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DIFFERENTIAL AND INTEGRAL EQUATIONS

An Error Estimate for the Finite Difference Scheme for One-Phase Obstacle Problem[1]

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Abstract—In this paper we consider the finite difference scheme approximation for one-phase obstacle problem and obtain an error estimate for this approximation.

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1. INTRODUCTION

The one-phase (or the classical) Obstacle Problem can be described as the problem of determination of the equilibrium state of an elastic membrane in the presence of obstacle.

Mathematically, if we assume that the membrane is spread over $\Omega \subset \mathbb{R}^n$ with boundary values φ, and the obstacle is given by $\{(x, y) \in \Omega \times \mathbb{R} : y \leq \psi(x)\} \subset \mathbb{R}^{n+1}$, then the Obstacle Problem is to minimize the energy integral

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx + \int_\Omega \lambda(x)u(x)dx$$

over the set of all admissible "deformations" of the membrane:

$$u \in \mathbb{K}_{\psi,\varphi} = \{v \in H^1(\Omega) : v - \varphi \in H_0^1(\Omega) \text{ and } v(x) \geq \psi(x) \text{ a.e. in } \Omega\}.$$

Here we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and $\varphi \in H^1(\Omega)$, $\lambda \in L^2(\Omega)$ and $\psi \in C^2(\Omega)$ are given functions.

The theoretical background of the one-phase Obstacle problem is well-studied by now (c.f. [1, 5, 7, 10, 16] and many others). Also, there is a vast literature devoted to numerical solutions of variational inequalities and, in particular, of the classical Obstacle Problem.

In this paper we focus on the Finite Difference Scheme (FDS) approximation to the solution of one-phase Obstacle problem.

The Finite Difference Scheme was extensively used for numerical solutions of variational inequalities, one-phase Obstacle problems of elliptic and parabolic type, and in particular, for American Option valuation problem (see, for example, [17]). This method is easily implementable and the rate of approximation is good enough in practice, so there is a theoretical and practical interest for investigation and mathematical justification of convergence. Surprisingly enough, there were no convergence and error estimate results up to 2006, when Cheng and Xue (see [3]) proved the quadratic convergence of the

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FDS for two-dimensional Classical Obstacle Problem under the condition, that the solution belongs to $C^4(\Omega)$. This is not a surprising result, since for $u \in C^4(\Omega)$, by the Taylor expansion, the local estimate

$$\Delta u(x) - \Delta_h u(x) = O(h^2)$$

is true (Δ_h is the FDS approximation to Δ, see below). But it is known (see [1]) that, in general, even for infinite differentiable obstacle ψ, the solution u of the Obstacle Problem belongs only to $C^{1,1}(\Omega)$. So, one cannot use the result of [3] in general.

Meanwhile, in 2009, the Finite Difference method has been applied for one-dimensional parabolic Obstacle Problem in connection with valuation of American type options (see [8]). It has been proved, that under some natural conditions, the Finite Difference Scheme converges to the exact solution and the rate of convergence is $o(\sqrt{h}+k)$. Here h and k are space- and time- discretization steps.

Here we use the technique of [8] (in fact, the idea goes back to Krylov (see [11, 12])) to prove the convergence of Finite Difference Scheme for one-phase Obstacle problem and obtain an error estimate.

2. THE FINITE DIFFERENCE SCHEME

It is well-known fact, that the classical (one-phase) Obstacle problem can be transformed to the minimization of the functional

$$\mathcal{J}(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx + \int_\Omega f(x)u(x)dx \tag{2.1}$$

over the set

$$\mathbb{K} = \{v \in H^1(\Omega) : v - g \in H_0^1,\ v(x) \geq 0 \text{ in } \Omega\} \tag{2.2}$$

where $f \in L^2(\Omega)$ and $g \in C(\overline{\Omega})$ are given functions, and the later can be formulated in fully nonlinear form as

$$\min\{-\Delta u(x) + f(x), u(x)\} = 0, \quad x \in \Omega.$$

If we denote

$$\mathcal{F}(v)(x) = \min\{-\Delta v(x) + f(x), v(x)\},$$

then the classical Obstacle problem (2.1)-(2.2) can be rewritten as

$$\begin{cases} \mathcal{F}(u) = 0 & in \ \ \Omega \\ u = g & on \ \ \partial\Omega. \end{cases} \tag{2.3}$$

Here the solution should be understood in the viscosity sense (see [2], [4] and references therein). It can be easily verified that the solution of the classical Obstacle problem solves (2.3) in the sense of viscosity solutions and a.e..

For numerical solution we consider only cases $n = 1$ or 2. In the case $n = 1$ we take $\Omega = (-1, 1)$ and if $n = 2$, we take $\Omega = (-1, 1) \times (-1, 1)$ for simplicity, keeping in mind that the method works also for more complicated domains.

