On Good-λ Inequalities for Couples of Measurable Functions

GRIGORI A. KARAGULYAN

ABSTRACT. We give a domination condition implying good-λ and exponential inequalities for couples of measurable functions. Those inequalities recover several classical and new estimations involving some operators in Harmonic Analysis. Among other corollaries we prove a new exponential estimate for Carleson operators. The main results of the paper are considered in a general setting, namely, on abstract measure spaces equipped with a ball-basis.

1. INTRODUCTION

A classical problem in the theory of singular operators is the control of a given operator by a maximal-type operator. A typical result in this study is the well-known Coifman-Fefferman [5] estimate of a Calderón-Zygmund operator by the Hardy-Littlewood maximal function.

Theorem A (Coifman-Fefferman, [5]). *Let T be a Calderón-Zygmund operator on $\mathbb{R}^n$ and M be the maximal operator. Then, for any weight w satisfying the Muckenhoupt A_∞ condition, the inequality*

$$\|T^* f\|_{L^p(w)} \le c\|Mf\|_{L^p(w)} \tag{1.1}$$

holds, where $0 < p < \infty$, and $c > 0$ is a constant depending on n, p, and w.

The original proof of this inequality is based on a special technique developed in the papers of Burkholder-Gundy [2] and Coifman [4]. Specifically, (1.1) can be easily deduced from the inequality

$$w\{x \in \mathbb{R}^n : |T^* f| > 2\lambda,\ |Mf| < \gamma\lambda\} \le c\gamma^\delta w\{|T^* f| > \lambda\}, \quad \lambda > 0,$$

Indiana University Mathematics Journal ©, Vol. 70, No. 6 (2021)

where $\gamma > 0$ is a sufficiently small number, and c and δ are constants. This kind of bounds are known as good-λ inequalities, and play a significant role in the study of norm estimates of singular operators. Similar estimations of the Hardy-Littlewood maximal function by the sharp maximal function were proved by Fefferman and Stein in [7] (see also [18, Chapter 4]).

In the present paper we give a general approach to good-λ inequalities. We provide domination conditions, which imply good-λ and exponential inequalities for couples of measurable functions. We work in abstract measure spaces equipped with a ball-basis. The concept of ball-basis was introduced in [12].

Definition 1.1. Let $(X, \mathfrak{M}, \mu)$ be a measure space. A family of sets $\mathfrak{B} \subset \mathfrak{M}$ is said to be a ball-basis if it satisfies the following conditions:

(B1) $0 < \mu(B) < \infty$ for any ball $B \in \mathfrak{B}$.
(B2) For any points $x, y \in X$ there exists a ball $B \ni x, y$.
(B3) If $E \in \mathfrak{M}$, then for any $\varepsilon > 0$ there exists a finite or infinite sequence of balls B_k, $k = 1, 2, \dots$, such that $\mu(E \triangle \bigcup_k B_k) < \varepsilon$.
(B4) For any $B \in \mathfrak{B}$ there is a ball $B^* \in \mathfrak{B}$ (called *the hull* of B) satisfying the conditions

$$\bigcup_{A \in \mathfrak{B}: \mu(A) \le 2\mu(B),\, A \cap B \neq \emptyset} A \subset B^*,$$
$$\mu(B^*) \le \mathcal{K} \mu(B),$$

where $\mathcal{K}$ is a positive constant.

One can check that the Euclidean balls (or cubes) in $\mathbb{R}^n$ form a ball-basis. Moreover, it was proved in [12] that if the family of metric balls in spaces of homogeneous type satisfies the density condition, then it is a ball-basis too. Other examples of ball-basis are the family of dyadic cubes in $\mathbb{R}^n$ and its martingale extensions (see [12] for other details).

Let $(X, \mathfrak{M}, \mu)$ be a measure space with a ball-basis $\mathfrak{B}$. Given measurable function f and ball $B \in \mathfrak{B}$ we denote

$$\mathrm{OSC}_{B,\alpha}(f) = \inf_{E \subset B: \mu(E) \ge \alpha\mu(B)} \mathrm{OSC}_E(f),$$
$$\mathrm{INF}_{B,\alpha}(f) = \inf_{E \subset B: \mu(E) \ge \alpha\mu(B)} \|f\|_{L^\infty(E)},$$
$$\mathrm{INF}_B(f) = \operatorname*{ess\,inf}_{y \in B} |f(y)|,$$

where $0 < \alpha < 1$ and

$$\mathrm{OSC}_E(f) = \sup_{x, x' \in E} |f(x) - f(x')|.$$

Definition 1.2. Let f and g be measurable functions. The function f is said to be weakly dominated by g if for any $0 < \alpha < 1$ there exists a number $\beta = c(\alpha) > 0$ such that the inequality

$$\mathrm{OSC}_{B,\alpha}(f) < \beta \cdot \mathrm{INF}_{B,1-\alpha}(g), \tag{1.2}$$

holds for every ball $B \in \mathfrak{B}$. If we have

$$\mathrm{OSC}_{B,\alpha}(f) < \beta \cdot \mathrm{INF}_B(g) \tag{1.3}$$

instead of (1.2), then we say f is strongly dominated by g.

Clearly, relation (1.3) yields (1.2). We will see below that if the ball-basis $\mathfrak{B}$ is doubling, then condition (1.2) yields a good-λ inequality for couples of measurable functions f and g.

Definition 1.3. We say that a ball-basis $\mathfrak{B}$ in a measure space $(X, \mathfrak{M}, \mu)$ is doubling if there is a constant $\eta > 2$ such that for any ball $A \in \mathfrak{B}$, $\mu(A) < \mu(X)/2$, one can find a ball $B \supset A$ satisfying

$$2\mu(A) \le \mu(B) \le \eta \cdot \mu(A).$$

Recall the definition of Muckenhoupt's A_∞-condition in the setting of general ball-bases.

Definition 1.4. Let $(X, \mathfrak{M}, \mu)$ be a measure space equipped with a ball-basis $\mathfrak{B}$. We say a positive measure w defined on the σ-algebra $\mathfrak{M}$ satisfies A_∞-condition if there are constants $\delta, \gamma > 0$ such that

$$\frac{w(E)}{w(B)} \le \gamma \cdot \left(\frac{\mu(E)}{\mu(B)}\right)^\delta \tag{1.4}$$

for every choice of a ball $B \in \mathfrak{B}$ and a measurable set $E \subset B$.

In the sequel, constants depending only on parameters $\mathcal{K}$ and η (if the ball-basis is doubling) will be called admissible constants. The relation $a \lesssim b$ ($a \gtrsim b$) will stand for the inequality $a \le c \cdot b$ ($a \ge c \cdot b$), where $c > 0$ is an admissible constant. The following statement is one of the main results of the present paper.

Theorem 1.5. *Let $(X, \mathfrak{M}, \mu)$ be a measure space with a doubling ball-basis $\mathfrak{B}$ such that $\mu(X) = \infty$, and let w be an A_∞ measure. If $0 < \alpha < 1$, $\beta > 0$, and measurable functions f, g satisfy (1.2), then we have the inequality*

$$\begin{aligned} \mu\left\{x \in X : |f(x)| > 2\lambda,\ |g(x)| \le \frac{\lambda}{\beta}\right\} &\lesssim \\ \lesssim \gamma(1-\alpha)^\delta \mu\{x \in X : |f(x)| > \lambda\}&, \quad \lambda > 0, \end{aligned} \tag{1.5}$$

where γ and δ are the constants from (1.4).

Applying a standard argument, well-known in classical situation, one can deduce from (1.5) the following result.

Corollary 1.6. *If a function f is weakly dominated by g, then for any measure w satisfying (1.4) we have the inequality*

$$\|f\|_{L^p(w)} \le c(p, \gamma, \delta)\|g\|_{L^p(w)}, \quad 0 < p < \infty,$$

where $c(p,\gamma,\delta) > 0$ *is a constant depending on* p *and the parameters* γ,δ *from* (1.4).

The functional $\mathrm{OSC}_{B,\alpha}(f)$ based on the classical Euclidean ball-basis in $\mathbb{R}^n$ was used in the definition of the local sharp maximal function given by Jawerth and Torchinsky in [11]. The original definition of this functional is slightly different, but it is equivalent to the above definition. Recall the definition of median from [11]. A median $m_f(B)$ of a measurable function f over a ball B is a real number (possibly not unique) satisfying

$$\begin{cases} \mu\{x \in B : f(x) > m_f(B)\} \le \dfrac{\mu(B)}{2}, \\ \mu\{x \in B : f(x) < m_f(B)\} \le \dfrac{\mu(B)}{2}. \end{cases} \tag{1.6}$$

Under the strong domination condition, in addition to (1.5) we also prove the following exponential estimate.

