical interpolation (solid line) and the rational interpolation for  $p = 1$  (dashed line) at the point  $x = 1$  when  $N = 16$ .

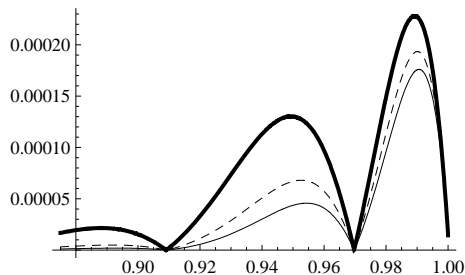


Figure 3: Plots of the absolute errors while interpolating (4) by the rational interpolation for  $p = 1$  (thick solid line),  $p = 2$  (dashed line) and  $p = 3$  (thin solid line) at the point  $x = 1$  when  $N = 16$ .

## References

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- [2] A. Nersessian and A. Poghosyan, *On a rational linear approximation of Fourier series for smooth functions*, Journal of Scientific Computing, **26**(1) (2006), 111–125.