Let $N \in \mathbb{N}$ be a natural, $h = 2/N$ and

$$x_i = -1 + ih,\ y_i = -1 + ih, \quad i = 0, 1, ..., N.$$

Denote

$$\mathcal{N} = \{i :\ 0 \leq i \leq N\} \quad \text{or} \quad \mathcal{N} = \{(i, j) :\ 0 \leq i, j \leq N\},$$

$$\mathcal{N}^o = \{i :\ 1 \leq i \leq N-1\} \quad \text{or} \quad \mathcal{N}^o = \{(i, j) :\ 1 \leq i, j \leq N-1\},$$

in one- or two- dimensional cases, respectively, and $\partial\mathcal{N} = \mathcal{N} \setminus \mathcal{N}^o$. Let $\Delta_h u$ be the Finite Difference Scheme approximation operator for Laplace operator, that is, for $n = 1$

$$\Delta_h u(x) = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2}$$

and for $n = 2$

$$\Delta_h u(x_1, x_2) = \frac{u(x_1 - h, x_2) + u(x_1 + h, x_2) + u(x_1, x_2 - h) + u(x_1, x_2 + h) - 4u(x_1, x_2)}{h^2}.$$

We denote $\mathcal{F}_h(v)(x) = \min\{-\Delta_h v(x) + f(x), v(x)\}$, and $x \in \Omega_h$ with $\Omega_h = \{\alpha \cdot h : \alpha \in \mathcal{N}^o\}$. We use also the following notations

$$\partial\Omega_h = \{\alpha \cdot h : \alpha \in \partial\mathcal{N}\} \quad \text{and} \quad \mathcal{H} = \{v = (v_\alpha) : v_\alpha \in \mathbb{R},\ \alpha \in \mathcal{N}\}.$$

We denote by u_h the solution of the following problem:

$$\begin{cases} \mathcal{F}_h(u_h) = 0 & in \quad \Omega_h \\ u_h = g & on \quad \partial\Omega_h. \end{cases} \tag{2.4}$$

The aim of this paper is to prove that $u_h \to u$ as $h \to 0$, where u is the solution of (2.3), and to obtain the approximation error estimate in terms of h. In the rest of the paper we assume that

$$\begin{aligned} &g \in C(\partial\Omega) \text{ and } g(x) > 0, x \in \partial\Omega; \\ &f \in C^3(\overline{\Omega}) \text{ and } f(x) > 0, x \in \Omega. \end{aligned} \tag{2.5}$$

The main result of the paper is the following

Theorem 2.1. *Let f and g satisfy* (2.5) *and u and u_h be the solutions of* (2.3) *and* (2.4)*, respectively. Then there exists a constant $C_0 > 0$, independent of h, such that*

$$|u(x) - u_h(x)| \le C_0 \cdot h^{4/11}, \quad x \in \Omega_h.$$

3. TECHNICAL LEMMAS

3.1. A comparison principle for continuous and discrete nonlinear equations

Lemma 3.1. *(Comparison Principle) Let Ω be a bounded domain and $v_1, v_2 \in W^{2,\infty}(\Omega)$. If $\mathcal{F}(v_1) \le \mathcal{F}(v_2)$ a.e. in Ω and $v_1 \le v_2$ on $\partial\Omega$, then $v_1 \le v_2$ in Ω.*

Proof. Denote

$$\Omega_1 = \{x \in \Omega |\ v_1(x) > v_2(x)\}.$$

If the set

$$\Omega_2 = \{x \in \Omega_1 : -\Delta v_1(x) > -\Delta v_2(x)\}$$

has positive Lebesgue measure, then we get a contradiction, since $\mathcal{F}(v_1)(x) > \mathcal{F}(v_2)(x)$ in Ω_2. Consequently, $-\Delta v_1(x) \le -\Delta v_2(x)$ a.e. in Ω_1. But in this case the weak maximum principle implies $v_2 \ge v_1$ in Ω_1, which is inconsistent with definition of Ω_1. Therefore, $\Omega_1 = \emptyset$. Lemma 3.1 is proved.

Lemma 3.2. *Suppose $v_1, v_2 \in \mathcal{H}$. If $\mathcal{F}_h(v_1) \le \mathcal{F}_h(v_2)$ in Ω_h and $v_1 \le v_2$ on $\partial\Omega_h$, then $v_1 \le v_2$ in Ω_h.*

For the proof of this lemma we refer to the paper [15], where the author proves the comparison principle for more general type of schemes called degenerate elliptic schemes.

3.2. Regularization and error estimate

Let $\beta \in C^\infty(\mathbb{R})$ be a function satisfying

$$\beta(z) = 0, \quad z \geq 0, \qquad \beta(z) = -z - 1, \; z \leq -2;$$

$$\beta'(z) \leq 0, \qquad \beta''(z) \geq 0, \; z \in \mathbb{R}$$

and $\beta_\varepsilon(z) = \beta\left(\frac{z}{\varepsilon}\right)$, $z \in \mathbb{R}$. We denote by u^ε the solution of the following auxiliary problem:

$$\begin{cases} -\Delta u^\varepsilon + f = \beta_\varepsilon(u^\varepsilon) & in \ \Omega, \\ u^\varepsilon = g & on \ \partial\Omega. \end{cases} \tag{3.1}$$

It follows from standard theory that there exists a unique solution u^ε to (3.1) and $u^\varepsilon \in C^4(\Omega)$. We denote

$$M = \max\left\{\max_{x\in\overline{\Omega}} f(x), 1\right\}.$$

Lemma 3.3. *If u^ε is the solution of* (3.1)*, then*

(i) $\beta_\varepsilon(u^\varepsilon(x)) \leq M$ *for all* $x \in \Omega$*;*

(ii) $u^\varepsilon(x) \geq -(M+1)\varepsilon$ *for all* $x \in \Omega$.