Theorem 1.7. *If the ball-basis* $\mathfrak{B}$ *in a measure space is doubling and measurable functions* f *and* g *satisfy strong domination condition* (1.3)*, then for any ball* $B \in \mathfrak{B}$ *we have*

$$\mu\{x \in B : |f(x) - m_f(B)| > \lambda|g(x)|\} \lesssim \exp(-c \cdot \lambda)\mu(B), \quad \lambda > 0, \tag{1.7}$$

where $c > 0$ *is an admissible constant.*

The inequality (1.7) in $\mathbb{R}^n$ can be deduced from a sparse domination theorem due to Lerner [15]. A basic idea applied in [15] (dyadic partition of cube) is not applicable in the case of general ball-basis. Our proof of Theorem 1.7 uses the technique of an exponential estimate for the Calderón-Zygmund operators proved in [13]. A bunch of estimates of exponential type, involving different operators of harmonic analysis, was proved by Ortiz-Caraballo, Pérez, and Rela [16]. However, the paper [16] still makes use of the dyadic partition technique along with the sparse domination theorem of Lerner [15]. In this context, one can also consider the recent paper of Canto and Pérez [6], where authors give two interesting extensions of the John-Nirenberg theorem in a weighted setting.

Inequalities (1.5) and (1.7) have a number of interesting applications in singular operators. Let U and V be operators on $L^r(X)$. We will say that the operator U is (strongly) dominated by V if Uf is (strongly) dominated by Vf for every $f \in L^r$. In Sections 4 and 5 we will discuss different examples of operators U and V satisfying the strong domination property. In view of Theorems 1.5 and 1.7, we will derive good-λ and exponential inequalities for those couples of operators. Among other corollaries we prove a new exponential estimate for Carleson operators.

2. Some Properties of Ball-bases

We will often use property (B4) of a ball-basis as follows. If for two balls $A, B \in \mathfrak{B}$ we have $A \cap B \neq \emptyset$ and $\mu(A) \le 2\mu(B)$, then $A \subset B^*$. The following Besicovitch-type covering lemma was proved in [12].

Lemma 2.1 *(*[12, Lemma 3.1]*). Let $(X, \mathfrak{M}, \mu)$ be a measure space with an arbitrary ball-basis $\mathfrak{B}$. If $E \subset X$ is a bounded measurable set (i.e., $E \subset B$ for some ball B) and $\mathcal{G}$ is a family of balls so that $E \subset \bigcup_{G \in \mathcal{G}} G$, then there exists a finite or infinite sequence of pairwise disjoint balls $G_k \in \mathcal{G}$ such that $E \subset \bigcup_k G_k^*$.*

Definition 2.2. For a measurable set $E \subset X$ a point $x \in E$ is said to be a density point if for any $0 < \gamma < 1$ there exists a ball $B \ni x$ such that $\mu(B \cap E) > \gamma\mu(B)$.

Lemma 2.3 *(*[12, Lemma 3.4]*). Almost all points of a measurable set $E \subset X$ are density points.*

Lemma 2.4. *Let $(X, \mathfrak{M}, \mu)$ be a measure space equipped with a ball-basis. Then, there exists a sequence of balls $G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots$ such that $X = \bigcup_k G_k$.*

Proof. Fix a point $x_0 \in X$ and let $\mathcal{A}$ be the family of balls containing x_0. Take a sequence $\eta_n \nearrow \eta = \sup_{A \in \mathcal{A}} \mu(A)$, where η can also be infinity. Let us see by induction that there is an increasing sequence of balls $A_n \in \mathcal{A}$ such that $\mu(A_n) > \eta_n$. The base of induction is obvious. Suppose we have already chosen the first elements A_k, $k = 1, 2, \dots, \ell$. There is a ball $B \in \mathcal{A}$ so that $\mu(B) > \eta_{\ell+1}$. Let C be the biggest among two balls B and A_ℓ and define $A_{\ell+1} = C^*$. According to property (B4) we have $B \cup A_\ell \subset C^* = A_{\ell+1}$, which implies $\mu(A_{\ell+1}) \ge \mu(B) > \eta_{\ell+1}$ and $A_{\ell+1} \supset A_\ell$. Once we have determined A_n, we can take $G_n = A_n^*$ as a desired sequence of balls. Indeed, let $x \in X$ be arbitrary. By the (B2) property there is a ball B containing both x_0 and x. In addition, for some n we have $\mu(B) \le 2\mu(A_n)$, and so by property (B4), $x \in B \subset A_n^* = G_n$. ◻

Lemma 2.5. *Let $(X, \mathfrak{M}, \mu)$ be a measure space equipped with a ball-basis $\mathfrak{B}$. If $\mu(X) < \infty$, then $X \in \mathfrak{B}$.*

Proof. Applying Lemma 2.4, one can find a ball B such that $\mu(B) > \mu(X)/2$. Consider the family of balls $\mathcal{A} = \{A \in \mathfrak{B} : A \cap B \neq \emptyset\}$. Focusing on (B2) and (B4), one can see that $X = \bigcup_{A \in \mathcal{A}} A \subset B^*$. Thus, we get $X = B^*$. ◻

Lemma 2.6. *Let $\mathfrak{B}$ be a doubling ball basis in $(X, \mathfrak{M}, \mu)$. If $\mu(F) < \mu(X)/4$, then for any density point $x \in F$ there exists a ball $B \ni x$ such that*

$$(2\eta\mathcal{K})^{-1}\mu(B^*) \le \mu(B^* \cap F) \le \frac{\mu(B^*)}{2}, \tag{2.1}$$

$$(2\eta)^{-1}\mu(B) \le \mu(B \cap F) \le \frac{\mu(B)}{2}. \tag{2.2}$$

Proof. Suppose we are given a measurable set F and a density point $x \in F$. Consider the family of balls

$$\mathcal{A} = \left\{ A \in \mathfrak{B} : x \in A,\ \mu(A \cap F) \geq \frac{\mu(A)}{2} \right\}.$$

Since x is a density point, $\mathcal{A}$ is nonempty. Besides, we have

$$r = \sup_{A \in \mathcal{A}} \mu(A) \leq 2\mu(F) < \frac{\mu(X)}{2}.$$

Choose an arbitrary $A_0 \in \mathcal{A}$ such that $\mu(A_0) > r/2$. According to the doubling property there is a ball $B \supset A_0$ such that $2\mu(A_0) \leq \mu(B) \leq \eta\mu(A_0)$. Since we get $\mu(B) > r$, neither B nor B^* are in $\mathcal{A}$, so the righthand sides of inequalities (2.1) and (2.2) hold. On the other hand, we have

$$\mu(B^* \cap F) \geq \mu(A_0 \cap F) \geq \frac{\mu(A_0)}{2} \geq \frac{\mu(B)}{2\eta} \geq (2\eta\mathcal{K})^{-1}\mu(B^*).$$

Similarly, one can also show the lefthand inequality in (2.2), so we are done. □

We say a ball B is well balanced with respect to a measurable set F if they satisfy (2.1) and (2.2). In the sequel the notation $A \subset B$ almost everywhere for two measurable sets $A, B \subset X$ will stand for the relation $\mu(B \setminus A) = 0$. The following balanced covering lemma is an extension of Lemma 2 from [13] to the abstract setting.