Proof. We treat the case $n = 2$, as the case $n = 1$ is similar.
(i) We have $u^\varepsilon(x) = g(x) > 0$ for $x \in \partial\Omega$. If $u^\varepsilon(x) \geq 0$ for every $x \in \Omega$, then by definition of β, we have $\beta_\varepsilon(u^\varepsilon) = 0 < M$. Otherwise, if

$$u^\varepsilon(x_0) = \min_{x\in\overline{\Omega}} u^\varepsilon(x) < 0,$$

then $D^2u^\varepsilon(x_0) \geq 0$ (i.e. the matrix $D^2u^\varepsilon(x_0)$ is positive semidefinite). Hence $\Delta u^\varepsilon(x_0) \geq 0$.
Now, using the fact, that β is decreasing, by definitions of u^ε and M, for any $x \in \Omega$ we get

$$\beta_\varepsilon(u^\varepsilon(x)) \leq \beta_\varepsilon(u^\varepsilon(x_0)) = -\Delta u^\varepsilon(x_0) + f(x_0) \leq f(x_0) \leq M.$$

(ii) Since $\beta(z) = -z - 1$ for $z \leq -2$, then $\beta\left(\frac{z}{\varepsilon}\right) = -\frac{z}{\varepsilon} - 1$ for $z \leq -2\varepsilon$, so $z = -\varepsilon\left(\beta_\varepsilon(z) + 1\right)$ for $z \leq -2\varepsilon$. This implies that $u^\varepsilon = -\varepsilon\left(\beta_\varepsilon(u^\varepsilon) + 1\right)$, if $u^\varepsilon \leq -2\varepsilon$, and using (i), we obtain $u^\varepsilon \geq -\varepsilon(M+1)$, if $u^\varepsilon \leq -2\varepsilon$. Otherwise, if $u^\varepsilon \geq -2\varepsilon$, we again obtain $u^\varepsilon \geq -\varepsilon(M+1)$, since $M \geq 1$ by definition. Hence, for all $x \in \Omega$, $u^\varepsilon(x) \geq -\varepsilon(M+1)$. The proof is complete.

Lemma 3.4. *For the function u^ε we have*

(i) $-(M+1)\varepsilon \leq \mathcal{F}(u^\varepsilon)(x) \leq 0$ *for all* $x \in \Omega$*;*

(ii) $0 \leq u(x) - u^\varepsilon(x) \leq \dfrac{3}{2}(M+1)\varepsilon$ *for all* $x \in \Omega$.

Proof: (i) First of all, by the definition of $\mathcal{F}$ and u^ε, we have

$$\mathcal{F}(u^\varepsilon) = \min\{-\Delta u^\varepsilon + f, u^\varepsilon\} = \min\{\beta_\varepsilon(u^\varepsilon), u^\varepsilon\}.$$

It is easy to see that $\mathcal{F}(u^\varepsilon) \leq 0$ for all $x \in \Omega$. Indeed,

- if $u^\varepsilon \geq 0$, then $\beta_\varepsilon(u^\varepsilon) = 0$, and hence $\mathcal{F}(u^\varepsilon) = \min\{0, u^\varepsilon\} = 0$;
- if $u^\varepsilon < 0$, then $\mathcal{F}(u^\varepsilon) = \min\{\beta_\varepsilon(u^\varepsilon), u^\varepsilon\} \leq u^\varepsilon < 0$.

On the other hand, using the inequality $\beta_\varepsilon(u^\varepsilon) \geq 0$ (since β is nonnegative everywhere) and (ii) of Lemma 3.3, we get

$$\mathcal{F}(u^\varepsilon) = \min\{\beta_\varepsilon(u^\varepsilon), u^\varepsilon\} \geq -(M+1)\varepsilon.$$

(ii) By (i) we have $\mathcal{F}(u^\varepsilon) \leq 0 = \mathcal{F}(u)$ in Ω and $u^\varepsilon = u$ on $\partial\Omega$. Hence, by Lemma 3.1, $u^\varepsilon \leq u$ in Ω. Denote

$$\ell(x) = \frac{(M+1)\varepsilon}{2n}\left(3n - |x|^2\right), \quad x \in \mathbb{R}^n.$$

Then

$$\Delta\ell(x) = -(M+1)\varepsilon, \quad x \in \mathbb{R}^n; \tag{3.2}$$

$$(M+1)\varepsilon \leq \ell(x) \leq \frac{3}{2}(M+1)\varepsilon, \quad x \in [-1,1]^n. \tag{3.3}$$

Now, by (3.2)

$$\mathcal{F}(u-\ell) = \min\{-\Delta u + \Delta\ell + f, u-\ell\} = \min\{-\Delta u - (M+1)\varepsilon + f, u-\ell\}$$

We consider two cases.

Case 1. If $u = 0$, i.e. u touches the obstacle, then $\min\{-\Delta u - (M+1)\varepsilon + f, u-\ell\} = \min\{-\Delta u - (M+1)\varepsilon + f, -\ell\} \leq -\ell \leq -(M+1)\varepsilon$.

Case 2. If $u > 0$, then $-\Delta u + f = 0$ and

$$\min\{-\Delta u - (M+1)\varepsilon + f, u-\ell\} = \min\{-(M+1)\varepsilon, u-\ell\} \leq -(M+1)\varepsilon.$$

This implies that $\mathcal{F}(u-\ell) \leq -(M+1)\varepsilon$, and combining with (i) of Lemma 3.3, we get $\mathcal{F}(u-\ell) \leq \mathcal{F}(u^\varepsilon)$, $\forall x \in \Omega$. If x belongs to the boundary $\partial\Omega$, we have (since $\ell(x) \geq (M+1)\varepsilon > 0$ in Ω)

$$u(x) - \ell(x) = g(x) - \ell(x) \leq g(x) = u^\varepsilon(x),$$

and using again Lemma 3.1 we obtain $u(x) - \ell(x) \leq u^\varepsilon(x)$, $\forall x \in \Omega$. So by (3.3),

$$u(x) - u^\varepsilon(x) \leq \ell(x) \leq \frac{3}{2}(M+1)\varepsilon, \quad x \in [-1,1]^n.$$

Lemma 3.4 is proved.

We also need the following Schauder estimate, which can be found in [9, p. 286].

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set and $f \in C^\alpha(\Omega)$ for some $\alpha \in (0,1)$. Then there exists a constant $C_1 > 0$ such that if v is a weak solution to $-\Delta v = f$ in Ω, then $v \in C^{2,\alpha}(\Omega)$, and*

$$\|v\|_{C^{2,\alpha}(\Omega)} \leq C_1 \cdot \left(\|f\|_{C^\alpha(\Omega)} + \|v\|_{L^2(\Omega)}\right). \tag{3.4}$$

Below we use the following weak version of this result.

Corollary 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set and $f \in C^1(\overline{\Omega})$. There exists a constant $C_2 > 0$ such that if v is a (classical) solution to $-\Delta v = f$ in Ω, then*

$$\|v\|_{C^2(\Omega)} \leq C_2 \cdot \left(\|f\|_{C^1(\Omega)} + \|v\|_{C^0(\Omega)}\right). \tag{3.5}$$

The next Theorem is an extension of the well-known interpolation type inequality of Landau-Kolmogorov for non-smooth domains and for functions of several variables, and the reader is referred to [13] (see also [14]).

Theorem 3.2. *Let $D \subset \mathbb{R}^n$ be a bounded domain having the exterior cone property (see [6]) and let $v \in L^\infty(D)$ and $D^2 v \in L^\infty(D)$. Then there exist constants $C_3 > 0$ and $C_4 > 0$ such that*

$$\|Dv\|_{C^0(D)} \leq C_3 \|D^2 v\|^{1/2}_{C^0(D)} \cdot \|v\|^{1/2}_{C^0(D)} + C_4 \|v\|_{C^0(D)}. \tag{3.6}$$

Corollary 3.2. *Under the conditions of Theorem 3.2 there exist constants $C_4, C_5 > 0$ such that*

$$\|Dv\|_{C^0(D)} \leq \delta\|D^2v\|_{C^0(D)} + \left(C_4 + \frac{C_5}{\delta}\right)\|v\|_{C^0(D)} \tag{3.7}$$

for any $\delta > 0$.

Now we proceed with our main tool in proving the convergence of the Finite Difference Scheme, namely, we need to estimate the forth-order derivatives of u^ε in terms of ε to control the difference between FD approximation of the Laplacian of u^ε and the Laplacian itself.

Lemma 3.5. *If the functions f and g satisfy* (2.5)*, and u^ε is the solution of* (3.1)*, then*

$$\left|\frac{\partial^4 u^\varepsilon(x)}{\partial x_i^4}\right| \leq \frac{C_6}{\varepsilon^{9/2}}, \quad \text{for any} \quad x \in \Omega, \quad i = 1, ..., n,$$

where the constant $C_6 > 0$ is independent of ε.

Proof. First note that we can assume $\varepsilon \in (0, 1)$, since we need to pass to the limit $\varepsilon \to 0+$.

During the proof, by C_i we denote positive constants, that *do not depend on ε*. In particular, let us denote $C_f = \|f\|_{C^3(\Omega)}$ and $C_\beta = \|\beta\|_{C^3(\mathbb{R})}$. Let u^ε be the solution to (3.1). It is known, that $u^\varepsilon \in C^4(\Omega)$. We denote $f_\varepsilon(x) = \beta_\varepsilon(u^\varepsilon(x)) - f(x)$. Then, $f_\varepsilon \in C^3(\Omega)$, and by (3.1),

$$-\Delta u^\varepsilon = f_\varepsilon \quad \text{in} \quad \Omega. \tag{3.8}$$

To obtain the fourth order partial derivative estimates for u^ε, we first differentiate twice both sides of (3.8) with respect to the variable x_i, and obtain

$$-\Delta \frac{\partial^2}{\partial x_i^2} u^\varepsilon = \frac{\partial^2}{\partial x_i^2} f_\varepsilon.$$

Hence, by (3.5),

$$\left\|\frac{\partial^2}{\partial x_i^2} u^\varepsilon\right\|_{C^2(\Omega)} \leq C_2\left(\left\|\frac{\partial^2}{\partial x_i^2} f_\varepsilon\right\|_{C^1(\Omega)} + \left\|\frac{\partial^2}{\partial x_i^2} u^\varepsilon\right\|_{C^0(\Omega)}\right) \leq$$

$$\leq C_2\left(\left\|\frac{\partial^2}{\partial x_i^2} f_\varepsilon\right\|_{C^1(\Omega)} + \|u^\varepsilon\|_{C^2(\Omega)}\right). \tag{3.9}$$

Now we need to estimate $\left\|\frac{\partial^2}{\partial x_i^2} f_\varepsilon\right\|_{C^1(\Omega)}$ and $\|u^\varepsilon\|_{C^2(\Omega)}$ to get estimate for $\left\|\frac{\partial^2}{\partial x_i^2} u^\varepsilon\right\|_{C^2(\Omega)}$, and particularly, for $\left\|\frac{\partial^4}{\partial x_i^4} u^\varepsilon\right\|_{C^0(\Omega)}$.