Lemma 2.7. *Let $\mathfrak{B}$ be a doubling ball-basis in a measure space $(X, \mathfrak{M}, \mu)$. If $\mu(F) < \mu(X)/4$ and a measurable set $F' \subset F$ is bounded, then there exists a sequence of balls B_k such that*

$$F' \subset \bigcup_k B_k \text{ almost everywhere}, \quad F' \cap B_k \neq \emptyset, \tag{2.3}$$

$$\sum_k \mu(B_k) \leq 2\eta\mathcal{K}\mu(F), \tag{2.4}$$

$$\mu(B_k \cap F) \leq \frac{\mu(B_k)}{2}. \tag{2.5}$$

Proof. Let $D \subset F$ be the density points set of F. According to Lemma 2.6, for any $x \in D$ there is a ball $G_x \ni x$, which is well balanced with respect to F. So from the right side of inequality (2.2) we obtain

$$\mu(G_x^*) \leq \mathcal{K}\mu(G_x) \leq 2\eta\mathcal{K}\mu(G_x \cap F). \tag{2.6}$$

Applying Lemma 2.1 to the set $D \cap F'$ and its covering $\mathcal{G} = \{G_x : x \in D \cap F'\}$, we find a sequence of pairwise disjoint balls G_k such that $D \cap F' \subset \bigcup_k G_k^*$. By

Lemma 2.3 we have $\mu(F \setminus D) = 0$, and so the sequence $B_k = G_k^*$ satisfies (2.3). Inequality (2.5) follows from the first balance condition (2.1). Finally, using (2.6), the second balance condition ((2.2)) for G_k and the disjointedness of the balls G_k, we get

$$\sum_k \mu(B_k) = \sum_k \mu(G_k^*) \le \mathcal{K} \sum_k \mu(G_k) \le 2\eta\mathcal{K} \sum_k \mu(G_k \cap F) \le 2\eta\mathcal{K}\mu(F),$$

which gives (2.4). ◻

3. Proofs of the Main Results

Proof of Theorem 1.5. We can suppose that $\frac{1}{2} < \alpha < 1$, since for the smaller numbers $0 < \alpha \le \frac{1}{2}$ inequality (1.5) trivially holds with a constant 2 on the right. Denote

$$F_\lambda = \{x \in X : |f(x)| > \lambda\}, \quad \lambda > 0. \tag{3.1}$$

We can suppose that $\mu(F_\lambda) < \infty$, since otherwise (1.5) is trivial. Thus, we have $\mu(F_\lambda) < \mu(X)/4 = \infty$. Let G be an arbitrary ball. Apply Lemma 2.7 with $F = F_\lambda$ and $F' = G \cap F_\lambda$. We find a sequence of balls B_k satisfying conditions (2.3), (2.4), and (2.5). We claim that

$$\mu\left\{x \in B_k : |f(x)| > 2\lambda,\ |g(x)| \le \frac{\lambda}{\beta}\right\} \le (1-\alpha)\mu(B_k) \tag{3.2}$$

for any $k = 1, 2, \ldots$. We can only focus on the balls B_k satisfying

$$\mu\left\{x \in B_k : |g(x)| \le \frac{\lambda}{\beta}\right\} \ge (1-\alpha)\mu(B_k), \tag{3.3}$$

since otherwise inequality (3.2) is obvious. Applying (1.2) and (3.3), one can find a set $E_k \subset B_k$ so that

$$\mu(E_k) \ge \alpha\mu(B_k) > \frac{\mu(B_k)}{2}, \tag{3.4}$$

$$\begin{aligned} \mathrm{OSC}_{E_k}(f) < \beta \cdot \mathrm{INF}_{B_k,1-\alpha}(f) &= \beta \inf_{E \subset B_k : \mu(E) \ge (1-\alpha)\mu(B_k)} \|g\|_{L^\infty(E)} \\ &\le \beta \sup_{x \in B_k : |g(x)| \le \lambda/\beta} |g(x)| \le \lambda. \end{aligned} \tag{3.5}$$

From (2.5) it follows that $\mu(B_k \setminus F_\lambda) \ge \mu(B_k)/2$. Combining it with (3.4), we obtain $E_k \setminus F_\lambda \ne \emptyset$, so there is a point $x_k \in E_k \setminus F_\lambda$. From (3.1) and (3.5) we conclude

$$|f(x_k)| \le \lambda, \quad |f(x) - f(x_k)| \le \mathrm{OSC}_{E_k}(f) < \lambda, \quad x \in E_k.$$

This implies $|f(x)| \le 2\lambda$ for all $x \in E_k$ and, once again using (3.4), we obtain

$$\mu\left\{x \in B_k : |f(x)| > 2\lambda, \quad |g(x)| \le \frac{\lambda}{\beta}\right\}$$
$$\le \mu(B_k \setminus E_k) \le (1-\alpha)\mu(B_k).$$

Once the validity of (3.2) is established, from the A_∞ condition of w we immediately get

$$w\left\{x \in B_k : |f(x)| > 2\lambda, \ |g(x)| \le \frac{\lambda}{\beta}\right\} \le \gamma \cdot (1-\alpha)^\delta w(B_k)$$

then, using also (2.3), (2.4), we obtain the inequality

$$\begin{aligned} &w\left\{x \in G : |f(x)| > 2\lambda, \ |g(x)| \le \frac{\lambda}{\beta}\right\} \\ &\quad \le \sum_k w\left\{x \in B_k : |f(x)| > 2\lambda, \ |g(x)| \le \frac{\lambda}{\beta}\right\} \\ &\quad \le \gamma(1-\alpha)^\delta w(B_k) \lesssim \gamma(1-\alpha)^\delta w(F_\lambda), \end{aligned}$$

which holds for any ball G. Choosing G to be one of the balls G_n from Lemma 2.7, and letting n to go to infinity, we will get (1.5). ◻

To prove Theorem 1.7 we need the following simple lemma.

Lemma 3.1. *Let B be a ball and let a measurable set $E \subset B$ satisfy $\mu(E) > \mu(B)/2$. Then, for any measurable function f on B we have* $\mathrm{INF}_E(f) \le m_f(B) \le \mathrm{SUP}_E(f)$.

Proof. Suppose to the contrary we have $m_f(B) < \mathrm{INF}_E(f)$. Then, by the definition of $m_f(B)$ (see (1.6)) we get

$$\begin{aligned} \mu(E) &\le \mu\{x \in B : \mathrm{INF}_E(f) \le f(x) \le \mathrm{SUP}_E(f)\} \\ &\le \mu\{x \in B : f(x) \ge m_f(B)\} \le \frac{\mu(B)}{2}, \end{aligned}$$

which is a contradiction. The case of $m_f(B) > \mathrm{SUP}_E(f)$ may be excluded similarly. ◻

Proof of Theorem 1.7. Given a ball A and a number $\frac{3}{4} < \alpha < 1$ we describe the following method.

Procedure. *We first fix a "good" set $E_A \subset A^*$ such that*

$$\mu(E_A) \ge \alpha\mu(A^*), \quad \mathrm{OSC}_{E_A}(f) \le 2\,\mathrm{OSC}_{A^*,\alpha}(f). \tag{3.6}$$

For the "bad" set $F = A^* \setminus E_A$ we have $\mu(F) < \mu(X)/4$. Thus, applying Lemma 2.7 to F and its subset $F' = A \setminus E_A$, we find a countable family of balls $\mathrm{ch}(A)$ *(children of A) such that*

$$A \setminus E_A \subset \bigcup_{G \in \mathrm{ch}(A)} G \text{ almost everywhere}, \quad A \cap G \neq \emptyset, \; G \in \mathrm{ch}(G), \tag{3.7}$$

$$\sum_{G \in \mathrm{ch}(A)} \mu(G) \leq 2\eta\mathcal{K}\mu(A^* \setminus E_A) \leq 2\eta\mathcal{K}(1-\alpha)\mu(A^*), \tag{3.8}$$

$$\mu(G \cap (A^* \setminus E_A)) \leq \frac{\mu(G)}{2}, \quad G \in \mathrm{ch}(A). \tag{3.9}$$

We first apply the procedure to the original ball B. We get E_B and a child balls collection $\mathfrak{U}_1$. Then, we do the same with each ball $A \in \mathfrak{U}_1$, getting the second generation of B denoted by $\mathfrak{U}_2$. Continuing this procedure to infinity we will get a ball family $\mathfrak{U}_k$ (kth generations of B) such that, for any ball $A \in \mathfrak{U} = \bigcup_{k \geq 0} \mathfrak{U}_k$, one has an attached set $E_A \subset A^*$, satisfying the relations (3.6)–(3.9) (where $\mathfrak{U}_0 = \{B\}$). For an admissible α closer to 1 the collection $\mathfrak{U}$ has two crucial properties. First,

$$\sum_{G \in \mathrm{ch}(A)} \mu(G) \leq \frac{\mu(A)}{4\mathcal{K}}, \quad A \in \mathfrak{U}, \tag{3.10}$$

which immediately follows from (3.8). Second,

$$E_A \cap E_G \neq \emptyset, \quad A \in \mathfrak{U}, \; G \in \mathrm{ch}(A). \tag{3.11}$$