By the definition of f_ε,

$$\frac{\partial^2}{\partial x_i^2} f_\varepsilon = \beta_\varepsilon''(u^\varepsilon)\left(\frac{\partial u^\varepsilon}{\partial x_i}\right)^2 + \beta_\varepsilon'(u^\varepsilon) \cdot \frac{\partial^2 u^\varepsilon}{\partial x_i^2} - \frac{\partial^2 f}{\partial x_i^2}.$$

Then

$$\left\|\frac{\partial^2}{\partial x_i^2} f_\varepsilon\right\|_{C^1(\Omega)} \leq \left\|\beta_\varepsilon''(u^\varepsilon)\left(\frac{\partial u^\varepsilon}{\partial x_i}\right)^2\right\|_{C^1(\Omega)} + \left\|\beta_\varepsilon'(u^\varepsilon) \cdot \frac{\partial^2 u^\varepsilon}{\partial x_i^2}\right\|_{C^1(\Omega)} + \left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_{C^1(\Omega)}. \tag{3.10}$$

First of all

$$\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_{C^1(\Omega)} \leq \|f\|_{C^3(\Omega)} = C_f. \tag{3.11}$$

Next

$$\left\|\beta'_\varepsilon(u^\varepsilon)\cdot\frac{\partial^2 u^\varepsilon}{\partial x_i^2}\right\|_{C^1(\Omega)} = \left\|\beta'_\varepsilon(u^\varepsilon)\cdot\frac{\partial^2 u^\varepsilon}{\partial x_i^2}\right\|_{C^0(\Omega)} + \left\|D\left(\beta'_\varepsilon(u^\varepsilon)\cdot\frac{\partial^2 u^\varepsilon}{\partial x_i^2}\right)\right\|_{C^0(\Omega)} \le$$

$$\le \left\|\beta'_\varepsilon(u^\varepsilon)\right\|_{C^0(\Omega)}\cdot\left\|\frac{\partial^2 u^\varepsilon}{\partial x_i^2}\right\|_{C^0(\Omega)} + \sum_{j=1}^n\left\|\beta''_\varepsilon(u^\varepsilon)\cdot\frac{\partial u^\varepsilon}{\partial x_j}\cdot\frac{\partial^2 u^\varepsilon}{\partial x_i^2}\right\|_{C^0(\Omega)} +$$

$$+\sum_{j=1}^n\left\|\beta'_\varepsilon(u^\varepsilon)\cdot\frac{\partial^3 u^\varepsilon}{\partial x_i^2\partial x_j}\right\|_{C^0(\Omega)} \le$$

$$\le \left\|\frac{1}{\varepsilon}\beta'\left(\frac{u^\varepsilon}{\varepsilon}\right)\right\|_{C^0(\Omega)}\cdot\left\|D^2u^\varepsilon\right\|_{C^0(\Omega)} + \sum_{j=1}^n\left\|\frac{1}{\varepsilon^2}\beta''\left(\frac{u^\varepsilon}{\varepsilon}\right)\right\|_{C^0(\Omega)}\cdot\left\|Du^\varepsilon\right\|_{C^0(\Omega)}\cdot\left\|D^2u^\varepsilon\right\|_{C^0(\Omega)} +$$

$$+\sum_{j=1}^n\left\|\frac{1}{\varepsilon}\beta'\left(\frac{u^\varepsilon}{\varepsilon}\right)\right\|_{C^0(\Omega)}\cdot\left\|D^3u^\varepsilon\right\|_{C^0(\Omega)},$$

and hence

$$\left\|\beta'_\varepsilon(u^\varepsilon)\cdot\frac{\partial^2 u^\varepsilon}{\partial x_i^2}\right\|_{C^1(\Omega)} \le \frac{C_\beta}{\varepsilon}\cdot\left\|D^2u^\varepsilon\right\|_{C^0(\Omega)} +$$

$$+\frac{C_7}{\varepsilon^2}\cdot\left\|Du^\varepsilon\right\|_{C^0(\Omega)}\cdot\left\|D^2u^\varepsilon\right\|_{C^0(\Omega)} + \frac{C_8}{\varepsilon}\cdot\left\|D^3u^\varepsilon\right\|_{C^0(\Omega)} \tag{3.12}$$

Similarly, for the first summand in the right-hand side of (3.10), we have

$$\left\|\beta''_\varepsilon(u^\varepsilon)\left(\frac{\partial u^\varepsilon}{\partial x_i}\right)^2\right\|_{C^1(\Omega)} \le \frac{C_\beta}{\varepsilon^2}\cdot\left\|Du^\varepsilon\right\|^2_{C^0(\Omega)} +$$

$$+\frac{C_9}{\varepsilon^3}\cdot\left\|Du^\varepsilon\right\|^3_{C^0(\Omega)} + \frac{C_{10}}{\varepsilon^2}\cdot\left\|Du^\varepsilon\right\|_{C^0(\Omega)}\cdot\left\|D^2u^\varepsilon\right\|_{C^0(\Omega)} \tag{3.13}$$

To proceed, we will successively obtain estimates for first, second and third order derivatives of u^ε. We emphasize, that this estimates are far from being optimal.

a. Estimate for $\|D^2u^\varepsilon\|_{C^0(\Omega)}$. By (3.8) and (3.5),

$$\|u^\varepsilon\|_{C^2(\Omega)} \le C_2\left(\|f_\varepsilon\|_{C^1(\Omega)} + \|u^\varepsilon\|_{C^0(\Omega)}\right). \tag{3.14}$$

First of all, it follows from (ii), Lemma 3.4, that there exists a constant $C_{11} > 0$, independent of ε, such that

$$\|u^\varepsilon\|_{C^0(\Omega)} \le C_{11}.$$

Now, by the definition of f_ε, and keeping in mind, that β_ε is zero for nonnegative arguments,

$$\|f_\varepsilon\|_{C^1(\Omega)} = \|f_\varepsilon\|_{C^0(\Omega)} + \|Df_\varepsilon\|_{C^0(\Omega)} \le \|\beta_\varepsilon(u^\varepsilon)\|_{C^0(\Omega)} + \|f\|_{C^0(\Omega)} +$$

$$+\|\beta'_\varepsilon(u^\varepsilon)\|_{C^0(\{u^\varepsilon<0\})}\cdot\|Du^\varepsilon\|_{C^0(\{u^\varepsilon<0\})} + \|Df\|_{C^0(\Omega)} \le$$

$$\le C_f + C_\beta\left(1 + \frac{1}{\varepsilon}\cdot\|Du^\varepsilon\|_{C^0(\{u^\varepsilon<0\})}\right) = C_{12} + \frac{C_\beta}{\varepsilon}\cdot\|Du^\varepsilon\|_{C^0(\{u^\varepsilon<0\})}.$$

Substituting into (3.14), we get

$$\|u^\varepsilon\|_{C^2(\Omega)} \leq C_{13} + \frac{C_{14}}{\varepsilon} \cdot \|Du^\varepsilon\|_{C^0(\{u^\varepsilon<0\})}, \tag{3.15}$$

and, in particular,

$$\|D^2u^\varepsilon\|_{C^0(\{u^\varepsilon<0\})} \leq C_{13} + \frac{C_{14}}{\varepsilon} \cdot \|Du^\varepsilon\|_{C^0(\{u^\varepsilon<0\})}. \tag{3.16}$$

Now we take $D = \{x \in \Omega : u^\varepsilon(x) < 0\}$ and $v = u^\varepsilon$ in (3.7) to obtain

$$\|Du^\varepsilon\|_{C^0(\{u^\varepsilon<0\})} \leq \delta\|D^2u^\varepsilon\|_{C^0(\{u^\varepsilon<0\})} + \left(C_4 + \frac{C_5}{\delta}\right)\|u^\varepsilon\|_{C^0(\{u^\varepsilon<0\})}.$$

Now, by (ii) of Lemma 3.3, $\|u^\varepsilon\|_{C^0(\{u^\varepsilon<0\})} \leq (M+1)\varepsilon$, hence

$$\|Du^\varepsilon\|_{C^0(\{u^\varepsilon<0\})} \leq \delta\|D^2u^\varepsilon\|_{C^0(\{u^\varepsilon<0\})} + (M+1)\left(C_4 + \frac{C_5}{\delta}\right)\varepsilon. \tag{3.17}$$

Now we take here $\delta = \dfrac{\varepsilon}{2C_5}$ and use this inequality in (3.16). The result is

$$\|D^2u^\varepsilon\|_{C^0(\{u^\varepsilon<0\})} \leq \frac{C_{15}}{\varepsilon} \tag{3.18}$$

for some constant C_{15}. Now going back to (3.17) with $\delta = \varepsilon$, by (3.18) we get

$$\|Du^\varepsilon\|_{C^0(\{u^\varepsilon<0\})} \leq C_{16} \tag{3.19}$$

for some constant C_{16}.
It follows from (3.15) that

$$\|u^\varepsilon\|_{C^2(\Omega)} \leq \frac{C_{17}}{\varepsilon} \tag{3.20}$$

for some constant C_{17}. Consequently,

$$\|D^2u^\varepsilon\|_{C^0(\Omega)} \leq \frac{C_{17}}{\varepsilon}. \tag{3.21}$$

b. Revised Estimate for $\|Du^\varepsilon\|_{C^0(\Omega)}$. From (3.21) follows that $\|Du^\varepsilon\|_{C^0(\Omega)} \leq \frac{C_{17}}{\varepsilon}$. But using estimate (3.6) of Theorem 3.2 with $v = u^\varepsilon$ and $D = \Omega$, then, by (3.20), we get

$$\|Du^\varepsilon\|_{C^0(\Omega)} \leq \frac{C_{18}}{\sqrt{\varepsilon}}. \tag{3.22}$$