To show (3.11), observe that (3.10) implies $\mu(G) \leq \mu(A)$, and so by (3.7) we have $G \subset A^*$. Hence, inequality (3.9) can be written in the form

$$\mu(G \cap E_A) \geq \frac{\mu(G)}{2}. \tag{3.12}$$

Thus, using (3.6) and (3.12), we get

$$\begin{aligned}
\mu(E_A \cap E_G) &\geq \mu((E_A \cap G) \cap (E_G \cap G)) \\
&= \mu(E_A \cap G) + \mu(E_G \cap G) - \mu((E_A \cap G) \cup (E_G \cap G)) \\
&\geq \frac{\mu(G)}{2} + \mu(G) - \mu(G^* \setminus E_G) - \mu(G) \\
&\geq \frac{\mu(G)}{2} - (1-\alpha)\mu(G^*) \\
&\geq \mu(G)\left(\frac{1}{2} - \mathcal{K}(1-\alpha)\right) > 0,
\end{aligned}$$

and so (3.11) follows. Denote

$$\Delta_k = \bigcup_{G \in \bigcup_{j \geq k} \mathfrak{U}_j} G, \quad k = 0, 1, \ldots .$$

Observe that $\{\Delta_k\}$ forms a decreasing sequence of measurable sets. Moreover, from (3.10) and the structure of $\mathfrak{U}$, it follows that

$$\mu(\Delta_k) \lesssim 4^{-k} \cdot \mu(B), \quad k = 1, 2, \dots, \; B \subset \bigcup_{k\geq 0} \Delta_k \text{ almost everywhere.} \tag{3.13}$$

Thus, for almost all $x \in B$ we have $x \in \Delta_{n-1} \setminus \Delta_n$ for some $n \geq 1$, so one can find a chain of balls $B_0 = B, B_1, \dots, B_{n-1}$ such that $B_j \in \text{ch}(B_{j-1})$ and $x \in E_{B_n}$. According to (3.11) there are $\xi_j \in E_{B_{j-1}} \cap E_{B_j}$, $j = 1, 2, \dots, n-1$. Set also $\xi_n = x$. Since $\xi_j, \xi_{j+1} \in E_{B_j}$, we have

$$|f(\xi_j) - f(\xi_{j+1}) \leq 2\,\text{OSC}_{B_j^*,\alpha}(f), \quad j = 1, 2, \dots, n-1. \tag{3.14}$$

In addition, we have $\mu(E_{B_0}) \geq \alpha\mu(B_0) \geq \mu(B)/2$ and $\xi_1 \in E_{B_0}$, and so by Lemma 3.1 we get

$$|f(\xi_1) - m_f(B_0)| \leq \text{OSC}_{E_0}(f) \leq 2\,\text{OSC}_{B_0^*,\alpha}(f). \tag{3.15}$$

Observe that $B_{k+1}^* \subset B_k^*$, since according to (3.10) we have

$$\mu(B_{k+1}^*) \leq \mathcal{K}\mu(B_{k+1}) \leq \frac{\mu(B_k)}{4} \leq \mu(B_k).$$

Hence, applying (1.3), (3.14), and (3.15), we obtain

$$\begin{aligned} |f(x) - m_f(B)| &= |f(\xi_n) - m_f(B_0)| \\ &= |f(\xi_1) - m_f(B_0)| + \sum_{j=1}^{n-1} |f(\xi_j) - f(\xi_{j+1})| \\ &\leq 2\sum_{j=0}^{n-1} \text{OSC}_{B_j^*,\alpha}(f) \leq 2n\beta(\alpha) \cdot |g(x)|. \end{aligned}$$

Finally, using (3.13), we get

$$\mu\{x \in B : |f(x) - m_f(B)| > 2n\beta(\alpha)|g(x)|\} \leq \mu(\Delta_n) \lesssim 4^{-n}\mu(B),$$

which completes the proof of the theorem. ∎

4. Estimates of Sharp Maximal Operators

Let $1 \leq r < \infty$ be fixed. For any function $f \in L^r(X)$ and a ball $B \in \mathfrak{B}$ we set

$$\langle f \rangle_B = \left(\frac{1}{\mu(B)} \int_B |f|^r\right)^{1/r}, \qquad \langle f \rangle_B^* = \sup_{A \in \mathfrak{B}: A \supseteq B} \langle f \rangle_A.$$

We will consider also the #-analogues of this quantities defined by

$$\langle f\rangle_{\#,B} = \left(\frac{1}{\mu(B)}\int_B |f - f_B|^r\right)^{1/r}, \quad \langle f\rangle^*_{\#,B} = \sup_{A\in\mathfrak{B}:A\supseteq B} \langle f\rangle_{\#,A}, \tag{4.1}$$

where $f_B = (1/\mu(B))\int_B f$. Recall the definitions of maximal and (#)-maximal functions

$$\mathcal{M}f(x) = \sup_{B\in\mathfrak{B}:B\ni x} \langle f\rangle_B, \quad \mathcal{M}_\# f(x) = \sup_{B\in\mathfrak{B}:B\ni x} \langle f\rangle_{\#,B}. \tag{4.2}$$

Observe the following standard properties of quantities (4.1). If $f \in L^r(X)$ and B is an arbitrary ball, then

$$\langle f\rangle_{\#,B} \le \langle f - c\rangle_B + |f_B - c| \le 2\langle f - c\rangle_B, \quad c \in \mathbb{R}, \tag{4.3}$$

$$\langle f\rangle_{\#,B} \le 2\langle f - f_{B^*}\rangle_B \le 2\left(\frac{1}{\mu(B)}\int_{B^*} |f - f_{B^*}|^r\right)^{1/r} \lesssim \langle f\rangle_{\#,B^*}, \tag{4.4}$$

$$|f_B - f_{B^*}| \le \langle f - f_{B^*}\rangle_B \lesssim \langle f\rangle_{\#,B^*}. \tag{4.5}$$

One can also check that $\mathcal{M}_\# f(x) \le 2\mathcal{M}f(x)$. The following theorem shows that this bound is somewhat convertible.

Theorem 4.1. *If $(X, \mathfrak{M}, \mu)$ is a measure space with an arbitrary ball-basis $\mathfrak{B}$, then for any $1 \le r < \infty$ the maximal operator $\mathcal{M}$ is strongly dominated by the operator $\mathcal{M}_\#$. Moreover, we have a bound*

$$\mathrm{OSC}_{B,\alpha}(\mathcal{M}f) \lesssim (1-\alpha)^{-1/r}\langle f\rangle^*_{\#,B}, \quad B \in \mathfrak{B}, \tag{4.6}$$

valid for any $0 < \alpha < 1$.

The following proposition shows that on the right side of (4.6) we can equivalently use the quantity $\mathrm{INF}_B(\mathcal{M}_\#(f))$.

Proposition 4.2. *Let $\mathfrak{B}$ be a ball-basis in a measure space $(X, \mathfrak{M}, \mu)$. For any ball $B \in \mathfrak{B}$ and a function $f \in L^r(X)$, it holds that*

$$\langle f\rangle^*_{\#,B} \le \mathrm{INF}_B(\mathcal{M}_\#(f)) \lesssim \langle f\rangle^*_{\#,B}. \tag{4.7}$$

Proof. The proof of the lefthand side of the inequality is straightforward. Let us prove the righthand side. For any $x \in B$ there exists a ball $B(x) \ni x$ such that

$$\langle f\rangle_{\#,B(x)} > \frac{\mathrm{INF}_B(\mathcal{M}_\#(f))}{2} = \lambda. \tag{4.8}$$

Applying Lemma 2.1, we find a sequence of pairwise disjoint balls $\{B_k\} \subset \{B(x) : x \in B\}$ such that $\bigcup_k B_k^* \supset B$. If some B_k satisfies $\mu(B_k) > \mu(B)$, then we have $B \subset B_k^*$ and, using (4.4), we get

$$\langle f \rangle^*_{\#,B} \geq \langle f \rangle_{\#,B_k^*} \gtrsim \langle f \rangle_{\#,B_k} > \frac{\lambda}{2}.$$

If $\mu(B_k) \leq \mu(B)$ for every k, then $\bigcup_k B_k \subset B^*$. Therefore by (4.3), (4.8), and the pairwise disjointness of B_k, we obtain

$$\begin{aligned}
\langle f \rangle^*_{\#,B} \geq \langle f \rangle_{\#,B^*} &\geq \left(\frac{1}{\mu(B^*)} \sum_k \int_{B_k} |f - f_{B^*}|^r\right)^{1/r} \\
&\geq \frac{1}{2}\left(\frac{1}{\mu(B^*)} \sum_k \int_{B_k} |f - f_{B_k}|^r\right)^{1/r} \\
&= \frac{1}{2}\left(\frac{1}{\mu(B^*)} \sum_k \mu(B_k)(\langle f \rangle_{\#,B_k})^r\right)^{1/r} \\
&\geq \frac{\lambda}{2}\left(\frac{1}{\mu(B^*)} \sum_k \mu(B_k)\right)^{1/r} \\
&\gtrsim \lambda\left(\frac{1}{\mu(B^*)} \sum_k \mu(B_k^*)\right)^{1/r} \geq \lambda.
\end{aligned}$$

□

Proof of Theorem 4.1. Let $f \in L^r(X)$ be a nontrivial function and B be an arbitrary ball. Set $g = (f - f_B) \cdot \mathbb{I}_{B^*}$ and $E_{B,\lambda} = \{y \in B : \mathcal{M}g(y) \leq \lambda\}$. According to the weak-$L^r$ bound of the maximal function $\mathcal{M}$ (see [12]), we have

$$\mu(B \setminus E_{B,\lambda}) = \mu\{y \in B : \mathcal{M}g(y) > \lambda\} \lesssim \frac{1}{\lambda^r} \cdot \int_{B^*} |g|^r.$$

Thus, for an appropriate number $\lambda \sim (1-\alpha)^{-1/r}\langle g \rangle_{B^*}$, we have $\mu(B \setminus E_{B,\lambda}) < (1-\alpha)\mu(B)$, and therefore, $\mu(E_{B,\lambda}) > \alpha\mu(B)$. Hence, applying (4.5), for the set $E = E_{B,\lambda} \subset B$ we get the relations

$$\mu(E) > \alpha\mu(B), \tag{4.9}$$

$$\begin{aligned}
\mathcal{M}g(y) &\lesssim (1-\alpha)^{-1/r}\langle g \rangle_{B^*} = (1-\alpha)^{-1/r}\langle f - f_B \rangle_{B^*} \\
&\leq (1-\alpha)^{-1/r}(\langle f \rangle_{\#,B^*} + |f_B - f_{B^*}|) \\
&\lesssim (1-\alpha)^{-1/r}\langle f \rangle^*_{\#,B}, \quad y \in E.
\end{aligned} \tag{4.10}$$

Take arbitrary points $x, x' \in E$. Without loss of generality we can suppose that $\mathcal{M}f(x) \geq \mathcal{M}f(x')$. For any $\delta > 0$ there is a ball $A \ni x$ such that

$$\mathcal{M}f(x) \leq \langle f \rangle_A + \delta.$$

If $\mu(A) > \mu(B)$, then $x' \in B \subset A^*$, and we have

$$\begin{aligned}(4.11)\quad \mathcal{M}f(x) - \mathcal{M}f(x') &\le \langle f\rangle_A - \langle f\rangle_{A^*} + \delta \\ &\le \langle f - f_{A^*}\rangle_A + |f_{A^*}| + \langle f - f_{A^*}\rangle_{A^*} - |f_{A^*}| + \delta \\ &\lesssim \langle f - f_{A^*}\rangle_{A^*} + \langle f - f_{A^*}\rangle_{A^*} + \delta \\ &\lesssim \langle f\rangle^*_{\#,B} + \delta.\end{aligned}$$

If $\mu(A) \le \mu(B)$, then $A \subset B^*$. Thus, using (4.10), we obtain

$$\begin{aligned}(4.12)\quad \mathcal{M}f(x) - \mathcal{M}f(x') &\le \langle f\rangle_A - \langle f\rangle_B + \delta \\ &\le \langle f - f_B\rangle_A + |f_B| + \langle f - f_B\rangle_B - |f_B| + \delta \\ &= \langle g\rangle_A + \langle f - f_B\rangle_B + \delta \\ &\le \mathcal{M}g(x) + \langle f\rangle^*_{\#,B} + \delta \\ &\lesssim (1-\alpha)^{-1/r}\langle f\rangle^*_{\#,B} + \langle f\rangle^*_{\#,B} + \delta \\ &\lesssim (1-\alpha)^{-1/r}\langle f\rangle^*_{\#,B} + \delta.\end{aligned}$$

Since δ can be arbitrary small, from (4.11) and (4.12) we conclude

$$|\mathcal{M}f(x) - \mathcal{M}f(x')| \lesssim (1-\alpha)^{-1/r}\langle f\rangle^*_{\#,B}, \quad x, x' \in E.$$

This implies

$$(4.13)\qquad \mathrm{OSC}_E(Mf) \lesssim (1-\alpha)^{-1/r}\langle f\rangle^*_{\#,B}.$$

Combining (4.9) and (4.13) we deduce (4.6), so the theorem is proved. □

Corollary 4.3. *Let $(X, \mathfrak{M}, \mu)$ be a measure space with a doubling ball-basis $\mathfrak{B}$ and $\mu(X) = \infty$. Then, for any functions $f \in L^r(X)$, $1 \le r < \infty$, and $\varepsilon > 0$ we have*

$$\begin{aligned}(4.14)\qquad \mu\{x \in X : \mathcal{M}f(x) > 2\lambda,\ \mathcal{M}_\# f(x) \le \varepsilon\lambda\} &\lesssim \\ \lesssim \varepsilon^r \cdot \mu\{x \in X : \mathcal{M}f(x) > \lambda\}, &\quad \lambda > 0.\end{aligned}$$

Proof. From (4.6) and (4.7) it follows that

$$\begin{aligned}\mathrm{OSC}_{B,\alpha}(\mathcal{M}f) &\lesssim (1-\alpha)^{-1/r} \cdot \mathrm{INF}_B(\mathcal{M}_\#(f)) \\ &\le (1-\alpha)^{-1/r} \cdot \mathrm{INF}_{B,1-\alpha}(\mathcal{M}_\#(f)),\end{aligned}$$

and so we can apply Theorem 1.5 with $\beta \sim (1-\alpha)^{-1/r}$. Then, the notation $\varepsilon = 1/\beta$ will give us the inequality (4.14). □

Combining Theorem 1.7 and Theorem 4.1, we can prove the following result.

Corollary 4.4. *Let $(X,\mathfrak{M},\mu)$ be a measure space with a doubling ball-basis. For any $f\in L^r(X)$ and a ball B, it holds that*

$$\mu\{x\in B:|\mathcal{M}f(x)-c_{B,f}|>t|\mathcal{M}_{\#}f(x)|\}\lesssim\exp(-c\cdot t)\cdot\mu(B),\quad t>0, \tag{4.15}$$

where $c_{B,f}$ is a median of function $\mathcal{M}f$ over B.

Along with operators (4.2) we will consider another maximal operator that was introduced by Jawerth and Torchinsky [11]. That is the local maximal sharp function operator

$$\mathcal{M}_{\#,\alpha}f(x)=\sup_{B\in\mathfrak{B}:B\ni x}\mathrm{OSC}_{B,\alpha}(f),\quad 0<\alpha<1.$$

The obvious inequality

$$\mathrm{OSC}_{B,\alpha}(f)\le\mathrm{INF}_B(\mathcal{M}_{\#,\alpha}(f))$$

yields a strong domination of any function $f\in L^r(X)$ by $\mathcal{M}_{\#,\alpha}(f)$. Thus, applying Theorem 1.7, we immediately get the following exponential estimate, which is an extension of John-Nirenberg's inequality.