c. Estimate for $\|D^3u^\varepsilon\|_{C^0(\Omega)}$. Differentiating both sides of equality (3.8) with respect to the variable x_i, we get

$$-\Delta\frac{\partial}{\partial x_i}u^\varepsilon = \frac{\partial}{\partial x_i}f_\varepsilon,$$

and by (3.5),

$$\left\|\frac{\partial}{\partial x_i}u^\varepsilon\right\|_{C^2(\Omega)} \leq C_{19}\left(\left\|\frac{\partial}{\partial x_i}f_\varepsilon\right\|_{C^1(\Omega)} + \left\|\frac{\partial}{\partial x_i}u^\varepsilon\right\|_{C^0(\Omega)}\right) \leq$$
$$C_{19}\left(\left\|\frac{\partial}{\partial x_i}f_\varepsilon\right\|_{C^1(\Omega)} + \|Du^\varepsilon\|_{C^0(\Omega)}\right). \tag{3.23}$$

By definition of f_ε,

$$\left\|\frac{\partial}{\partial x_i}f_\varepsilon\right\|_{C^1(\Omega)} = \left\|\beta'_\varepsilon(u^\varepsilon)\cdot\frac{\partial}{\partial x_i}u_\varepsilon - \frac{\partial}{\partial x_i}f\right\|_{C^1(\Omega)} \leq \left\|\beta'_\varepsilon(u^\varepsilon)\cdot\frac{\partial}{\partial x_i}u_\varepsilon\right\|_{C^1(\Omega)} + \left\|\frac{\partial}{\partial x_i}f\right\|_{C^1(\Omega)} \leq$$

$$\leq \left\|\beta_\varepsilon'(u^\varepsilon)\cdot\frac{\partial}{\partial x_i}u_\varepsilon\right\|_{C^0(\Omega)} + \left\|D\left(\beta_\varepsilon'(u^\varepsilon)\cdot\frac{\partial}{\partial x_i}u_\varepsilon\right)\right\|_{C^0(\Omega)} + \|D^2 f\|_{C^0(\Omega)} \leq$$

$$\leq \left\|\frac{1}{\varepsilon}\beta'\left(\frac{u^\varepsilon}{\varepsilon}\right)\right\|_{C^0(\Omega)}\cdot\|Du_\varepsilon\|_{C^0(\Omega)} +$$

$$+\sum_{j=1}^{n}\left(\left\|\beta_\varepsilon''(u^\varepsilon)\cdot\frac{\partial u_\varepsilon}{\partial x_j}\cdot\frac{\partial u_\varepsilon}{\partial x_i}\right\|_{C^0(\Omega)} + \left\|\beta_\varepsilon'(u^\varepsilon)\cdot\frac{\partial^2 u_\varepsilon}{\partial x_j\partial x_i}\right\|_{C^0(\Omega)}\right) + \|D^2 f\|_{C^0(\Omega)} \leq \frac{C_{20}}{\varepsilon^3},$$

and since (3.23) is true for any $i = 1, ..., n$, we conclude that

$$\|D^3 u^\varepsilon\|_{C^0(\Omega)} \leq \frac{C_{20}}{\varepsilon^3}. \tag{3.24}$$

Finally, combining (3.9)-(3.13), (3.20), (3.21), (3.23) and (3.24), we obtain

$$\left\|\frac{\partial^4 u^\varepsilon}{\partial x_i^4}\right\|_{C^0(\Omega)} \leq \frac{C_6}{\varepsilon^{9/2}}, \quad i = 1, ..., n.$$

The proof is complete.

Lemma 3.6. *There exists a constant $C > 0$, independent of ε, such that*

$$|\Delta u^\varepsilon(x) - \Delta_h u^\varepsilon(x)| \leq \frac{C}{\varepsilon^{9/2}}h^2, \quad x \in \Omega. \tag{3.25}$$

Proof. The proof follows from Taylor expansion by standard method using Lemma 3.5.

Lemma 3.7. *Let $C > 0$ be the constant in* (3.25)*,*

$$\varphi_\varepsilon(x) = C\cdot\frac{h^2}{\varepsilon^{9/2}}\cdot\left(1 - \frac{|x|^2}{2n}\right), \quad x \in \mathbb{R}^n$$

and

$$\psi_\varepsilon(x) = K\cdot\varepsilon\cdot\sum_{i=1}^{n}(\cos x_i - \cos 1.5), \quad x \in \Omega$$

with $K = \dfrac{M+1}{n\cdot(\cos 1 - \cos 1.5)}$. If we denote

$$\Phi_h^\pm(x) = u^\varepsilon(x) \pm (\varphi_\varepsilon(x) + \psi_\varepsilon(x)), \quad x \in \mathbb{R}^n$$

then for small $h > 0$,

$$\Phi_h^-(x) \leq u_h(x) \leq \Phi_h^+(x), \quad x \in \Omega_h.$$

Proof. First of all, let us prove, that for sufficiently small $h > 0$

$$\mathcal{F}_h(\Phi_h^-) \leq 0 = \mathcal{F}_h(u_h) \leq \mathcal{F}_h(\Phi_h^+), \quad x \in \Omega_h. \tag{3.26}$$