Corollary 4.5. *Let $(X,\mathfrak{M},\mu)$ be a measure space with a doubling ball-basis. For any $f\in L^r(X)$ and a ball B, it holds that*

$$\begin{aligned}\mu\{x\in B:|f(x)-m_f(B)|>t\cdot\mathcal{M}_{\#,\alpha}f(x)\}\\ \lesssim\exp(-c\cdot t)\cdot\mu(B),\quad t>0.\end{aligned}$$

This inequality is the extension of analogous inequalities of papers [16], [13] to general ball-bases. Specifically, Ortiz-Caraballo, Pérez, and Rela [16] proved the same inequality (4.16) in $\mathbb{R}^n$ equipped with Euclidean balls. Observe that

$$\begin{aligned}\alpha\cdot\mathcal{M}_{\#,\alpha}f(x)\le\mathcal{M}_{\#}f(x)\le 2\mathcal{M}f(x),\\ |f(x)|\le\mathcal{M}f(x)\quad\text{almost everywhere,}\end{aligned}$$

where the last inequality follows from the density property. Focusing on these bounds, one can see a difference between inequalities (4.15) and (4.16).

5. BOUNDED OSCILLATION OPERATORS

Let $1\le r<\infty$, $(X,\mathfrak{M},\mu)$ be a measure space and $L^0(X)$ be the linear space of real functions on X. An operator $T:L^r(X)\to L^0(X)$ is said to be subadditive if

$$\begin{aligned}|T(\lambda\cdot f)(x)|=|\lambda|\cdot|Tf(x)|,\quad\lambda\in\mathbb{R},\\ |T(f+g)(x)|\le|Tf(x)|+|Tg(x)|.\end{aligned}$$

Recall the definition of bounded oscillation (BO) operators from [12].

Definition 5.1. Let $(X, \mathfrak{M}, \mu)$ be a measure space with a doubling ball-basis $\mathfrak{B}$. We say that a subadditive operator $T : L^r(X) \to L^0(X)$ is a bounded oscillation operator with respect to $\mathfrak{B}$ if we have the bound

$$\sup_{f \in L^r(X),\, B \in \mathfrak{B}} \frac{\mathrm{OSC}_B(T(f \cdot \mathbb{I}_{X \setminus B^*}))}{\langle f \rangle_B^*} = \mathcal{L}(T) < \infty, \tag{5.1}$$

called the *localization* property. The family of all bounded oscillation operators with respect to a ball-basis $\mathfrak{B}$ will be denoted by $\mathrm{BO}_{\mathfrak{B}}$ or simply BO.

In fact, the paper [12] gives the definition of BO operators in the setting of general ball-bases without the doubling condition. In such a general definition, along with (5.1), the so-called *connectivity* property was assumed. It was proved in [12] that if a ball-basis is doubling, then the *localization* property implies the *connectivity*. It was also established that the class of BO operators involves the Calderón-Zygmund operators on general homogeneous spaces and their truncations, the maximal function, martingale transforms (nondoubling case), and the Carleson-type operators. Finally, the above paper recovers many standard estimates of classical operators for general BO operators. Those include some sharp weighted-norm estimates that were recently investigated in series of papers.

Proposition 5.2. *Let $\mathfrak{B}$ be a ball-basis satisfying the doubling property. If a $\mathrm{BO}_{\mathfrak{B}}$ operator T satisfies the weak-L^r inequality, then*

$$\mathrm{OSC}_{B,\alpha}(|Tf|) \lesssim c \cdot \langle f \rangle_B^*, \tag{5.2}$$

where $c = \mathcal{L}(T) + (1-\alpha)^{-1/r} \cdot \|T\|_{L^r \to L^{r,\infty}}$.

Proof. Let T be a BO operator. Given function $f \in L^r(X)$ and ball B, denote

$$E_{B,\lambda} = \{x \in B : |T(f \cdot \mathbb{I}_{B^*})(x)| \le \lambda\}.$$

The weak-L^r inequality of T implies

$$\mu(B \setminus E_{B,\lambda}) \le \frac{\|T\|_{L^r \to L^{r,\infty}}^r}{\lambda^r} \cdot \int_{B^*} |f|^r.$$

Thus, for an appropriate number

$$\lambda \sim (1-\alpha)^{-1/r} \cdot \|T\|_{L^r \to L^{r,\infty}} \cdot \langle f \rangle_{B^*}$$

and for $E = E_{B,\lambda}$, we have $\mu(E) > \alpha\mu(B)$ and

$$|T(f \cdot \mathbb{I}_{B^*})(y)| \lesssim (1-\alpha)^{-1/r} \cdot \|T\|_{L^r \to L^{r,\infty}} \cdot \langle f \rangle_{B^*}, \quad y \in E.$$

Take $x, x' \in E \subset B$ and suppose that $|Tf(x)| \ge |Tf(x')|$. By the definition of BO operators we have

$$\begin{aligned} |Tf(x)| - |Tf(x')| &\le |T(f\cdot \mathbb{I}_{X\setminus B^*})(x)| + |T(f\cdot \mathbb{I}_{B^*})(x)| \\ &\qquad - |T(f\cdot \mathbb{I}_{X\setminus B^*})(x')| + |T(f\cdot \mathbb{I}_{B^*})(x')| \\ &\lesssim |T(f\cdot \mathbb{I}_{X\setminus B^*})(x) - T(f\cdot \mathbb{I}_{X\setminus B^*})(x')| \\ &\qquad + (1-\alpha)^{-1/r}\cdot \|T\|_{L^r\to L^{r,\infty}}\cdot \langle f\rangle_{B^*} \\ &\le \mathcal{L}(T)\langle f\rangle_B^* + (1-\alpha)^{-1/r}\cdot \|T\|_{L^r\to L^{r,\infty}}\cdot \langle f\rangle_{B^*} \\ &\le (\mathcal{L}(T) + (1-\alpha)^{-1/r}\cdot \|T\|_{L^r\to L^{r,\infty}})\langle f\rangle_B^*. \end{aligned}$$

Clearly, this all implies (5.2). ∎

Proposition 5.3. *Let $\mathfrak{B}$ be a ball-basis in a measure space $(X, \mathfrak{M}, \mu)$. For any ball $B \in \mathfrak{B}$ and a function $f \in L^r(X)$, it holds that*

$$\langle f\rangle_B^* \le \mathrm{INF}_B\, \mathcal{M}(f) \lesssim \langle f\rangle_B^*. \tag{5.3}$$

Proof. The lefthand side of (5.3) is clear. To prove the righthand side, we denote $\lambda = \inf_{y\in B} \mathcal{M}f(y)/2$. For any $x \in B$ there exists a ball $B(x) \ni x$ such that $\langle f\rangle_{B(x)} > \lambda$. Applying Lemma 2.1, we find a sequence of pairwise disjoint balls $\{B_k\} \subset \{B(x) : x \in B\}$ such that $\bigcup_k B_k^* \supset B$. If some ball B_k satisfies $\mu(B_k) > \mu(B)$, then we have $B \subset B_k^*$, and then

$$\langle f\rangle_B^* \ge \langle f\rangle_{B_k^*} \gtrsim \langle f\rangle_{B_k} > \lambda.$$

This implies (5.3). Hence, we can suppose that $\mu(B_k) \le \mu(B)$, and so $B_k \subset B^*$ for any k. Therefore,

$$\begin{aligned} \langle f\rangle_B^* \ge \langle f\rangle_{B^*} &\ge \left(\frac{1}{\mu(B^*)}\sum_k \int_{B_k} |f|^r\right)^{1/r} \ge \lambda \left(\frac{1}{\mu(B^*)}\sum_k \mu(B_k)\right)^{1/r} \\ &\gtrsim \lambda \left(\frac{1}{\mu(B^*)}\sum_k \mu(B_k^*)\right)^{1/r} \ge \lambda. \end{aligned}$$ ∎

Corollary 5.4. *Let $(X, \mathfrak{M}, \mu)$ be a measure space equipped with a doubling ball-basis, and let T be a* BO *operator on X satisfying the weak-L^r bound, $1 \le r < \infty$. Then, for any function $f \in L^r(X)$ and ball B such that* $\operatorname{supp} f \subset B$*, we have*

$$\mu\{x \in B : |Tf(x)| > t\cdot \mathcal{M}f(x)|\} \lesssim c_T \cdot \exp(-c\cdot t)\mu(B), \quad t > 0, \tag{5.4}$$

where $c_T > 0$ is a constant depending on T.

Proof. Applying Theorem 1.7 along with (5.2) and (5.3), we will get a slightly different inequality

(5.5)
$$\mu\{x \in B : |Tf(x) - m_{T(f)}(B)| > t \cdot \mathcal{M}f(x)|\} \lesssim \exp(-c \cdot t)\mu(B), \quad t > 0.$$

Then, we denote

$$E = \{x \in B : |Tf(x)| \le \lambda \cdot \langle f \rangle_B\}, \quad \lambda = 2||T||^r_{L^r \to L^{r,\infty}}.$$

From a weak-L^r estimate we get $\mu(E) > \mu(B)/2$. By Lemma 3.1 we have

$$\mathrm{INF}_E(T(f)) \le m_{T(f)}(B) \le \mathrm{SUP}_E(T(f)),$$

which implies

$$|m_{T(f)}(B)| \le \lambda \cdot \langle f \rangle_B \le 2||T||^r_{L^r \to L^{r,\infty}} \cdot \mathcal{M}f(x), \quad x \in B. \tag{5.6}$$

From (5.5) and (5.6) one can easily obtain (5.4). ◻

Corollary 5.4 implies the following good-λ inequality.

Corollary 5.5. *Let $(X, \mathfrak{M}, \mu)$ be a measure space with a doubling ball-basis $\mathfrak{B}$, and let T be a* BO *operator on X. Then, for any function $f \in L^r(X)$, $1 \le r < \infty$, and for any $0 < \varepsilon < \varepsilon_T$, we have*

$$\begin{aligned} \mu\{x \in X : |Tf(x)| > \lambda,\ \mathcal{M}f(x) \le \varepsilon\lambda\} \lesssim \\ \lesssim c_T \exp\left(-\frac{c}{\varepsilon}\right) \cdot \mu\{x \in X : |Tf(x)| > \lambda\}, \quad \lambda > 0, \end{aligned} \tag{5.7}$$

where ε_T is a number depending on the operator T.

Proof. We can suppose that the set

$$F_\lambda = \{x \in X : |Tf(x)| > \lambda\}, \quad \lambda > 0$$

has a finite measure. We have either $\mu(F_\lambda) \ge \mu(X)/4$ or $\mu(F_\lambda) < \mu(X)/4$. In the first case, we get $\mu(X) < \infty$, and so by Lemma 2.5 we have $X \in \mathfrak{B}$. Applying Corollary 5.4 with $B = X$, we obtain

$$\begin{aligned} &\mu\{x \in X : |Tf(x)| > 2\lambda,\ \mathcal{M}f(x) \le \varepsilon\lambda\} \\ &\quad \le \mu\left\{x \in X : |Tf(x)| > \frac{\mathcal{M}f(x)}{\varepsilon}\right\} \\ &\quad \lesssim c_T \exp\left(-\frac{c}{\varepsilon}\right) \mu(X) \\ &\quad \lesssim c_T \exp\left(-\frac{c}{\varepsilon}\right) \cdot \mu\{x \in X : |Tf(x)| > \lambda\}. \end{aligned}$$

Now let us suppose that $\mu(F_\lambda) < \mu(X)/4$, and let G be an arbitrary ball. Apply Lemma 2.7 to $F = F_\lambda$ and $F' = G \cap F_\lambda$. We find balls B_k satisfying conditions (2.3), (2.4), and (2.5). We claim that

$$\mu\{x \in B_k : |Tf(x)| > 2\lambda,\ \mathcal{M}f(x) \le \varepsilon\lambda\} \le c_T \exp\left(-\frac{c}{\varepsilon}\right) \cdot \mu(B_k). \tag{5.8}$$

We can suppose that $\mathcal{M}f(\xi_k) \le \varepsilon\lambda$ for some $\xi_k \in B_k$, since otherwise (5.8) is trivial. This implies $\langle f\rangle^*_{B_k} \le \lambda\varepsilon$. Given ball B_k consider the functions

$$f_k = f \cdot \mathbb{I}_{B_k^*}, \quad g_k = f - f_k = f \cdot \mathbb{I}_{X\setminus B_k^*}.$$

From Corollary 5.4 it follows that

$$\begin{aligned}
\mu\left\{x \in B_k : |Tf_k(x)| > \frac{\lambda}{3},\ \mathcal{M}f(x) \le \varepsilon\lambda\right\} \qquad\qquad (5.9)\\
\le \mu\left\{x \in B_k^* : |Tf_k(x)| > \frac{\lambda}{3},\ \mathcal{M}f_k(x) \le \varepsilon\lambda\right\}\\
\le \mu\left\{x \in B_k^* : |Tf_k(x)| > \frac{\mathcal{M}f_k(x)}{\varepsilon}\right\} \lesssim c_T \exp\left(-\frac{c}{\varepsilon}\right) \cdot \mu(B_k).
\end{aligned}$$

Since T is a BO operator, for $0 < \varepsilon < \mathcal{L}(T)/3$ we have

$$\mathrm{OSC}_{B_k}(T(g_k)) \le \mathcal{L}(T) \cdot \langle f\rangle^*_{B_k} \le \lambda\varepsilon\mathcal{L}(T) < \frac{\lambda}{3}. \tag{5.10}$$

Applying weak-L^r inequality with $t = 3\lambda\varepsilon\|T\|_{L^r\to L^{r,\infty}}$, we have

$$\begin{aligned}
\mu\{x \in B_k : |Tf_k(x)| > t\} &\le \frac{\|T\|_{L^r\to L^{r,\infty}}}{t}\int_{B_k^*}|f|\\
&\le \frac{\|T\|_{L^r\to L^{r,\infty}}}{t}\langle f\rangle^*_{B_k} \cdot \mu(B_k^*)\\
&\lesssim \frac{\lambda\varepsilon\|T\|_{L^r\to L^{r,\infty}}}{t} \cdot \mu(B_k) < \frac{\mu(B_k)}{2}.
\end{aligned}$$

Combining this bound with (2.5), we now find a point $\eta_k \in B_k \setminus F_\lambda$ such that $|Tf_k(\eta_k)| \le t$ and $|Tf(\eta_k)| < \lambda$. Hence, by the additivity of T for $0 < \varepsilon < (9\|T\|_{L^r\to L^{r,\infty}})^{-1}$, we get

$$Tg_k(\eta_k) \le |Tf_k(\eta_k)| + |Tf(\eta_k)| \le t + \lambda < \frac{4\lambda}{3}.$$

Thus, applying (5.10), we get

$$|Tg_k(x)| \le |Tg_k(x) - Tg_k(\eta_k)| + |Tg_k(\eta_k)| \le \frac{5\lambda}{3} \quad \text{for all } x \in B_k,$$

and so by (5.9) we conclude that

$$\begin{aligned}&\mu\{x \in B_k : |Tf(x)| > 2\lambda,\ \mathcal{M}f(x) \le \varepsilon\lambda\}\\ &\quad\le \mu\left\{x \in B_k : |Tf_k(x)| > \frac{\lambda}{3},\ \mathcal{M}f(x) \le \varepsilon\lambda\right\}\\ &\quad\lesssim c_T \exp\left(-\frac{c}{\varepsilon}\right)\cdot\mu(B_k).\end{aligned}$$

Once we have (5.8), applying (2.3) and (2.4), we obtain the bound

$$\begin{aligned}&\mu\{x \in G : |Tf(x)| > 2\lambda,\ \mathcal{M}f(x) \le \varepsilon\lambda\}\\ &\quad\le \sum_k \mu\{x \in B_k : |Tf(x)| > 2\lambda,\ \mathcal{M}f(x) \le \varepsilon\lambda\}\\ &\quad\lesssim c_T \exp\left(-\frac{c}{\varepsilon}\right)\cdot\sum_k \mu(B_k) \lesssim c_T \exp\left(-\frac{c}{\varepsilon}\right)\cdot\mu(F_\lambda),\end{aligned}$$

valid for an arbitrary ball G. Choosing G to be one of the balls G_n in Lemma 2.7, and letting n go to infinity, we will get (5.7). ◻

Note that exponential inequality (5.4) for the classical Calderón-Zygmund operators on $\mathbb{R}^n$ was proved in [13]. The partial sums operators in Walsh and rearranged Haar systems was established in [14]. The Calderón-Zygmund operator version of inequality (5.7) was proved by Buckley [1]. The Hilbert transform case of this inequality goes back to the work of Hunt [9].