Since

$$-\Delta_h\varphi_\varepsilon(x) = C\cdot\frac{h^2}{\varepsilon^{9/2}},$$

then by (3.25), we have

$$-\Delta_h\varphi_\varepsilon(x) \geq |\Delta u^\varepsilon(x) - \Delta_h u^\varepsilon(x)|.$$

This means that

$$-\Delta u^\varepsilon(x) \leq -\Delta_h\Big(u^\varepsilon(x) + \varphi_\varepsilon(x)\Big) = -\Delta_h\Big(\Phi_h^+ - \psi_\varepsilon\Big) = -\Delta_h\Phi_h^+ + \Delta_h\psi_\varepsilon \tag{3.27}$$

and

$$-\Delta u^\varepsilon(x) \geq -\Delta_h\Big(u^\varepsilon(x) - \varphi_\varepsilon(x)\Big) = -\Delta_h\Big(\Phi_h^- + \psi_\varepsilon\Big) = -\Delta_h\Phi_h^- - \Delta_h\psi_\varepsilon. \tag{3.28}$$

Hence,

$$\mathcal{F}(u^\varepsilon) + \psi_\varepsilon = \min\{-\Delta u^\varepsilon + f, u^\varepsilon\} + \psi_\varepsilon = \min\{-\Delta u^\varepsilon + \psi_\varepsilon + f, u^\varepsilon + \psi_\varepsilon\} =$$

$$= \min\{-\Delta u^\varepsilon - \Delta_h\psi_\varepsilon + \psi_\varepsilon + \Delta_h\psi_\varepsilon + f, u^\varepsilon + \psi_\varepsilon\}.$$

It is easy to verify, that

$$\psi_\varepsilon + \Delta_h\psi_\varepsilon \leq 0 \qquad \text{in} \quad \mathbb{R}^n \tag{3.29}$$

for small $h > 0$. Using (3.27) and (3.29) and definition of Φ_h^+, we get

$$\mathcal{F}(u^\varepsilon) + \psi_\varepsilon = \min\{-\Delta u^\varepsilon - \Delta_h\psi_\varepsilon + \psi_\varepsilon + \Delta_h\psi_\varepsilon + f, u^\varepsilon + \psi_\varepsilon\} \leq$$

$$\leq \min\{-\Delta u^\varepsilon - \Delta_h\psi_\varepsilon + f, u^\varepsilon + \psi_\varepsilon\} \leq \min\{-\Delta_h\Phi_h^+ + f, u^\varepsilon + \psi_\varepsilon\} =$$

$$= \min\{-\Delta_h\Phi_h^+ + f, \Phi_h^+ - \varphi_\varepsilon\} \leq \min\{-\Delta_h\Phi_h^+ + f, \Phi_h^+\} = \mathcal{F}_h\left(\Phi_h^+\right).$$

In the same way one can prove that

$$\mathcal{F}(u^\varepsilon) - \psi_\varepsilon \geq \mathcal{F}_h\left(\Phi_h^-\right).$$

From the last inequality, using (i) of Lemma 3.4, we get

$$\mathcal{F}_h\left(\Phi_h^-\right) \leq \mathcal{F}(u^\varepsilon) - \psi_\varepsilon \leq -\psi_\varepsilon \leq 0,$$

and

$$\mathcal{F}_h\left(\Phi_h^+\right) \geq \mathcal{F}(u^\varepsilon) + \psi_\varepsilon \geq -(M+1)\varepsilon + \psi_\varepsilon =$$

$$= -(M+1)\varepsilon + K\cdot\varepsilon\cdot\sum_{i=1}^n(\cos x_i - \cos 1.5) \geq -(M+1)\varepsilon + K\cdot\varepsilon\cdot\sum_{i=1}^n(\cos 1 - \cos 1.5) = 0$$

for all $x \in \Omega$.

Thus we have proved (3.26). Now, using nonnegativity of φ_ε and ψ_ε on $\partial\Omega_h$, we get for $x \in \partial\Omega_h$

$$\Phi_h^-(x) \leq u_h(x) \leq \Phi_h^+(x).$$

It remains to use Lemma 3.2 to complete the proof of the Lemma 3.7.

Corollary 3.3. *There exist constants $M_1, M_2 > 0$ such that for small $h > 0$*

$$|u^\varepsilon(x) - u_h(x)| \leq M_1\frac{h^2}{\varepsilon^{9/2}} + M_2\varepsilon, \quad x \in \Omega_h. \tag{3.30}$$

Proof. By Lemma 3.7,

$$|u^\varepsilon(x) - u_h(x)| \leq \varphi_\varepsilon(x) + \psi_\varepsilon(x), \quad x \in \Omega_h.$$

4. PROOF OF THEOREM 2.1

By (3.30) and Lemma 3.4, (ii)

$$|u(x) - u_h(x)| \leq |u(x) - u^\varepsilon(x)| + |u^\varepsilon(x) - u_h(x)| \leq \frac{3}{2}(M+1)\varepsilon + M_1\frac{h^2}{\varepsilon^{9/2}} + M_2\varepsilon.$$

Taking $\varepsilon = h^{4/11}$, we obtain

$$|u(x) - u_h(x)| \leq C_0\cdot h^{4/11}, \quad x \in \Omega_h$$

with $C_0 = \frac{3}{2}(M+1) + M_1 + M_2$.

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