Now suppose we are given a family of functions

$$\Phi = \{\varphi_a \in L^\infty(\mathbb{R}^n) : \|\varphi_a\|_\infty \le 1\}_{a\in A}$$

and a Calderón-Zygmund operator T acting from $L^r(\mathbb{R}^n)$ to $L^{r,\infty}(\mathbb{R}^n)$. Let us consider the Carleson-type maximal modulated singular operator defined by

$$T^\Phi f(x) = \sup_{a\in A} |T(\varphi_a \cdot f)(x)|. \tag{5.11}$$

It was proved in [13] that T^Φ is a BO operator. Thus, from Corollary 5.4 we obtain the following result.

Corollary 5.6. *Let T^Φ be an operator of the form (5.11) acting from $L^r(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$, and let $\mathcal{M}$ be the maximal function on $\mathbb{R}^n$. Then, for any function $f \in L^r(\mathbb{R}^n)$ and ball B the inequalities*

$$\mu\{x \in B : |T^\Phi f(x)| > \lambda\cdot\mathcal{M}f(x)|\} \le c_T\cdot\exp(-c\cdot\lambda)\mu(B), \quad \lambda > 0 \tag{5.12}$$

and

$$\begin{aligned}&\mu\{x \in X : |T^\Phi f(x)| > \lambda,\ \mathcal{M}f(x) \le \varepsilon\lambda\}\\ &\quad\lesssim c_T \exp\left(-\frac{c}{\varepsilon}\right)\cdot\mu\{x \in X : |T^\Phi f(x)| > \lambda\}, \quad \lambda > 0\end{aligned} \tag{5.13}$$

hold, where $c_T > 0$ is a constant depending on T.

As we saw above, (5.12) implies (5.13). Note that inequality (5.13) with a rate of decay ε^{cr} instead of $\exp(-c/\varepsilon)$ was proved by Grafakos, Martell, and Soria in [8]. The classical example of maximal modulated singular operators is the Carleson operator

$$Cf(x) = \sup_{a\in\mathbb{R}} \left| \text{p.v.} \int_{\mathbb{T}} \frac{e^{2\pi i a t}}{2\tan(x-t)/2} f(t)\, \mathrm{d}t \right|.$$

It is well known that C is bounded on L^r for all $1 < r < \infty$ ([3], [10]). Thus, the inequalities (5.12) and (5.13) hold also for the Carleson operator. Specifically, we have the following result.

Corollary 5.7. *If C is the Carleson operator and $\mathcal{M}$ is the maximal function on unit circle $\mathbb{T}$, then for any function $f \in L^r(\mathbb{T})$ we have*

$$|\{x \in \mathbb{T} : |Cf(x)| > \lambda \cdot \mathcal{M}f(x)|\} \le c_r \cdot \exp(-c \cdot \lambda), \quad \lambda > 0, \tag{5.14}$$

and

$$\begin{aligned} \mu\{x \in \mathbb{T} : |Cf(x)| > \lambda,\ \mathcal{M}f(x) \le \varepsilon\lambda\} \\ \le c_r \exp\left(-\frac{c}{\varepsilon}\right) \cdot \mu\{x \in \mathbb{T} : |Tf(x)| > \lambda\}, \quad \lambda > 0. \end{aligned} \tag{5.15}$$

In the particular case of $f \in L^\infty(\mathbb{T})$ we have the inequality

$$\mu\{x \in \mathbb{T} : |Cf(x)| > t\} \lesssim \exp\left(-c \cdot \frac{t}{\|f\|_\infty}\right), \quad t > 0,$$

because of Sjölin [17]. Estimates analogous to (5.14), (5.15) are also valid for the Walsh-Carleson operator.

Acknowledgments Research for this paper was supported by the Science Committee of Armenia (grant no. 18T-1A081).

Many thanks are due to the referee for the valuable remarks.

REFERENCES

[1] STEPHEN M. BUCKLEY, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc. **340** (1993), no. 1, 253–272. http://dx.doi.org/10.2307/2154555. MR1124164

[2] DOUGLAS L. BURKHOLDER and RICHARD F. GUNDY, *Distribution function inequalities for the area integral*, Studia Math. **44** (1972), 527–544. http://dx.doi.org/10.4064/sm-44-6-527-544. MR340557

[3] LENNART CARLESON, *On convergence and growth of partial sums of Fourier series*, Acta Math. **116** (1966), 135–157. http://dx.doi.org/10.1007/BF02392815. MR199631

[4] RONALD R. COIFMAN, *Distribution function inequalities for singular integrals*, Proc. Nat. Acad. Sci. U.S.A. **69** (1972), 2838–2839. http://dx.doi.org/10.1073/pnas.69.10.2838. MR303226

[5] RONALD R. COIFMAN and CHARLES R. FEFFERMAN, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250. http://dx.doi.org/10.4064/sm-51-3-241-250. MR358205

[6] JAVIER CANTO and CARLOS PÉREZ, *Extensions of the John-Nirenberg theorem and applications*, Proc. Amer. Math. Soc. **149** (2021), no. 4, 1507–1525. http://dx.doi.org/10.1090/proc/15302. MR4242308

[7] CHARLES L. FEFFERMAN and ELIAS M. STEIN, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3–4, 137–193. http://dx.doi.org/10.1007/BF02392215. MR447953

[8] LOUKAS GRAFAKOS, JOSÉ MARÍA MARTELL, and FERNANDO SORIA, *Weighted norm inequalities for maximally modulated singular integral operators*, Math. Ann. **331** (2005), no. 2, 359–394. http://dx.doi.org/10.1007/s00208-004-0586-2. MR2115460

[9] RICHARD A. HUNT, *An estimate of the conjugate function*, Studia Math. **44** (1972), 371–377, Collection of Articles Honoring the Completion by Antoni Zygmund of 50 Years of Scientific Activity, IV. http://dx.doi.org/10.4064/sm-44-4-371-377. MR0338667

[10] ______, *On the convergence of Fourier series*, Orthogonal Expansions and their Continuous Analogues (Proc. Conf., Edwardsville, Ill., 1967), Southern Illinois Univ. Press, Carbondale, Ill., 1968, pp. 235–255. MR0238019

[11] BJÖRN JAWERTH and ALBERT TORCHINSKY, *Local sharp maximal functions*, J. Approx. Theory **43** (1985), no. 3, 231–270. http://dx.doi.org/10.1016/0021-9045(85)90102-9. MR779906

[12] GRIGORI A. KARAGULYAN, *An abstract theory of singular operators*, Trans. Amer. Math. Soc. **372** (2019), no. 7, 4761–4803. http://dx.doi.org/10.1090/tran/7722. MR4009440

[13] ______, *Exponential estimates for the Calderón-Zygmund operator and related problems of Fourier series*, Mat. Zametki **71** (2002), no. 3, 398–411 (Russian, with Russian summary); English transl., Math. Notes **71** (2002), no. 3–4, 362–373. http://dx.doi.org/10.1023/A:1014850924850. MR1913610

[14] ______, *Exponential estimates for partial sums of Fourier series in the Walsh system and the rearranged Haar system*, Izv. Nats. Akad. Nauk Armenii Mat. **36** (2001), no. 5, 23–34 (2002) (Russian, with English and Russian summaries); English transl., J. Contemp. Math. Anal. **36** (2001), no. 5, 19–30 (2002). MR1964580

[15] ANDREI K. LERNER, *A pointwise estimate for the local sharp maximal function with applications to singular integrals*, Bull. Lond. Math. Soc. **42** (2010), no. 5, 843–856. http://dx.doi.org/10.1112/blms/bdq042. MR2721744

[16] CARMEN ORTIZ-CARABALLO, CARLOS PÉREZ, and EZEQUIEL RELA, *Exponential decay estimates for singular integral operators*, Math. Ann. **357** (2013), no. 4, 1217–1243. http://dx.doi.org/10.1007/s00208-013-0940-3. MR3124931

[17] PER SJÖLIN, *Convergence almost everywhere of certain singular integrals and multiple Fourier series*, Ark. Mat. **9** (1971), 65–90. http://dx.doi.org/10.1007/BF02383638. MR336222

[18] ELIAS M. STEIN, *Harmonic Analysis*: *Real-variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series; Monographs in Harmonic Analysis, III, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy. MR1232192

Faculty of Mathematics and Mechanics
Yerevan State University
Alex Manoogian, 1
0025, Yerevan, Armenia
E-MAIL: g.karagulyan@ysu.am

KEY WORDS AND PHRASES: Good-λ inequality, Calderón-Zygmund operator, maximal function, sharp function, ball-basis.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 42C05, 42C10, 42C20.

Received: August 21, 2019.